THE FISCAL THEORY OF THE PRICE LEVEL IN OVERLAPPING GENERATIONS MODELS*

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Abstract. We demonstrate that the Fiscal Theory of the Price Level (FTPL) cannot be used to determine the price level uniquely in the overlapping generations (OLG) model. We provide two examples of OLG models, one with three 3-period lives and one with 62-period lives. Both examples are calibrated to an income profile chosen to match the life-cycle earnings process in U.S. data estimated by Guvenen et al. (2015). In both examples, there exist multiple steady-state equilibria. Our findings challenge established views about what constitutes a good combination of fiscal and monetary policies. As long as the primary deficit or the primary surplus is not too large, the fiscal authority can conduct policies that are unresponsive to endogenous changes in the level of its outstanding debt. Monetary and fiscal policy can *both* be active at the same time.

1. Introduction

This paper is about the way that monetary policy interacts with fiscal policy to determine the price level. For macro theory to act as a guide to good policy in the real world, it must first be understood in a macro model that realistically captures those features of the real world that we care about. We demonstrate that the overlapping generations (OLG) model has very different implications from the representative agent (RA) model for the determination of the price level and for the dynamics of government debt and we argue that the OLG model deserves serious consideration as an an alternative framework for policy analysis.

In Farmer and Zabczyk (2018) we showed, in a two-generation OLG model, that equilibrium debt dynamics can be self-stabilizing. In this paper we extend our previous analysis to a calibrated 62-generation OLG model. In our 62-period calibrated example, a fiscal policy that does not respond to endogenous fluctuations in debt, can safely be pursued, at least for small

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values of the primary deficit, without the fear that a policy of this kind will lead to an exploding debt level as a fraction of GDP. Our paper is the first, of which we are aware, to provide a long-lived example of a calibrated economy with an indeterminate equilibrium when monetary and fiscal policy are active.

Following a paper by Eric Leeper (1991), it is common to classify monetary and fiscal policies into *active* and *passive*.[1] In the New Keynesian (NK) model typically used to study these issues, uniqueness of equilibrium requires either that monetary policy is active and fiscal policy is passive, or that monetary policy is passive and fiscal policy is active. The fact that a passive monetary policy in combination with an active fiscal policy leads to a unique price level is referred to as the fiscal theory of the price level (FTPL).

It has been known for some time that the price level is indeterminate in rational expectations models if the monetary authority is not sufficiently aggressive in the way it adjusts the interest rate in response to observed inflation.[2] Advocates of the FTPL claim that the price level is nevertheless uniquely determined even in the extreme case when the monetary authority sets an interest rate peg. Their argument rests on a reinterpretation of the government's debt accumulation equation which is seen, not as a budget constraint, but as a debt valuation equation. The FTPL argument for price-level uniqueness is typically made in the context of the infinitely-lived representative RA model. We show that in general equilibrium models with multiple cohorts and finite lives, the debt-valuation equation has multiple solutions. We conclude that some other mechanism must be introduced into these models to completely determine prices and quantities.

At the heart of our argument is the question: Is government debt net wealth? Government bonds represent a financial asset to their holders which is offset by a financial liability of present and future taxpayers. In the special case of the RA model, the taxpayers and the owners of government liabilities are the same people. As a consequence, government debt is not net wealth, and the entire path of future real interest rates is independent of the initial level of government debt. The equivalence of debt and lump-sum taxation in models of this class is known as *Ricardian Equivalence*.[3]

[1] If the central bank raises the interest rate more than one-for-one in response to inflation, monetary policy is said to be *active*. If it raises the interest rate less than one-for-one in response to inflation, monetary policy is *passive*. If the fiscal authority borrows to finance an arbitrary path of expenditure and taxes, fiscal policy is said to be *active*. If the fiscal authority adjusts its expenditures and the tax rate to ensure fiscal solvency for all possible paths of the real interest rate, fiscal policy is *passive*.

[2] Sargent and Wallace (1975); McCallum (1981).

[3] Ricardian Equivalence (Ricardo, 1888) describes the idea that, to a first approximation, a given stream of government expenditure will have the same impact on the private economy independently of whether it is financed by debt or taxes (see also Barro (1974)). O'Driscoll (1977) has argued that Ricardian Equivalence is

In the overlapping generations model, Ricardian Equivalence is violated because those holding government debt are not the same people who repay debt through future taxes. As a consequence, a higher level of initial debt makes its holders better off by redistributing resources to current generations from future generations. In OLG models, the real interest rate may adjust to equilibrate the demand and supply for government bonds and thus provide an automatic stabilizing mechanism that restores fiscal balance that is absent from Ricardian models. In Farmer and Zabczyk (2018) we showed how this mechanism works in a two-generation example. This paper extends our previous analysis to more realistic calibrated examples.

In the remainder of the paper we establish the conditions for price-level and interest rate determinacy in a general T-generation OLG model and we present two examples of overlapping generations economies in which the FTPL fails to determine the price level uniquely. In both of our examples, fiscal policy is active. We derive equilibrium equations that describe the evolution of the real value of government debt and we show that both examples contain locally stable and dynamically efficient equilibria in which government debt converges to a positive number for arbitrary values of the initial price level.[4] Our results are established by studying the local dynamics of a linearized version of our model and they are not driven by global or non-linear dynamics as in the work of Benhabib et al. (2001, 2002).

2. The Relationship of our Work to Previous Literature

Following the 2008 financial crisis, the Federal Reserve System, the European Central Bank and the Bank of England, maintained passive interest rate rules with a constant nominal interest rate peg for more than a decade. When the central bank pegs the interest rate, standard economic theory predicts that the price level is indeterminate (Sargent and Wallace, 1975; McCallum, 1981). This is one reason why the Fiscal Theory of the Price Level that began with seminal papers by Leeper (1991), Sims (1994) and Woodford (1994, 1995, 2001) has received considerable attention in recent years. But although it has been promoted to replace monetarism as a determinant of the price level, the theory is not uncontroversial. It has attracted prominent critics, including Buiter (2002, 2017) and McCallum (1999, 2001, 2003) as well as vocal defenders (Cochrane, 2005, 2018).

A 2016 conference organized by the Becker Friedman Institute at the University of Chicago featured presentations, among others, by Eric Leeper, John Cochrane, Christopher Sims and

inconsistent with the actual views of Ricardo and a large body of empirical evidence (Poterba and Summers, 1987; Summers et al., 1987) demonstrates that Ricardian Equivalence is violated in data.

[4]An equilibrium is said to be *dynamically efficient* if the interest rate is greater than or equal to the growth rate.

Narayana Kocherlakota.[5] Speaking in a conference video, available on YouTube, Eric Leeper voiced the opinion that:

> There are basically two tasks that macro policy has to accomplish before it can do anything else: it's got to determine the price level, and the second is, it's got to make sure that fiscal policy, debt, is sustainable. Eric Leeper. (Becker Friedman Institute, 2016)

We concur. But not all researchers agree on the boundaries of discourse as they apply to the FTPL. For example Bennett McCallum and Edward Nelson, writing in their (2005) survey paper, restrict attention to economic models where Ricardian Equivalence holds. They point out, correctly, that overlapping generations models have very different properties from RA models. We have nothing to add to their discussion of the representative agent case, but we take a rather more inclusive view of the Fiscal Theory of the Price Level and we agree with Leeper's statement. A theory that guides macro policy must explain the price level and it must explain which fiscal policies are sustainable and which are not.

We take the position that for macro theory to act as a guide to good policy in the real world, it must first be understood in a macro model that realistically captures those features of the real world that we care about. Our main argument rests on the fact that overlapping generations models typically contain multiple steady-state equilibria. Importantly, the local dynamics of equilibrium price paths close to any given steady state may be indeterminate (Kehoe and Levine, 1985).

Previous examples of indeterminate equilibria in overlapping generations models have been restricted to two-generation or three-generation models in which indeterminacy was associated with the absence of money (Samuelson, 1958), negative money (Gale, 1973; Farmer, 1986), or unrealistic calibrations that were thought to be theoretical curiosities unrelated to the real world Azariadis (1981); Farmer and Woodford (1997); Kehoe and Levine (1983). Farmer (2018) constructs a version of Blanchard's (1985) perpetual youth model with an indeterminate monetary steady-state, but he assumes that monetary and fiscal policy are both passive. Our paper is the first, of which we are aware, to provide a long-lived example of a calibrated economy with an indeterminate equilibrium when monetary and fiscal policy are active.

An extensive literature uses OLG models to answer questions of political economy (Auerbach, 2003; Auerbach and Kotlikoff, 1987; Ríos-Rull, 1996), but almost all of it either ignores

[5] A YouTube presentation featuring discussions the views of each of these prominent academics can be found at Becker Friedman Institute (2016).

the possibility of indeterminacy or calibrates models to explicitly rule it out. We suspect this oversight may be related to an influential paper by Aiyagari (1985) who showed that, under some assumptions, the OLG model becomes close to the RA model as the length of life increases. Importantly, that result requires the endowments of agents to be bounded away from zero, an assumption we find to be too strong if one wishes to consider the possibility that people die. Pietro Reichlin (1992) has shown that, when one drops that assumption, OLG models display very rich behaviours even when people live potentially forever, but face a probability of death each period as in the perpetual youth model of Blanchard (1985).

There is an extensive literature that studies the wealth distribution in stochastic overlapping generations models with and without complete securities markets including work by Ríos-Rull and Quadrini (1997), Castañeda et al. (2003) and Kubler and Schmedders (2011). Our work is peripherally related to that literature but we study a different question. A number of authors have studied the existence of non-fundamental equilibria in overlapping generations models. A non-exhaustive list, following Tirole's seminal contribution (Tirole, 1985), would include papers by Martin and Ventura (2011, 2012) Miao and Wang (2012), Miao et al. (2012) and Azariadis et al. (2015). Our models contain what Tirole would call 'bubbly equilibria' but, unlike these papers, our core model is the standard complete markets overlapping generations model without the credit constraints introduced by these authors.

Eggertsson et al. (2019) study a long-lived overlapping generations model with sticky prices and they show that a calibrated model with 56 generations delivers a steady-state equilibrium with a negative steady-state real interest rate. Like the Eggertsson et al. (2019) paper, our 62-generation example contains a dynamically inefficient steady-state equilibrium with a negative steady-state real interest rate. In contrast to their work, our model has flexible prices and we focus on dynamic equilibria around a dynamically efficient steady-state. In our 62-period example, the real interest rate may be negative for decades at a time, even when the steady state equilibrium is dynamically efficient.

