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Stationary state structure of a random copying mechanism over a complex network

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Abstract

An analytical approach to network dynamics is used to show that when agents copy their state randomly, the network arrives to a stationary regime in which the distribution of states is independent of the degree. The effects of network topology on the process are characterized introducing a quantity called *influence* and studying its behavior for scale-free and random networks. We show that for this model degree averaged quantities are constant in time regardless of the number of states involved.

Keywords: Complex networks; State dynamics; Sociophysics; Social dynamics; Opinion formation

1. Introduction

Complex network theory has flourished as an effort to explain the main characteristics of discrete interacting systems. The first studies were able to describe and explain their topology by showing that different thermodynamical quantities are needed to characterize a particular network, such as its degree distribution [1–5],

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community structures [6,7], resilience [8] and motifs [9]. Recent efforts have also studied quantities spreading on complex networks, such as computer viruses [10–14] in technological networks and information in their social counterpart [15].

On the other hand, state dynamics has been studied disguised as opinion formation processes, which have been modelled using approaches including cellular automata [16], spin chains [17,18] and opinion drifts over a continuous opinion space [19], without incorporating topology. Recently, network topology has become more relevant and computer simulations have been performed in this direction [20–24], though a systematic way to evaluate the topological effects on the dynamical processes taking place on networks is still missing.

In this paper, we study the effects of topology on a copy mechanism taking place over a network with a non-trivial topology and an arbitrary number of states. To achieve this, we group agents with similar topological characteristics and study the density of the evolving quantities over these groups by considering the interactions among them.

We will first introduce the model together with its associated notation and then solve it analytically. We will then study how network topology affects the simple dynamical process introduced below and show that the system arrives to a configuration in which the distribution of states is independent of the degree.

2. The model

We consider a process in which at each time step an agent changes its state by copying one randomly from one of its immediate neighbors. The probability that this occurs is the product between the probability that the agent is chosen, times the fraction of neighbors in a given state, including the agent, that are acquainted by it. We will consider a directed network with an adjustable topology as the substrate in which these dynamical processes will occur, and attempt to characterize the influence of topology upon them. We assume that changes in the network topology take place at larger timescales than the relaxation times of the system.

2.1. *Definitions*

We begin by grouping agents according to their degree into aggregates we call *guilds*.[1] This reduces the problem to one as big as the number of guilds used to calculate the approximation, this depends on the desired degree resolution,[2] but it is always considerably smaller than the number of agents. Agents in a specific guild can be in any of the available *states*. We introduce the density $\rho_{i\alpha}$ as the number of agents in the ith guild which pertain to the α state, over the number of agents in the

[1] Agents pertaining to the same guild do not have to be connected. Guilds are abstractions in which agents with equal or similar topological characteristics are replaced by an effective agent density.

[2] The definition of guilds can be relaxed to include agents with similar degree instead of the ones with necessarily the same.

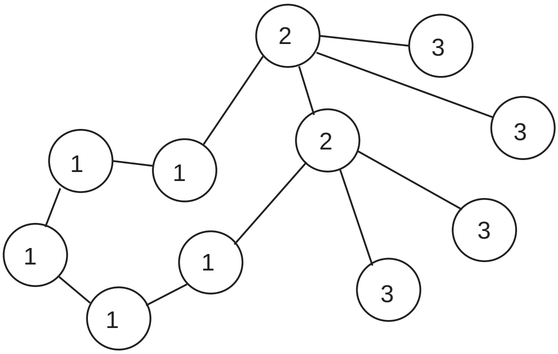

Fig. 1. Undirected network with 11 nodes in which 3 guilds are distinguishable. Nodes are labeled by the guild they pertain. The first guild has 5 agents with a degree equal to 2. The second guild is formed by 2 agents with a degree equal to 4. The third guild is formed by 4 agents with a degree equal to 1.

network. We define *guild size* $\rho_i = \sum_\alpha \rho_{i\alpha}$ as the fraction of agents constituting that guild and *fractional abundance* as the fraction of agents on a particular state in the entire network $\rho_\alpha = \sum_i \rho_{i\alpha}$. From now on, latin indices will refer to guilds, while greek indices will label states.

We assume that the linking patterns of agents in the same guild are similar. In other words, we will model a network in which agents from the same guild are linked to a similar number of agents in other guilds, although not necessarily the same. This motivates us to introduce P_{ij} as the probability that an agent in the ith guild is linked to an agent in the jth guild. We will refer to these quantities as *linking probabilities* and say that someone is linked to someone else if the linking agent has a *personal*- or *media*-based knowledge of the linked one.

An example is shown in Fig. 1, three equal size guilds are clearly distinguishable and acquaintanceship probabilities are given by

$$P = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} .$$

In this case we reduce the problem by working with the $3 \times 3 P$ matrix instead of the 11×11 adjacency matrix used to traditionally represent a graph.