3. The Fiscal Theory of the Price Level

In this section we outline the idea behind the Fiscal Theory of the Price Level and we explain why it may fail to hold in overlapping generations models.

3.1. **The debt accumulation equation.**

The government purchases g_t units of a consumption good which it finances with dollar-denominated pure discount bonds and lump-sum taxes, τ_t. Let B_t be the quantity of pure-discount bonds each of which promises to pay \$1 at date $t+1$

and let Q_t be the date t dollar price of a discount bond. Let p_t be the date t dollar price of a consumption good. Using these definitions, government debt accumulation is represented by the following equation,

$$Q_t B_t + p_t \tau_t = B_{t-1} + p_t g_t.$$

Let i_t be the net nominal interest rate from period t to period $t+1$, and let Π_{t+1}, be the gross inflation rate. These variables are defined as follows,

$$i_t \equiv \frac{1}{Q_t} - 1 \qquad \text{and,} \quad \Pi_{t+1} \equiv \frac{p_{t+1}}{p_t}.$$

Further, let

$$b_t \equiv \frac{B_{t-1}}{p_t}, \tag{1}$$

be the real value of government debt maturing in period t and define the *real primary deficit* as

$$d_t \equiv g_t - \tau_t$$

where the negative of d_t is the *real primary surplus.* Letting R_{t+1} represent the gross real return from t to $t+1$, which from the Fisher-parity condition equals

$$R_{t+1} \equiv \frac{1+i_t}{\Pi_{t+1}}, \tag{2}$$

we can combine these definitions to rewrite the government budget equation in purely real terms

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{3}$$

Although Equation (3) is expressed in terms of real variables, the debt instrument issued by the treasury is nominal. It follows that the real value of debt in period 1 is determined by the period 1 price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

Advocates of the FTPL argue that Equation (3) is not a budget equation in the usual sense; it is a debt valuation equation. To understand their argument, let $\mathcal{Q}_t^k$,

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^{k} \frac{1}{R_j}, \qquad \mathcal{Q}_t^t = 1,$$

be the relative price at date t of a commodity for delivery at date k. Now, iterate Equation (3) forwards to write the current real value of debt outstanding as the present value of all future surpluses,

$$\frac{B_0}{p_1} = -\sum_{t=1}^{\infty} \mathcal{Q}_1^t d_t + \lim_{T\to\infty} \mathcal{Q}_1^T b_T. \tag{4}$$

By additionally requiring

$$\lim_{T\to\infty} \mathcal{Q}_1^T b_T \leq 0,$$

we turn Equation (4) into an inequality that some have interpreted as a budget constraint.

If the government were to be treated in the same way as other agents, Equation (4) would act as a constraint on feasible paths for the sequence of surpluses, $-\{d_t\}_{t=1}^{\infty}$, that would be required to hold for all paths of $\{\mathcal{Q}_1^t\}_{t=1}^{\infty}$ and all initial price levels, p_1. We show, in Section 9, that in New-Keynesian models in which the central bank follows a passive monetary policy, the initial price level would be indeterminate if the government were constrained to balance its budget in this way.

Advocates of the FTPL argue that government should be treated differently from other agents in a general equilibrium model. When monetary policy is passive, Equation (4) should, they claim, be treated as a debt valuation equation that determines the value of p_1 as a function of the *specific* path of primary surpluses $-\{d_t\}_{t=1}^{\infty}$ chosen by the treasury. All initial price levels, other than the specific value of p_1 that satisfies Equation (4) are infeasible since they lead to paths of government debt that eventually become unbounded.

In contrast, in the examples we construct in Sections 6 and 7, the real value of debt can remain bounded for multiple initial values of b_1. This multiplicity is possible because under the equilibrium dynamics of the real interest rate, Equation 3 defines sequences for the real value of government debt that converge to a well defined number which represents the steady-state real value of debt. Our examples imply that the logic behind the FTPL cannot be extended to models that violate Ricardian Equivalence and they suggest that indeterminacy may be more prevalent in realistically calibrated OLG models than previously believed.

4. Equilibria in an Overlapping Generations Model

Sections 4 and 5 contain our theoretical results. The reader who is interested in the practical application of our work to calibrated examples is invited to skip ahead to section 6 and 7 where we provide a 3-generation and a 62-generation model that illustrate our key findings.

To characterize equilibria we derive two sets of equations. The first set, which we refer to as *non-generic equations*, characterize market clearing in the first $T-1$ periods. The second set, which we refer to as *generic equations*, characterize market clearing in all other periods.

The first period of the model is populated by a young *generic generation* who live for T periods and by $T-1$ *non-generic generations* who live for shorter periods of time. The non-generic generations have horizons that vary from 1 period to $T-1$ periods. They can be thought of as people born before the initial period.

For example, in the two-generation OLG model there is an initial young person who lives for 2 periods and an initial old person who lives for 1 period. In this example there is one non-generic generation: the initial old. In the three-generation OLG model there is an initial young person who lives for 3 periods, an initial middle-aged person who lives for 2 periods and an initial old person who lives for 1 period. In this example there are 2 non-generic generations: the initial middle-aged and the initial old. The T-generation model generalizes this concept and, in the T-period-lived overlapping generations model, there are $T-1$ non-generic generations.

The existence of non-generic generations gives rise to a set of non-generic equations that characterize equilibrium in the first $T-2$ periods. These non-generic equations represent constraints on the initial state for a vector-valued difference equation that describes the time path of the real interest rate and the real value of government debt for all periods $T-1$ and later. We refer to the equations that make up this vector-valued difference equation as the generic equilibrium equations. To describe the generic equilibrium equations we turn first to the decision problem faced by a generic consumer.

4.1. **The Generic Consumer's Problem.** We refer to a person born in period t as *generation* t and we use a superscript on a variable to denote generation. A subscript denotes calendar time. For example, c_τ^t is consumption of generation t in period τ. We assume endowments are independent of calendar time and we index them by age. If t is generation, τ is calendar time and s is age then s is related to t and τ by the identity $s \equiv \tau - t + 1$.

Generation t has a utility function defined over consumption in periods $\{t, \ldots, t+T-1\}$ and it solves the problem

$$\max_{\{c_t^t, \ldots, c_{t+T-1}^t\}} U^t(c_t^t, c_{t+1}^t, \ldots, c_{t+T-1}^t),$$

such that

$$\sum_{k=t}^{t+T-1} \mathcal{Q}_t^k (c_k^t - \tilde{w}_s) \leq 0, \quad s = 1 - t + k,$$

where recall that

$$\mathcal{Q}_t^k \equiv \prod_{j=t+1}^{k} \frac{1}{R_j}, \qquad \mathcal{Q}_t^t = 1,$$

are present-value prices.[6]

The solution to this problem is characterized by a set of T excess demand functions, one for each period of life;

$$c^t_k - \tilde{w}_s \equiv \tilde{x}^t_k(R_{t+T-1}, R_{t+T-2}, \ldots, R_{t+1}), \quad \text{for} \quad k \in \{t, \ldots, t+T-1\}.$$

To smooth their consumption, generation t trade in the asset markets by buying or selling one-period (nominal) government bonds. Corresponding to the T excess demand functions there are $T-1$ savings functions $\{s^t_k\}^{t+T-2}_{k=t}$ which are computed recursively from the excess demand functions using the following expressions,

$$\left.\begin{array}{lcl} s^t_t & = & 0, \\ s^t_{k+1} & = & R_{k+1}s^t_k + \tilde{w}_{k-t+1} - \tilde{x}^t_k, \quad k = t, \ldots, t+T-2. \end{array}\right\} \tag{5}$$

There are $T-1$, rather than T of these savings functions because we assume no bequest motive and hence the optimal amount to save in the T'th period of life is zero. We express the dependence of the savings functions on the sequence of interest factors by the notation,

$$s^t_{k+1}(R_{t+T-1}, R_{t+T-2}, \ldots, R_{t+1}), \quad k = t, \ldots, t+T-2. \tag{6}$$

Next, we turn to the generic equilibrium equations which we derive by adding up the savings functions of all people alive at date t and equating aggregate asset demand to the real value of government bonds outstanding.

4.2. **The Generic Equilibrium Equations.** There are two generic equilibrium equations. The first generic equilibrium equation is an expression for the private demand for government bonds that we characterize by a function $f(\cdot)$. This function depends on interest factors in the past and interest factors in the future. By equating this expression to the supply of government bonds, (this equals the real value of maturing debt plus the primary fiscal deficit), we arrive at the equation.

$$b_t + d_t = f(R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+4}, R_{t-T+3}). \tag{7}$$

The function $f(\cdot)$ that appears in Equation (7) represents the sum of savings functions, defined in Equation (6), of all generations except that of age $s = T$, which is in the final period of life and does not participate in asset trade. Those generations that do participate include the cohort of age $s = T-1$ which has 1 more period to live and the generation of age $s = 1$ which

[6]To simplify the exposition we assume perfect foresight; this cuts on notation and is without loss of generality as long as asset markets are complete in the corresponding stochastic setup.

has T more periods to live. Asset market participants may be borrowers or lenders depending on their age, their endowment pattern and their preferences.[7]

The second generic equilibrium equation is an expression for the evolution of the supply of government bonds. We met this expression in Section 3.1, Equation (3), where we referred to it as the debt accumulation equation. We reproduce Equation (3) below for completeness,

$$b_{t+1} = R_{t+1}(b_t + d_t). \tag{8}$$

4.3. **Steady State Equilibria.** The perfect foresight competitive equilibria of a T-generation OLG model are fully characterized by sequences of interest factors and government debt that satisfy equations (7) and (8), along with corresponding conditions that are derived from the behaviour of non-generic generations in the first $T-2$ periods. To characterize these sequences we follow much of the literature by studying the local properties of equilibria around a steady state.

The steady state equilibria of the model are constant equilibrium sequences of interest factors and government debt. More precisely, a steady state equilibrium is a non-negative real number $\bar{R}$ and a (possibly negative) real number $\bar{b}$ that solve the steady state equations

$$\bar{b} + d = f(\bar{R}, \bar{R}, \ldots, \bar{R}), \quad \bar{b}\,(1 - \bar{R}) = \bar{R}\, d.$$

This is a highly non-linear system of equations that generically has at least two solutions. In the important special case when $d = 0$ one of these solutions is always given by $\bar{R} = 1$ and the others are solutions to the equation $f(\bar{R}, \bar{R}, \ldots, \bar{R}) = 0$.[8]

4.4. **Local Dynamic Equilibria.** Let $\{\bar{R}, \bar{b}\}$ be a steady state equilibrium and let

$$\tilde{R}_t \equiv R_t - \bar{R}, \quad \text{and} \quad \tilde{b}_t \equiv b_t - \bar{b},$$

represent deviations of b_t and R_t from their steady state values. Define a vector

$$X_t \equiv [R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+4}, b_{t+1}]^{\top},$$

[7] Those of age $T-1$ were born in period $\tau = t - T + 2$. These people form plans that depend on the real interest factors from R_{t-T+3} to R_t. They have the longest dependence of saving behaviour on the past and they are responsible for the term R_{t-T+3} in the aggregate savings function $f(\cdot)$. Those of age 1 were born in period t. These people form plans that depend on the real interest factors from R_{t+1} to R_{t+T-1}. They have the longest dependence of saving behaviour on the future and they are responsible for the term R_{t+T-1} in the savings function $f(\cdot)$.

[8] In a classic paper on the overlapping generations model David Gale referred to the first of these solutions as the golden-rule steady state and the second as a generationally autarkic steady states (Gale, 1973).

and a function $F(\cdot)$,

$$F(X_t, X_{t-1}) \equiv \begin{bmatrix} b_t + d_t & - & f(R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+3}) \\ b_{t+1} & - & R_{t+1}(b_t + d_t) \end{bmatrix},$$

and let J_1 and J_2 represent the partial derivatives of this function with respect to X_t and X_{t-1}.

Using these definitions, consider the following matrix expressions,

$$J_1 X_t \equiv \overbrace{\begin{bmatrix} -f_{t+T-1} & -f_{t+T-2} & \cdots & -f_{t+2} & \cdots & -f_{t-T+5} & -f_{t-T+4} & 1 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}}^{J_1} \overbrace{\begin{bmatrix} \tilde{R}_{t+T-1} \\ \tilde{R}_{t+T-2} \\ \vdots \\ \tilde{R}_{t+2} \\ \vdots \\ \tilde{R}_{t-T+5} \\ \tilde{R}_{t-T+4} \\ \tilde{b}_{t+1} \end{bmatrix}}^{\tilde{X}_t},$$

$$J_2 X_{t-1} \equiv \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & f_{t-T+3} & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \ddots & \vdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \bar{b} + d & \cdots & 0 & 0 & \bar{R} \end{bmatrix}}^{J_2} \overbrace{\begin{bmatrix} \tilde{R}_{t+T-2} \\ \tilde{R}_{t+T-3} \\ \vdots \\ \tilde{R}_{t+1} \\ \vdots \\ \tilde{R}_{t-T+4} \\ \tilde{R}_{t-T+3} \\ \tilde{b}_t \end{bmatrix}}^{\tilde{X}_{t-1}}.$$

In these expressions, f_k is the partial derivative of the function f with respect to R_k evaluated at the steady state $\{\bar{R}, \bar{b}\}$. Using this notation, the local dynamics of equilibrium $\{R_t, b_t\}$ sequences close to this steady state can be approximated as solutions to the linear difference equation

$$\begin{aligned} J_1 \tilde{X}_t &= J_2 \tilde{X}_{t-1}, \quad t = T-1, \ldots \\ \tilde{X}_{T-2} &= \bar{X}_{T-2}. \end{aligned} \tag{9}$$

The local stability of these equations depends on the eigenvalues of the matrix

$$J \equiv J_1^{-1} J_2.$$

In Section 5 we will show how to derive a set of restrictions on the vector of initial values $\bar{X}_{T-2}$ that eliminates the influence of unstable eigenvalues of J on the behaviour of the equilibrium sequence $\{X_t\}$ generated by Equation (9). Next, we turn to a description of the problems faced, in period 1, by the non-generic cohorts.

4.5. **The Non-Generic Consumer's Problem.** Let j be an index that runs from 1 to $T-1$. We assume that in period 1, government has an outstanding liability of B_0 dollars and that generation $1-j$ has assets or liabilities of μ_{1-j} dollars. We will also need notation for the initial real wealth of agent $1-j$ and for the real value of government liabilities. We use the symbols ν_{1-j} and b_1 for these variables,

$$\nu_{1-j} \equiv \frac{\mu_{1-j}}{p_1}, \quad \text{and} \quad b_1 \equiv \frac{B_0}{p_1}.$$

We refer to the vector that characterizes the initial wealth distribution with the symbol $\mathcal{N}$,

$$\mathcal{N} \equiv \{\nu_0, \nu_{-1}, \ldots, \nu_{2-T}, b_1\}.$$

We further assume that

$$\sum_{j=1}^{T-1} \mu_{1-j} = B_0, \quad \text{and} \quad \sum_{j=1}^{T-1} \nu_{1-j} = b_1.$$

For example, when $T = 3$, the sum of the dollar denominated assets of the middle-aged and the old equals the dollar denominated liability of the government and the period 1 real value of private wealth is equal to the period 1 real value of government debt,

$$\mu_0 + \mu_{-1} = B_0, \quad \nu_0 + \nu_{-1} = b_1.$$

Using these definitions we proceed to show how the initial wealth distribution enters the model by influencing the decision problems of the non-generic generations. These generations solve the problem,

$$\max_{\left\{c_1^{1-j}, \ldots, c_{1-j+T-1}^{1-j}\right\}} U^{1-j}\left(c_1^{1-j}, c_2^{1-j}, \ldots, c_{1-j+T-1}^{1-j}\right),$$

such that

$$\sum_{k=1}^{(1-j)+T-1} \mathcal{Q}_1^k (c_k^{1-j} - \tilde{w}_s) \le \nu_{1-j}, \quad s = 1 - (1-j) + k,$$

where s is the age of a member of generation $1-j$ in period k. The solution to each generation's maximization problem is characterized by a set of excess demand functions, one for each remaining period of their lives. These excess-demand functions depend not just on the sequence of interest factors but also on initial asset positions. In the next section, we explain how the non-generic equilibrium equations place restrictions on the initial condition of the function $F(\cdot)$.

4.6. **The Non-Generic Equilibrium Equations.** For each period $1, \ldots, T-2$ there are two sets of non-generic equilibrium equations. The first set are aggregate asset market equilibrium conditions. The second set are expressions for the government debt accumulation equations. The debt accumulation equations are the same as those that characterize debt accumulation in the generic periods. The non-generic aggregate asset demand equations are, however, different from the generic case.

The non-generic aggregate asset demand equations are characterized by families of functions $g_{t+1}^{T}(\cdot)$ for $t \in \{1, \ldots, T-2\}$, one family for each value of T. Unlike the function $f(\cdot)$ that characterizes the private demand for government bonds in periods $T-1$ and later, the functions $g_{t+1}^{T}(\cdot)$ are different for every t. The arguments of these functions are sequences of interest factors $\{R_t\}_{t=2}^{2T-3}$ and initial wealth holdings $\{\nu_{1-j}\}_{j=1}^{2-T}$ and the specific elements of these sequences that enter the functions $g_{t+1}^{T}(\cdot)$ are different for every value of t and T.

For example, when $T=3$ there is a single non-generic function $g_2^3(\cdot)$ that characterizes the private demand for government bonds in period 1,

$$b_1 + d = g_2^3\left(R_3, R_2, \nu_0\right),$$

and a single debt accumulation equation

$$b_2 = R_2\left(b_1 + d\right).$$

These two equations place two restrictions on the initial unknowns that we collect together into a vector Z_0,

$$Z_0 \equiv \left[R_3, R_2, b_2, b_1\right]^{\top} \equiv \left[X_{T-2}, Y_{T-2}\right]^{\top}.$$

X_{T-2} is the vector of initial conditions to the difference equation $F(X_t, X_{t-1}) = 0$ and Y_{T-2} is a vector of real values of government debt in the first $T-2$ periods. For the case $T=3$, Y_{T-2} consists of a single element,

$$Y_{T-2} = b_1 \equiv \frac{B_0}{p_1}.$$

The variable b_1 is an initial condition of the model and although we have stated it in terms of the real variable b_1, it is free to be determined in period 1 by adjustments in the price level, p_1. By stating the initial condition in terms of b_1 rather than the nominal variable B_0 we are able to model the case when $b_1 = 0$. Choosing p_1 as an initial condition would be problematic when $b_1 = 0$ because $p_1 \equiv \frac{B_0}{b_1}$ is either infinite, if $B_0 \neq 0$, or undefined if $B_0 = 0$.

We represent the initial restrictions placed on Z_0 by the non-generic equilibrium equations by defining a function $G(\cdot)$ and equating this function to 0,

$$G(Z_0) \equiv \begin{bmatrix} b_1 + d & - & g_2^3 (R_3, R_2, \nu_0) \\ b_2 & - & R_2 (b_1 + d) \end{bmatrix} = 0.$$

When $T = 3$, the equation $G(\cdot) = 0$ place 2 restrictions on the four elements of Z_0.

Consider next the case when $T = 4$. The vector Z_0 for this example is

$$Z_0 \equiv [R_5, R_4, R_3, R_2, b_3, b_2, b_1]^\top \equiv [X_{T-2}, Y_{T-2}]^\top,$$

and Y_{T-2} is the vector

$$Y_{T-2} \equiv [b_2, b_1]^\top.$$

When $T = 4$ there are two non-generic asset demand equations and two non-generic debt accumulation equations, one for each of the two initial periods. These equations are described by the expression

$$G(Z_0) \equiv \begin{bmatrix} b_1 + d & - & g_2^4 (R_4, R_3, R_2, \nu_0, \nu_{-1}) \\ b_2 + d & - & g_3^4 (R_5, R_4, R_3, R_2, \nu_0) \\ b_2 & - & R_2 (b_1 + d) \\ b_3 & - & R_3 (b_2 + d) \end{bmatrix} = 0.$$

The function $g_2^4(\cdot)$, which appears in the first row of $G(\cdot)$ when $T = 4$, has additional arguments to the function g_2^3 because people who live for 3 periods only care about 2 interest factors, R_2 and R_3, whereas people who live for 4 periods also care about R_4. In addition to containing an extra interest factor, the function g_2^4 includes an additional initial wealth variable to the function $g_2^3(\cdot)$. This is because generations 0 and -1 both participate in the period 1 asset market in contrast to the $T = 3$ model where only generation 0 participates in this market. When $T = 4$, the equation $G(\cdot) = 0$ place 4 restrictions on the 7 elements of Z_0.

These examples can be generalized. For the T-generation model, Z_0 contains $3T - 5$ elements, equal to the sum of the terms in the over-braces of the following expression,

$$Z_0 \equiv [\overbrace{R_{2T-3}, R_{2T-4}, \ldots, R_2, b_{T-1}}^{2T-3}, \overbrace{b_{T-2}, \ldots, b_2, b_1}^{T-2}]^\top \equiv [X_{T-2}, Y_{T-2}]^\top.$$

The function $G(\cdot)$ contains $2T-4$ rows,

$$G(Z_0) \equiv \begin{bmatrix} b_1+d & - & g_2^T\left(R_{1+T-1}, R_{1+T-2}, \ldots, R_2, \nu_0, \nu_{-1}, \ldots, \nu_{3-T}\right) \\ \vdots & & \vdots \\ b_{T-2}+d & - & g_{T-1}^T\left(R_{2T-3}, R_{2T-4}, \ldots, R_2, \nu_0\right) \\ \cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_2 & - & R_2\left(b_1+d\right) \\ \vdots & & \vdots \\ b_{T-1} & - & R_{T-1}\left(b_{T-2}+d\right) \end{bmatrix} = 0,$$

and the equation $G(\cdot) = 0$ places $2T-4$ restrictions on the $3T-5$ elements of the unknown vector Z_0. Subtracting the number of these restrictions from the number of initial variables leaves $T-1$ non-predetermined elements of X_{T-2}. In the following section we describe how the non-generic equilibrium equations may be combined with the assumption that equilibrium sequences $\{R_t, b_t\}$ must remain bounded to characterize the determinacy properties of equilibria.

5. The Determinacy Properties of Equilibria

In this section we linearize the function $G(Z_0)$ in the neighbourhood of the steady state $\{\bar{R}, \bar{b}\}$ and we use the linearized function to find a set of linear restrictions on the initial vector X_{T-2}. We combine these restrictions with a second set of linear restrictions on the vector X_{T-2} that arise from the assumption that equilibrium sequences of interest factors and government debt must remain bounded.

5.1. **Restrictions on X_{T-2} from the Non-Generic Equilibrium Equations.** The function $G(\cdot)$ depends not only on X_{T-2} but also on the initial wealth distribution $\mathcal{N}$. For arbitrary values of $\mathcal{N}$ there is no guarantee that the restrictions placed on X_{T-2} by the equation $G(Z_0) = 0$ are consistent with the model remaining in the neighbourhood of the steady state $\{\bar{R}, \bar{b}\}$. To ensure that we are indeed close to a steady state, we choose the initial wealth distribution by selecting

$$\nu_{1-j} = \bar{R} \cdot s_1^{1-j}(\bar{R}, \bar{R}, \ldots, \bar{R}), \quad j = 1, \ldots, T-1,$$

where s_1^{1-j} is the savings function defined in Equation (6) evaluated at the steady state. This choice ensures that the initial generation of age s in period 1 starts off with the same wealth as an arbitrary generation of age s in the steady state equilibrium $\{\bar{R}, \bar{b}\}$.

Let G_X, G_Y and $G_{\mathcal{N}}$ be the Jacobians of the function $G(\cdot)$ with respect to X_{T-2}, Y_{T-2} and $\mathcal{N}$ evaluated at the steady state equilibrium $\{\bar{R}, \bar{b}\}$ and let $\tilde{X}_{T-2}$, $\tilde{Y}_{T_2}$ and $\tilde{\mathcal{N}}$ be deviations of these vectors from their steady state values. Using this notation, the predetermined equilibrium conditions place the following $2T-4$ restrictions on the $3T-5$ unknowns $Z_0 \equiv [X_{T-2}, Y_{T-2}]$.

$$G_X \tilde{X}_{T-2} + G_Y \tilde{Y}_{T-2} + G_{\mathcal{N}} \tilde{\mathcal{N}} = 0. \tag{10}$$

By perturbing the vector $\mathcal{N}$ we can study the return to the steady state from an arbitrary initial wealth distribution.

5.2. **Restrictions on X_{T-2} from the Generic Equilibrium Equations.** In Section 4.4 we approximated sequences of interest factors and government debt that satisfy the equation $F(\cdot) = 0$ by a linear approximation that holds close to the steady state. We reproduce that approximation below

$$\tilde{X}_{t-1} = J\tilde{X}_{t-1}, \quad t = T-1, \ldots, \infty, \tag{11}$$

with

$$\tilde{X}_{T-2} = \bar{X}_{T-2}.$$

If one or more roots of the matrix J is outside the unit circle there is no guarantee that sequences of interest factors and government debt generated by Equation (11) will remain bounded. To ensure stability, we must choose initial conditions that place $\tilde{X}_{T-2}$ in the linear subspace associated with the stable eigenvalues of J. These restrictions can be expressed theoretically by exploiting the properties of the Jordan decomposition of J,

$$J = Q\Lambda Q^{-1},$$

where Λ is an upper triangular matrix with the eigenvalues of J on the diagonal and Q is the matrix of left eigenvectors of J. The restrictions that prevent X_t from becoming unbounded are found by premultiplying X_{T-2} by the rows of Q^{-1} associated with the unstable eigenvalues of J and equating the product to zero. We refer to the matrix that contains these rows as Q_u^{-1}.[9]

Let K be the number of eigenvalues of J that lie outside the unit circle. The requirement that equilibrium sequences remain bounded places K linear restrictions on the initial vector X_{T-2} which we express with the matrix equation

$$Q_u^{-1} X_{T-2} = 0.$$

[9]In practice, the Jordan decomposition is numerically unstable but there are several good computational methods to compute Q_u^{-1} that are implemented in all modern programming languages. The reader is referred to Golub and VanLoan (1996) for a description of the algorithms used to solve problems of this kind. Farmer (1999) provides an accessible introduction to solution methods for rational expectations models with indeterminate equilibria.

Q_u is a $K \times 2T-3$ matrix.

The uniqueness of equilibrium then comes down to the question of the number of solutions to the equations

$$\overset{(2T-4+K)\times(3T-5)}{\begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix}} \overset{(3T-5)\times 1}{\begin{bmatrix} \tilde{X}_{T-2} \\ \tilde{Y}_{T-2} \end{bmatrix}} = \overset{(2T-4+K)\times(3T-5)}{\begin{bmatrix} -G_{\mathcal{N}} \\ 0 \end{bmatrix}} \overset{(3T-5)\times 1}{\begin{bmatrix} \tilde{\mathcal{N}} \\ 0 \end{bmatrix}}. \tag{12}$$

It follows immediately that a sufficient condition for the existence of a unique equilibrium is that the matrix

$$\mathcal{M} \equiv \begin{bmatrix} G_X & G_Y \\ Q_u^{-1} & 0 \end{bmatrix},$$

is square and non singular.

If $K > T-1$ then $\mathcal{M}$ has more rows than columns and, except for the special case where the rows of $\mathcal{M}$ are linearly dependent, Equation (12) has no solution.

If $K = T-1$ then $\mathcal{M}$ is square. If $\mathcal{M}$ also has full row rank, there exists a unique solution to this equation given by the expression

$$\begin{bmatrix} X_{T-2} \\ Y_{T-2} \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} -G_{\mathcal{N}}\tilde{\mathcal{N}} \\ 0 \end{bmatrix}.$$

Finally, if $K \leq T-1$ then $\mathcal{M}$ has fewer rows than columns. In this case we are free to append a $(T-1-K) \times (3T-5)$ matrix of linear restrictions to the Equation (12). For example, if $T-1-K = 1$ we can append a row that sets

$$b_1 = \bar{b}_1.$$

In words, one degree of indeterminacy allows us to choose an arbitrary initial debt position and by implication, an arbitrary initial price level. By setting d equal to a constant, (possibly zero), we have imposed the assumption that fiscal policy is active. It follows that, if the equilibrium has at least one degree of indeterminacy, we cannot use the fiscal theory of the price level to pin down the price level.

If the degree of indeterminacy is greater than 1, we are free also to pick an arbitrary initial real rate of interest from the initial vector $R_2, R_3, \ldots, R_{2T-3}$. In this case the economy exhibits both real and nominal indeterminacy. In sections 6 and 7 we will present realistically calibrated economies in which there exist steady state equilibria with both one and two degrees of indeterminacy. Our discussion can be summarized in the following proposition.[10]

Proposition 1. *Assume that $\mathcal{M}$ has full row rank. Let K denote the number of eigenvalues of J with modulus greater than 1.*

- *If $K > T-1$ there are no bounded sequences that satisfy the non-generic equilibrium conditions in the neighbourhood of $\bar{X}$. In this case equilibrium does not exist.*
- *If $K = T-1$ there is a unique bounded sequence that satisfies the non-generic equilibrium equations. Further, this sequence converge asymptotically to the steady state $\bar{X}$. In this case the steady state equilibrium $\bar{X}$ is determinate.*
- *If $K \in \{0, \ldots, T-2\}$ there is a $T-1-K$ dimensional subspace of initial conditions that satisfy the non-generic equilibrium equations. All of these initial conditions are associated with sequences that converges asymptotically to the steady state $\bar{X}$. In this case the steady state equilibrium $\bar{X}$ is indeterminate with degree of indeterminacy equal to $T-1-K$.*

6. A three-generation example

In this section we present a three-generation example of an economy in which the FTPL breaks down. For the particular calibration we highlight here there are two steady states, both of which are indeterminate. The first steady state displays one degree of indeterminacy and steady-state government debt of zero. The second steady state displays two degrees of indeterminacy and steady-state government debt of 5% of GDP.

6.1. **Multiplicity and Determinacy.**

Our three generation example is inspired by Kehoe and Levine (1983) who provide a three-generation OLG model with CES preferences, an endowment profile of $[3, 15, 2]$ and utility weights on the three periods of life of $[2, 2, 1]$. Their example displays four steady states, two of which display one degree of indeterminacy, one of which is determinate and one of which displays second degree indeterminacy.

To see if the Kehoe-Levine example might provide a plausible explanation of a real-world economy, we tried calibrating the income profile to U.S. data and modifying the preference weights to allow for a constant rate of discounting. The key features of their example are the

[10] These are the Blanchard-Kahn conditions, (Blanchard and Kahn, 1980). An updated solution method for rational expectations models that allows either J_1 or J_2 or both to be singular can be found in Sims (2001). Solution methods for models with indeterminacy are provided in Farmer et al. (2015).

hump-shaped income profile and a coefficient of relative risk aversion of 5 which is well within the bounds of calibrated models in the macro-finance literature.

In our three-generation example, people maximize the utility function,

$$u\left(c_t^t, c_{t+1}^t, c_{t+2}^t\right) = \sum_{i=1}^{3} \beta^{i-1} \left(\frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right).$$

The period length is 20 years, with the first two periods corresponding to active employment, ages $20 - 39$ and ages $40 - 60$, and the final period, ages $61 - 79$, representing retirement. We chose the after-tax endowment profile to be $[1, 1.61, 0.08]$. The first two of these numbers are based on the U.S. income profile estimated in Guvenen et al. (2015) and the last is calibrated to Supplemental Security Income payments.[11] We set the annual discount factor equal to 0.9 and a per-period value for β of 0.9 raised to the power 20. The curvature parameter α equals -4, which implies an elasticity of substitution of $1/(1-\alpha) = 0.2$ or a coefficient of relative risk aversion of 5 which coincides with the Kehoe-Levine example.

For these parameter values, Figure 1 graphs the steady-state excess demand for goods as a function of the log of $\bar{R}$ in the upper panel and government debt at the steady state as a percentage of GDP in the lower panel. We see from this figure that our model admits two steady states; one with a negative interest rate and one with an interest rate of 0. The fact that the interest rate is negative in the first steady state implies that this equilibrium is dynamically inefficient.

In Table 1 we list the two steady-state values of $\bar{R}$ together with the real value of government bonds in the steady state and the number of unstable roots. As Table 1 makes clear, the first steady state is first-degree indeterminate and the second one is second-degree indeterminate. It is the second steady state, that we focus on in the remainder of this section.

Equilibrium Real Interest rates					
Type	Value of R	Value of b	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Autarkic Steady-State	0.002	0	1	2	1
Golden-rule Steady-State	1	5% of GDP	0	2	2

Table 1. Steady States of our Three-Generation Example

[11]This is an aggregated example of the 62-generation profile that we discuss in more depth in Section 7.

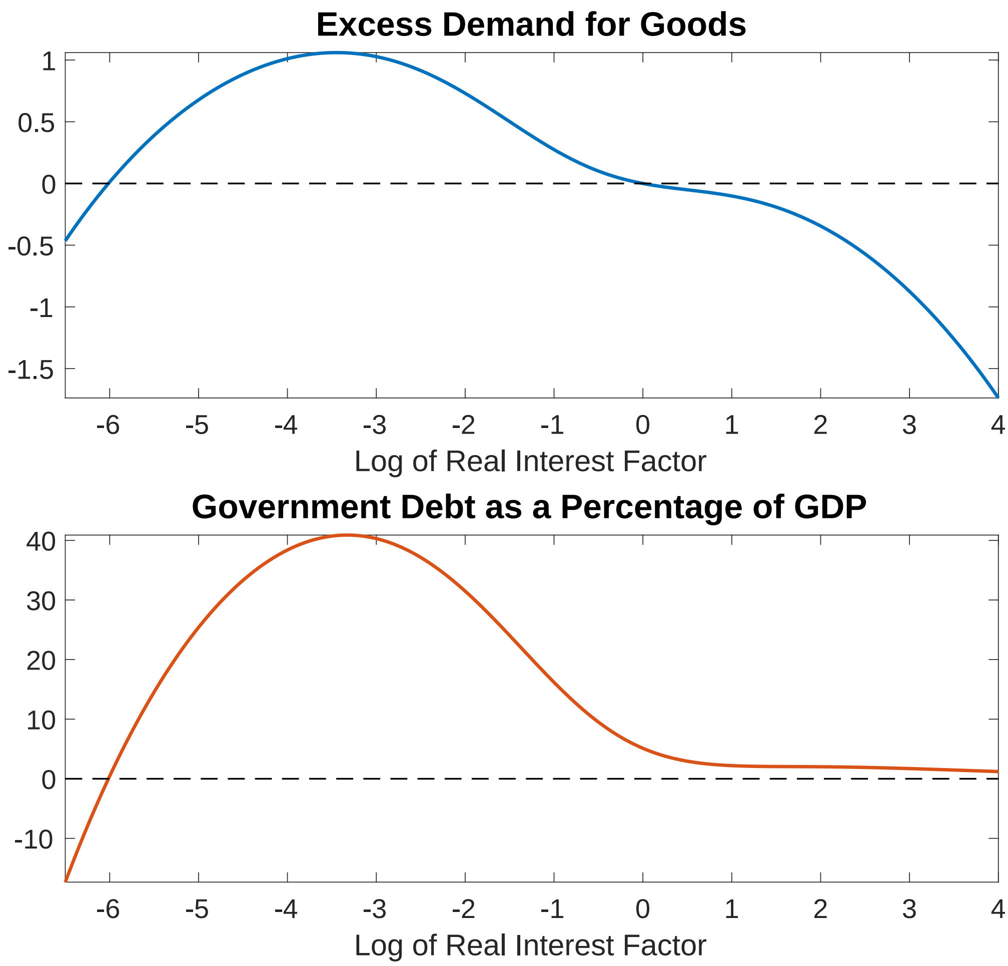

Figure 1. Steady States in the Three-Generation Example

In the three-generation model there are two constraints on the initial conditions of the generic difference equation that begins in period 2. These are given by the period 1 bond-market clearing equation,

$$b_1 + d = g_2^3(R_3, R_2, \nu_0)$$

and the period 1 debt-accumulation equation

$$b_2 = R_2(b_1 + d).$$

The variable b_1 depends on the value of B_0 and the endogenous initial period price level through the definition

$$b_1 \equiv \frac{B_0}{p_1}.$$

These equations place two restrictions on the vector

$$Z_0 \equiv [R_3, R_2, b_2, b_1] \equiv [X_{T-2}, Y_{T-2}] \equiv [X_1, Y_1],$$

leaving two of the variables R_3, R_2, b_2 and b_1 to be freely chosen. In representative agent models, the matrix J in the difference equation

$$X_t = JX_{t-1},$$

would have two unstable roots thus providing an additional two initial conditions to completely determine Z_0. In our calibrated example, there are *no* additional restrictions of this kind since all of the roots of J are inside the unit circle when evaluated at the steady state $\bar{R} = 1$. Because the monetary steady state of our model displays two degrees of indeterminacy, unlike its Ricardian counterpart, we cannot appeal to the debt valuation equation to pin down the initial value of the price level.

6.2. **Graphical illustration of the results.** In this subsection, we illustrate the properties of the model by plotting graphs of the paths of debt and real interest rates for two different initial conditions.

In our first initial condition, we highlight the persistence of the initial period wealth distribution. We pick the initial real interest rate R_2 to equal the steady state value $\bar{R}$ and we pick the price level so that the initial real value of debt, b_1, is equal to its steady state value $\bar{b}$. Instead of starting off the model in the steady state, we display the paths of the real interest factor and government debt for a world where the initial middle-aged begin life with wealth that is 3% greater than their counterpart middle-aged cohort in the steady state. Since b_0 is held constant at its steady-state value this implies that the initial old must start out with wealth that is 3% less than their counterpart old cohort in the steady state. Specifically, we chose

$$\bar{\nu}_0 \equiv \bar{R} \cdot s_1^0(\bar{R}, \bar{R}, \bar{b}) \times 1.03.$$

In Figure 2 we illustrate the time paths of the real interest rate and real government debt for this initial condition. Even though we initialize the model by setting b_1 and R_2 to their steady-state values, the deviations in initial asset holdings of the middle-aged, at the expense of the initial old, have large persistent implications for the path of market-clearing real interest rates, and for the path of government debt. The figure shows that an initial deviation of the initial wealth of just 3% from its steady state value causes the real interest rate to fluctuate by

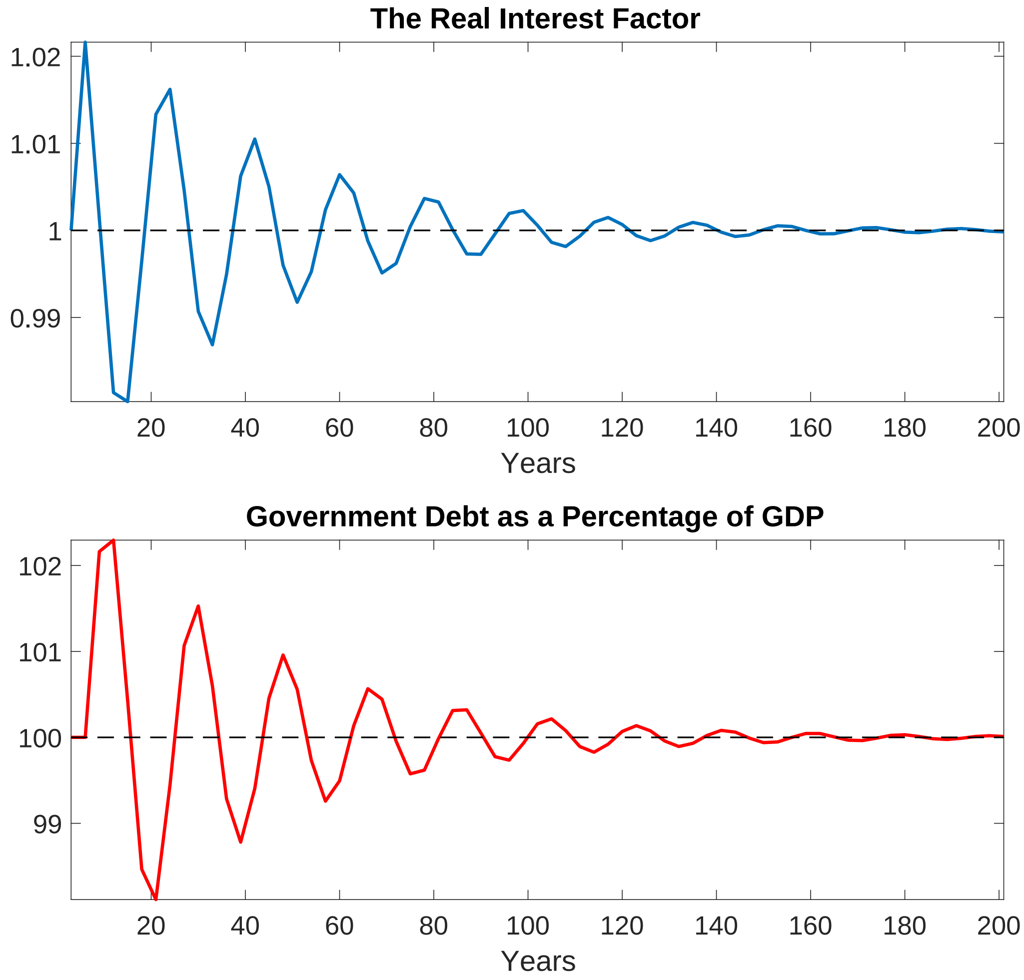

Figure 2. The impact of the initial period debt of the middle-aged exceeding its steady state by 3%

3% twenty years later and still has notable effects after 80 years. Our three-generation model is capable of generating a very high degree of endogenous persistence in the real interest rate.

For our second experiment we feature the implications of indeterminacy for the failure of the FTPL. The paths that we plot in the top and bottom panels of Figure 3 correspond to a deviation in the wealth distribution caused by a 3% deviation in the initial price level. This is similar to a wealth shock caused by a change in ν_0; but instead of holding b_0 fixed we allow a change in the price level to alter the entire vector of initial wealth positions. Importantly, the equilibrium paths of real interest rates and government debt illustrated in Figure 3 are bounded and both variables converge back to their steady state values. The boundedness of real debt for alternative nominal initial conditions implies that we cannot appeal to the FTPL to determine the initial price level even when fiscal policy is active.

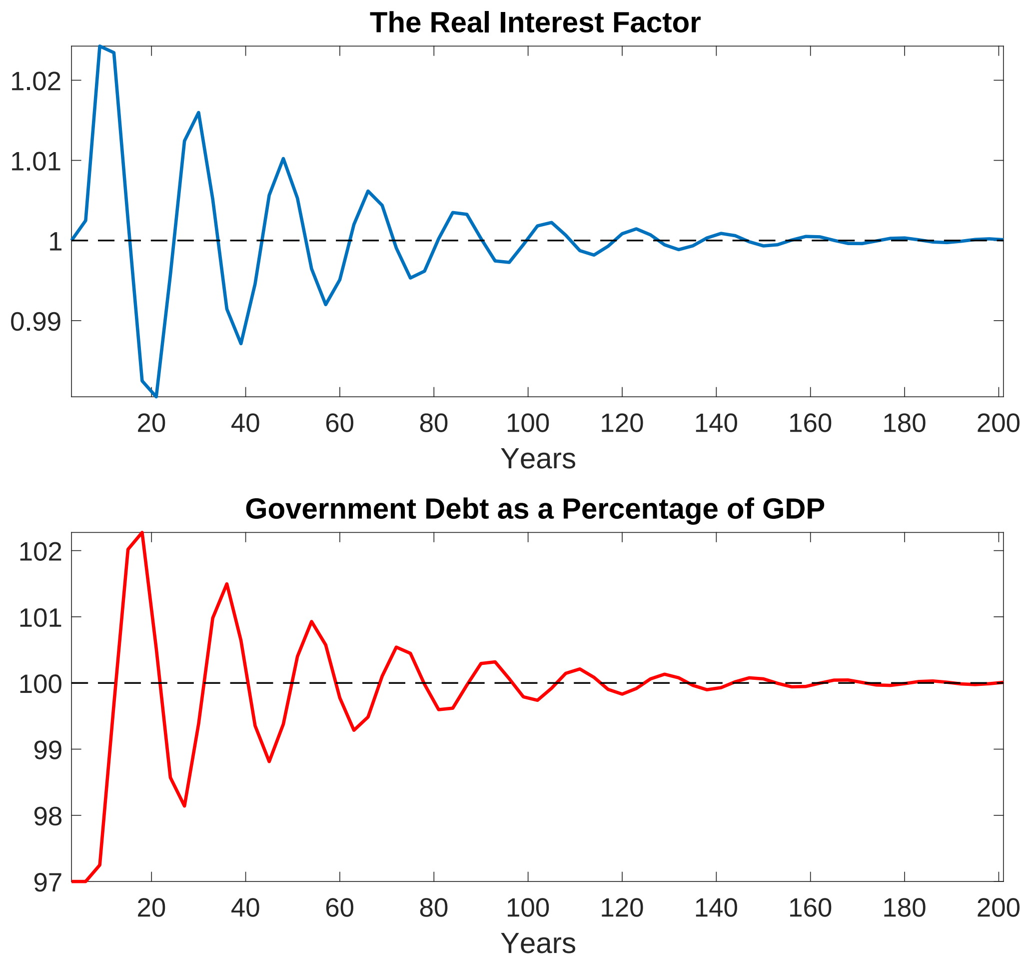

FIGURE 3. The impact of a 3% increase in the initial price level keeping R_2 constant

If examples like this are pathological then perhaps they should be dismissed. Perhaps, for example, risk aversion or intertemporal smoothing parameters taken from quarterly models may not adequately describe behaviour when the horizon is 20 years? To address these criticisms, in Section 7 we show that a 62-generation OLG model displays similar features to the three-generation model illustrated here. Our results suggest that models with indeterminate equilibria should be taken seriously as potential descriptions of real world phenomena.

7. A Sixty-Two Generation Example

In this section we construct a 62-generation model where each cohort begins its economic life at age 18 and in which a period corresponds to one year. The representative person receives an income profile when working that we calibrate to U.S. micro data and an income profile when retired that we calibrate to U.S. Supplemental Security Income.

In our 62-generation example, people maximize the utility function,

$$u\left(c_t^t, \ldots, c_{t+61}^t\right) = \sum_{i=1}^{62} \beta^{i-1} \left(\frac{[c_{t+i-1}^t]^\alpha - 1}{\alpha} \right).$$

Explicit formulas for the excess demand functions and the savings functions for this functional form and MATLAB code to solve the model are available in an online Technical Appendix.

We graph our calibrated income profile in Figure 4. Our representative cohort enters the labor force at age 18, retires at age 66, and lives to age 79. We chose the lifespan to correspond to current U.S. life expectancy at birth and we chose the retirement age to correspond to the age at which a U.S. adult becomes eligible for social security benefits. For the working-age portion of this profile we use data from Guvenen et al. (2015) which is available for ages 25 to 60. The working-age income profiles for ages 18 to 24 and for ages 61 to 66, were extrapolated to earlier and later years using log-linear interpolation. For the retirement portion we used data from the U.S. Social Security Administration.

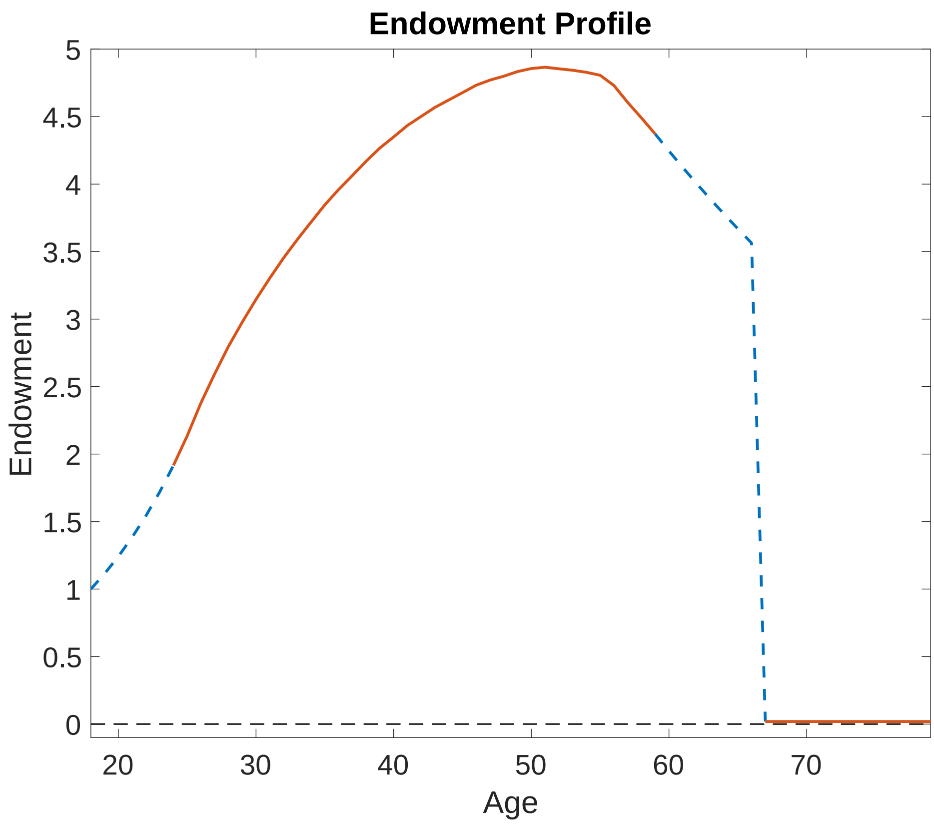

FIGURE 4. Normalized endowment profile. U.S. data in solid red: interpolated data in dashed blue.

U.S. retirement income comes from three sources; private pensions, government social security benefits, and Supplemental Security Income. We treat private pensions and government social security benefits as perfect substitutes for private savings since the amount received in retirement is a function of the amount contributed while working. To calibrate the available retirement income that is independent of contributions, we used Supplementary Security Income which, for the U.S., we estimate at 0.137% of GDP.[12]

For the remaining parameters of our model we chose an annual discount rate of 0.953 and an elasticity of substitution of 0.17. This corresponds to $\alpha = -5$ and a corresponding measure of Arrow-Pratt risk aversion of 6. For the calibrated income profile depicted in Figure 4 and for this choice of parameters, our model exhibits four steady state equilbria. In Section 8 we explore the robustness of the properties of our model to alternative choices for the discount parameter and for the risk aversion parameter.

In Figure 5 we graph the steady-state equilibria of our model. The upper panel of this figure plots the logarithm of the gross real interest rate on the horizontal axis and the steady-state excess demand for goods on the vertical axis. The lower panel plots government debt as a percentage of GDP at the steady state. We see from the upper panel that the excess demand function crosses the horizontal axis four times. And we see from the lower panel that three of these crossings are associated with steady-state equilibria in which steady-state government debt is equal to zero.

The three equilibria in which steady-state debt equals zero are examples of what Gale (1973) refers to as autarkic steady-state equilibria: in these equilibria there is no possibility of trade with future, unborn generations. The fourth steady-state is what Gale refers to as the golden-rule equilibrium. This equilibrium always exists in OLG models and in models with population growth it has the property that the real interest rate equals the rate of population growth. But although the golden-rule equilibrium always exists, *it is not true* that the golden-rule value of $\bar{b}$ is always non-negative.

For our model to provide a realistic theory of the value of money, it must be true that the golden-rule steady state is associated with a positive value of $\bar{b}$. To check that this property does indeed hold, the reader is invited to compare the upper and lower panels of Figure 5. The lower panel of this figure depicts the value of steady-state government debt. By inspecting this panel it is apparent that the golden-rule steady state, which occurs when the logarithm of

[12]From Table 2 of the March 2018 Social Security Administration Monthly Statistical Snapshot we learn that the average monthly Supplemental Security Income for recipients aged 65 or older equalled \$447 (with 2,240,000 claimants), which implies that total monthly nominal expenditure on Supplemental Security Income equalled \$1,003 million. This compares to seasonally adjusted wage and salary disbursements (A576RC1 from FRED) in February 2018 of \$8,618,700 million per annum, or \$718,225 million per month. Back of the envelope calculations suggest that Supplemental Security Income in retirement equalled 0.137% of total labor income.

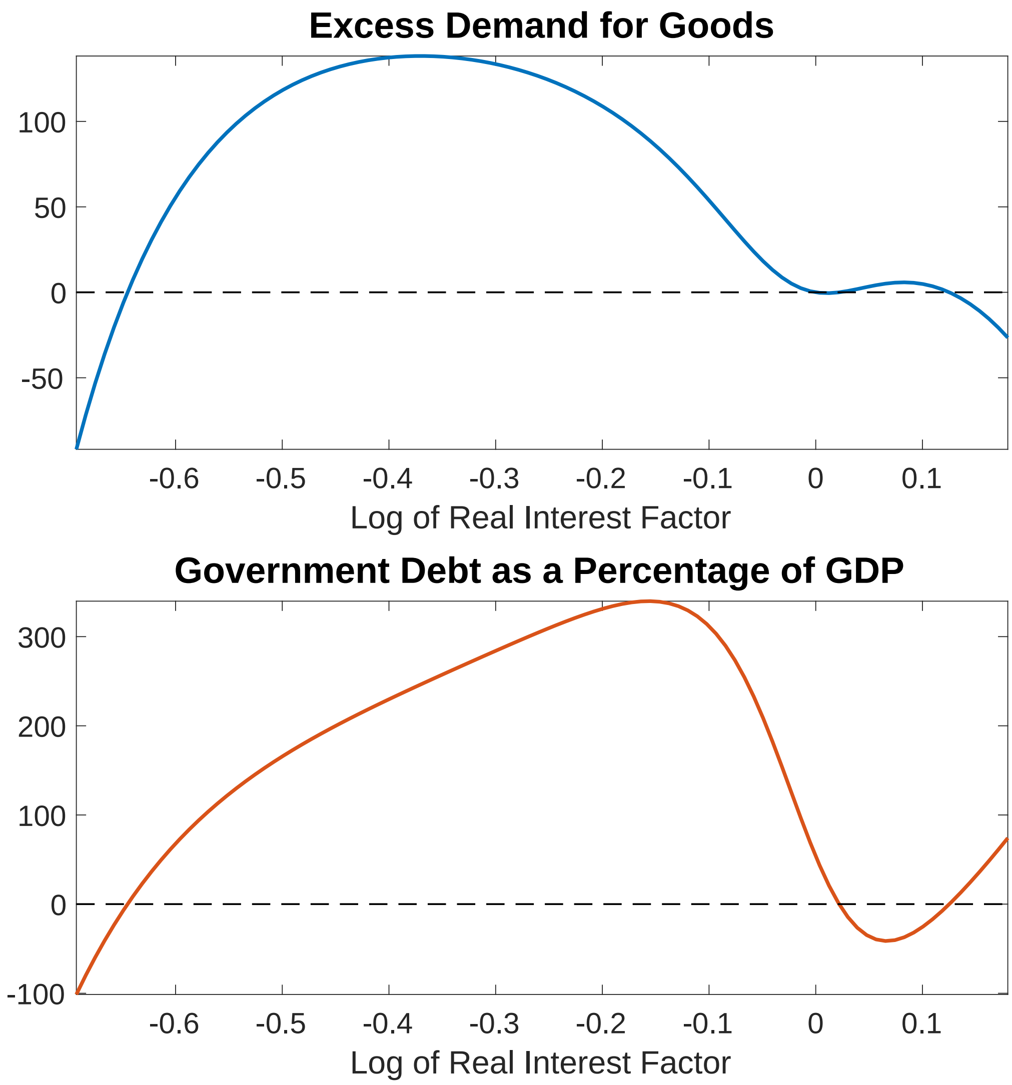

FIGURE 5. Steady States in the 62-Generation Model

the real interest factor equals zero, is indeed associated with positive valued government debt. Since debt is denominated in dollars, that fact implies that money has positive value in the golden-rule steady-state.

An equilibrium in which the interest rate is greater than or equal to the growth rate is said to be dynamically efficient. A steady-state equilibrium in which the interest rate is less than the growth rate is said to be dynamically inefficient. In the representative agent model, dynamically inefficient equilbria cannot exist because they imply that the wealth of the representative agent is unbounded. But in the OLG model, dynamically inefficient equilibria are common and, in many examples of OLG models, dynamic inefficiency is associated with indeterminacy. In our 62-generation model however, there exist indeterminate equilibria that are dynamically efficient.

The values and properties of all four steady-state equilibria are reported in Table 2. We refer to the autarkic steady states as Steady-State A, Steady-State C and Steady-State D and to the golden-rule equilibrium as Steady-State B. We see from this table that steady states B, C and D are associated with a non-negative interest rate and are therefore dynamically efficient. Steady-State A is associated with a negative interest rate and is therefore dynamically inefficient.

Equilibrium Discount Factors					
Type	Value of Real Rate	Value of b	# Unstable Roots	# Free Initial Conditions	Degree of Indeterminacy
Steady-State A	-52.5%	0	60	61	1
Steady-State B	0%	53.7% of GDP	59	61	2
Steady-State C	2.2%	0	60	61	1
Steady-State D	13.3%	0	61	61	0

Table 2. Steady States of the 62-generation Model

The 62-generation model with a calibrated income profile is similar in many respects to the more stylized 3-generation model. In both examples, the golden-rule steady-state equilibrium displays second degree indeterminacy. And in both examples, the steady-state price level is positive and the initial price level is indeterminate even when fiscal policy is active. Importantly, because the monetary steady-state is second-degree indeterminate, indeterminacy of the price level holds even when both monetary and fiscal policy are active.

In Figure 6 we show the result of an experiment in which we perturb the initial value of b_1 by 3% and we perturb the real value of the initial wealth of all of the non-generic generations by the same amount. We restrict R_2 to equal its steady state value but all other elements of Z_0 are allowed to respond to the shock to keep the path of interest rates and debt on a convergent path back to the steady state. We refer to this shock as a 3% shock to the initial price level. Figure 6 demonstrates that the 62-generation model preserves the feature from the 3-generation example that the return to the steady state from an arbitrary initial condition is extremely slow.

We also see from Figure 6 that our model can explain persistent periods of negative real interest rates. The upper panel of this figure plots the path by which the real interest rate returns to its steady state value and the lower panel plots the return path of the real value of government debt expressed as a percentage of GDP. The figure demonstrates that small deviations of initial conditions from their steady state values may be highly persistent and, during the adjustment to the steady state, the real interest rate may be negative for periods well in excess of ten years.

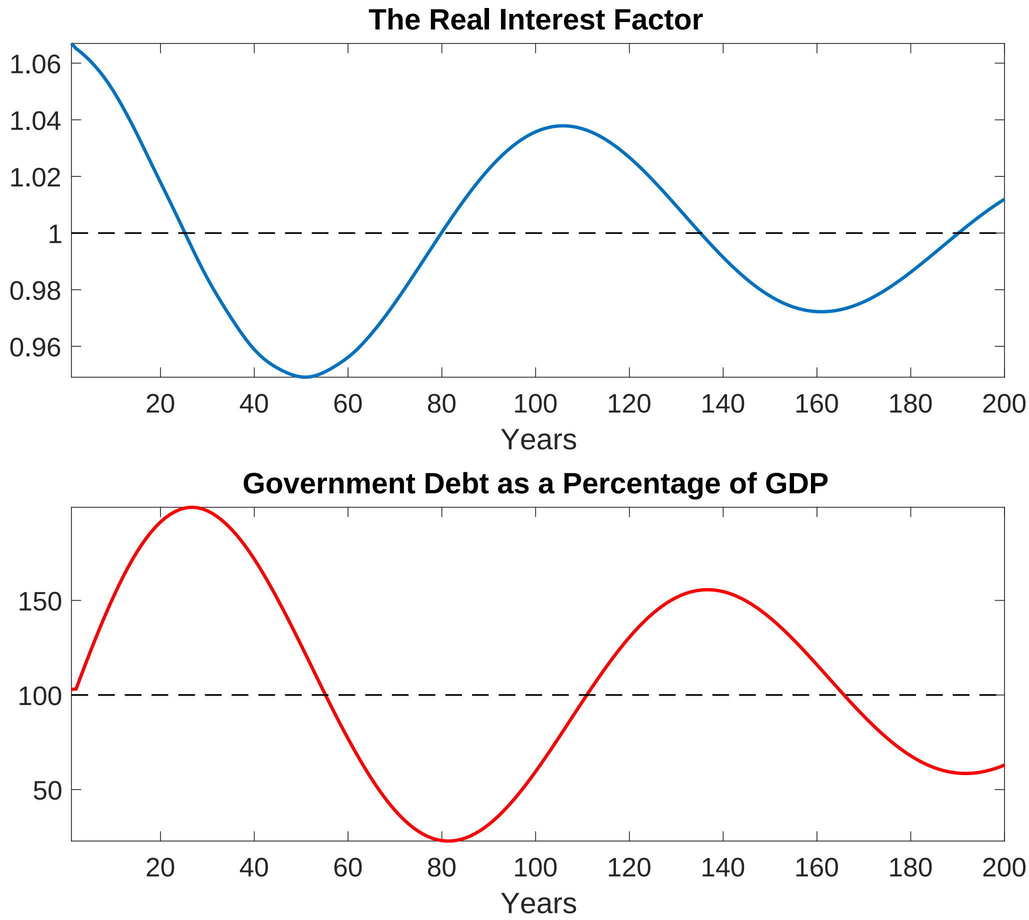

FIGURE 6. The impact of changes in the initial price level and the real interest rate: p_1 exceeds its steady state values by 3%

8. Robustness to Different Calibrations

We have presented an example of a 62-generation OLG model in which there is a golden-rule steady state with two degrees of indeterminacy. To explore the robustness of our findings to alternative calibrations, in Table 3 we record the properties of our model for different values of the annual discount rate and the coefficient of relative risk aversion. The example we featured in Section 7 had two-degrees of indeterminacy and positive valued debt at the monetary steady state. Table 3 demonstrates that that property is not particularly special.

Table 3 provides 40 different parameterizations of our model with risk aversion parameters ranging from 5 to 9 and discount rates ranging from 0.91 to 0.98. In all of these parameterizations we maintained the calibrated income profile illustrated in Figure 4. For each calibration Table 3 displays the number of steady states and the number of degrees of indeterminacy at the golden-rule steady state. There are 15 parameterizations in which the golden-rule steady state displays

		Annual Discount Factor							
		0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98
Risk Aversion									
RA = 5	# Steady States	4	4	4	4	4	2	2	2
	Degree of Indeterminacy	1	1	1	1	1	2	0	0
	Value of Debt	-2.8	-2.2	-1.5	-0.8	-0.2	0.5	1.1	1.8
RA = 6	# Steady States	4	4	4	4	4	4	2	2
	Degree of Indeterminacy	1	1	1	1	2	2	0	0
	Value of Debt	-1.9	-1.3	-0.7	-0.2	0.4	0.9	1.5	2.0
RA = 7	# Steady States	4	4	4	4	4	4	2	2
	Degree of Indeterminacy	1	1	1	2	2	0	0	0
	Value of Debt	-1.2	-0.7	-0.2	0.3	0.8	1.2	1.7	2.2
RA = 8	# Steady States	4	4	4	4	4	4	4	2
	Degree of Indeterminacy	1	1	2	2	2	0	0	0
	Value of Debt	-0.6	-0.2	0.2	0.6	1.0	1.5	1.9	2.3
RA = 9	# Steady States	4	4	4	4	4	4	4	4
	Degree of Indeterminacy	1	2	2	2	2	0	0	0
	Value of Debt	-0.2	0.1	0.5	0.9	1.3	1.6	2.0	2.4

Table 3. Robustness of Indeterminacy to Alternative Calibrations

one degree of indeterminacy and 12 in which it displays two degrees of indeterminacy. In all twelve of these parameterizations, money has positive value in the steady state.

In Section 7, we focused on the golden-rule steady where $\beta = 0.953$ and $\rho = 6$. An example with two-degrees of indeterminacy is interesting because it is not only the price level that is free to be determined by the beliefs of market participants; it also the real rate of interest. We want to stress, however, that only one degree of indeterminacy is required for violations of the FTPL. And that occurs more frequently in our model than second degree violations.

If we hold constant the degree of risk aversion and increase the discount rate, the number of unstable eigenvalues decreases initially from 60 to 59 and then changes abruptly to 61. We see this behaviour in Table 3 by moving along a typical row and observing that we pass from one degree of indeterminacy to two degrees of indeterminacy and then jump abruptly to 0 degrees of indeterminacy. At this last transition, a pair of complex roots crosses the unit circle, a phenomenon associated with a Hopf Bifurcation and the creation of a limit cycle.[13] We have not explored the phenomenon in this paper, but it is likely that for discount rates close to 1, this model displays endogenous limit cycles that are second-degree indeterminate.

[13]See Guckenheimer and Holmes (1983) for a discussion of the Hopf Bifurcation.

9. Fiscal and Monetary Policy

In the OLG models that we described in Sections 6 and 7, we assumed that fiscal policy is active and monetary policy is passive. The assumption of an active fiscal policy follows from our formulation of a debt accumulation equation in which there is an outstanding stock of debt but no attempt to adjust spending or taxes to ensure that debt is repaid. The assumption of a passive monetary policy follows from our assumption that nominal treasury liabilities are willingly held, but that these liabilities pay a zero money interest rate. In this section we discuss what would happen if we were to relax either of these assumptions.

9.1. **Passive Fiscal policy.** Consider first what would happen to our model if we were to assume that fiscal policy is passive. Suppose, for example, that the treasury raises taxes τ_t in proportion to the real value of outstanding debt,

$$\tau_t = \delta b_t,$$

where $\delta \geq 0$ is a debt repayment parameter. Combining this assumption with the definition of the government debt accumulation equation leads to the amended debt accumulation equation

$$b_{t+1} = [R_{t+1} - \delta] b_t. \tag{13}$$

For values of $[\bar{R} - \delta] < 1$ the effect of this switch to a passive from an active fiscal policy is to introduce an additional stability mechanism that increases the degree of indeterminacy at each of the four steady states for which δ is large enough. A passive fiscal policy makes indeterminacy more likely.

9.2. **Active Monetary Policy.** We model an active Taylor Rule with the equation, (Taylor, 1999),

$$1 + i_t = \bar{R} \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\eta} \Pi_t. \tag{14}$$

Here, $\bar{\Pi}$ is the inflation target, and $\bar{R}$ is the steady state real interest rate. An active Taylor rule is represented by $\eta > 0$ and a passive rule by $\eta < 0$. This rule also encompasses the special case of an interest-rate peg for which $\eta = -1$.

In the T-generation OLG model, we have shown that the equilibrium real interest rate is independent of the path of the inflation rate. It is fully characterized by the bond market clearing equations

$$b_t + d_t = f(R_{t+T-1}, R_{t+T-2}, \ldots, R_{t-T+4}, R_{t-T+3}),$$

and the debt accumulation equation,

$$b_{t+1} = R_{t+1}(b_t + d_t).$$

Conditional on a path for the real interest rate, the equilibrium inflation rate is determined by combining the Taylor Rule, Equation (14), with the Fisher parity condition, Equation (2), which we reproduce below,

$$R_{t+1} \equiv \frac{1 + i_t}{\Pi_{t+1}}.$$

Combining these equations leads to the following difference equation in inflation,

$$\Pi_{t+1} = \left(\frac{\bar{R}}{R_{t+1}} \right) \left(\frac{\Pi_t}{\bar{\Pi}} \right)^{\eta} \Pi_t, \quad \text{for all} \quad t \geq 1. \tag{15}$$

When monetary policy is active, Equation (15) is unstable if solved backwards for Π_{t+1} as a function of Π_t and R_{t+1}. But because the initial price level is free to be chosen, the equation can be solved forwards to find the unique initial inflation rate, Π_2, as a function of the future path of real interest rates. Written in this way, the equation determines the unique initial value for Π_2, consistent with a bounded path for all future inflation rates. If we anchor the first period Taylor rule by choosing p_0 to be an initial condition of the model, the initial value of Π_2 also determines the initial price level, p_1.

But although an active monetary policy is consistent with a unique initial price level, even when fiscal policy is also active, it does not uniquely determine the path of real interest rates. We have provided two examples of models, each of which display *two degrees* of indeterminacy in the neighbourhood of the golden-rule steady state. That fact implies that conditional on a given value of p_1, and therefore an initial value of b_1, the model still admits multiple equilibrium paths for the real rate of interest, all of which converge back to the golden-rule steady state. These different real rate paths are associated with different inflation paths. Real indeterminacy implies nominal indeterminacy.

10. Conclusions

We have demonstrated an important difference between an infinitely lived representative agent model and its overlapping generations counterpart. In the RA model, government debt is both an asset and a liability of the representative agent. Because these two aspects exactly offset each other, the representative agent is indifferent about the quantity of debt she holds and in the simplest case the real interest rate in the corresponding model reflects time preferences and the evolution of the endowment.

In the OLG model, the situation is different. Because the stock of government debt is unlikely to be fully paid off during the lifetime of any generation, the assets and liabilities of the treasury do not cancel each other out as they would in the representative agent model. As a consequence, changes in real interest rates are redistributive across cohorts, and they may lead to fluctuations in the demand for government bonds that are self-stabilizing.

In our calibrated model the golden-rule steady state equilibrium is both dynamically efficient and second-degree indeterminate. As long as the primary deficit or the primary surplus is not too large, the fiscal authority can conduct policies that are unresponsive to endogenous changes in the level of its outstanding debt. Monetary and fiscal policy can *both* be active at the same time.

Our findings challenge established views about what constitutes a good combination of fiscal and monetary policies. Our agents are rational and have rational expectations. Nevertheless, the price level and the real interest rate are not uniquely determined by what most economists would recognize as economic fundamentals, even when the central bank and the treasury both pursue active policies. These features of our model lead to very different conclusions from those of the RA approach.

If the FTPL holds, a benevolent monetary policy maker who pursues an interest rate peg might rely on fiscal policy to anchor the price level. In the OLG model we studied here that is no longer possible. Giving up on active monetary policy implies that the policy maker has also abandoned the ability of active monetary policy to provide a nominal anchor.

Our model also leads to non-standard advice to fiscal policy makers. In an RA model, the fiscal policy maker must raise taxes or lower expenditures in response to recessions, however they are caused. In Farmer and Zabczyk (2018) we showed, in a two-generation OLG model, that equilibrium debt dynamics can be self-stabilizing. In this paper we have extended our previous analysis to a calibrated 62-generation OLG model. In our 62-period calibrated example, a fiscal policy that does not respond to endogenous fluctuations in debt, can safely be pursued, at least for small values of the primary deficit, without the fear that a policy of this kind will lead to an exploding debt level as a fraction of GDP.

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