2.2. Topologically considerate mean-field approach

In the large network limit the rate of change for a state in a given guild is equal to the difference between the probabilities of wining and losing an agent. The probability that an agent in the ith guild turns into the αth state is

$$\Pi^+_{i\alpha} = \frac{\sum_j P_{ij}\rho_{j\alpha} \sum_{\mu \neq \alpha} \rho_{i\mu}}{\sum_{\mu,j} P_{ij}\rho_{j\mu}} , \tag{1}$$

while the probability that an agent in the ith guild and the αth state changes its state to a different one is

$$\Pi^{-}_{i\alpha} = \frac{\rho_{i\alpha} \sum_{\mu \neq \alpha, j} P_{ij}\rho_{j\mu}}{\sum_{\mu,j} P_{ij}\rho_{j\mu}} \ . \tag{2}$$

We simplify the notation introducing Υ as the matrix representing the expected fraction of agents in a particular state known by an agent in a particular guild on the entire network,

$$\Upsilon_{i\alpha} = \sum_{j} P_{ij}\rho_{j\alpha} \ . \tag{3}$$

Combining Eqs. (1)–(3) we can write the temporal variation of state abundance for a given guild as

$$\partial_t \rho_{i\alpha} = \frac{1}{\Upsilon_i}\left(\Upsilon_{i\alpha}\sum_{\mu}\rho_{i\mu} - \rho_{i\alpha}\sum_{\mu}\Upsilon_{i\mu}\right) , \tag{4}$$

where $\Upsilon_i = \sum_{\mu} \Upsilon_{i\mu}$ represents the fraction of agents acquainted by the ith guild.[3] The extension of the restricted sums in Eqs. (1) and (2) to include all states has no effect on the system dynamics because the additional terms cancel out when we place them in Eq. (4). Time units represent the network's natural update time, which is the number of time steps equal to the population size in a random or sequential updating processes.

2.3. Analytical solution

An analytical solution can be obtained by re-writing Eq. (4) and performing the substitution $\rho_{i\alpha} = U_{i\alpha}\mathrm{e}^{-t}$. This transforms Eq. (4) into an eigenvalue equation for the vector $\vec{U}_{\alpha} = (U_{1\alpha}, U_{2\alpha}, \ldots)$. The problem then reduces to finding the solution of

$$\partial_t \vec{U}_{\alpha} = M\vec{U}_{\alpha} , \tag{5}$$

where $M_{ij} = (\rho_i/\Upsilon_i)P_{ij}$. From Eq. (5) it follows that the eigenfunctions of M depend exponentially on time with a decay rate λ, where λ is the associated eigenvalue, this agrees with the results presented in [27,28]. A general solution can be written in terms of this basis as

$$\rho_{i\alpha} = \sum_{\eta} C^{\eta}_{i\alpha} V^{\eta}_{i} \mathrm{e}^{(\lambda_\eta - 1)t} , \tag{6}$$

in which V^{η}_{i} is the ith element of the ηth eigenvector of the matrix M and λ_η is its associated eigenvalue. It is important to notice that M does not depend on the state, implying that the state dependence is introduced through the initial conditions only, represented by $C^{\eta}_{i\alpha}$.

[3]From now on we will assume that quantities with a missing index have been added over it, for example: $\rho_\alpha = \sum_i \rho_{i\alpha}$.

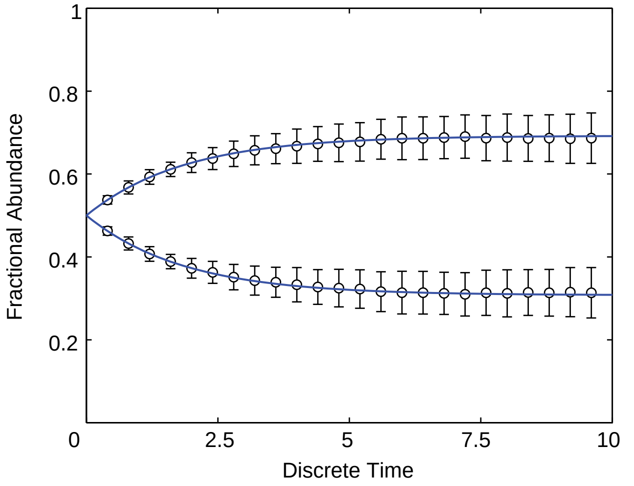

Fig. 2. The continuous line represents the mean-field solution of a two guild network with 1000 agents, while circles represent an average of 50 stochastic simulations. Error bars indicate one standard deviation. In each simulation the network was chosen randomly constrained to satisfy the conditions presented in the text.

2.3.1. *System equilibria*

The steady states are characterized by the eigenvector associated with the eigenvalue $\lambda = 1$. By direct calculation one can show that M always has an eigenvalue equal to 1 associated with guild sizes as its eigenvector. One has

$$\sum_j M_{ij}\rho_j = \sum_j \frac{\rho_i P_{ij}\rho_j}{\Upsilon_i} = \rho_i \sum_j \frac{P_{ij}\rho_j}{\Upsilon_i} = \rho_i \ .$$

We have observed numerically that the remaining eigenvalues have a real part smaller than 1 and lie inside the unit circle of the complex plane, a formal demonstration of this remains open.

The mean-field approach presented above was compared with a stochastic simulation on a 1000 agents network in which two equal size guilds were distinguishable (Fig. 2). The acquaintanceship probabilities were chosen such that all agents acquainted everyone in their respective guild, while the ones in the first guild acquainted an average of $\frac{3}{5}$ of the agents in the second guild. On the contrary, agents in the second guild acquainted $\frac{1}{5}$ of the agents in the first one. The simulation shows how accurate the mean field model is and reveals that the standard deviation tends to stabilize as the system relaxes to equilibrium.

3. Degree distribution

We define the *in* degree of an agent pertaining to a specific guild as the fraction of agents in the total sample that it knows. It, therefore represents the fraction of the network exerting a direct influence on it. On the other hand, an agent's *out* degree is the fraction of the entire network that knows the agent, these are the ones directly influenced by it. This definition of degree implies that peer influence flows opposite to acquaintanceship. According to this, the *in* and *out* degree's in our model are represented by

$$k_i^{in} = N \sum_j P_{ij}\rho_j = N\Upsilon_i, \quad k_i^{out} = N \sum_j P_{ji}\rho_j \,, \tag{7}$$

thus degree distribution can be imposed by considering that the fraction of agents with the described degree is equal to guild size ρ_i. We can approach a complex network with an *out* degree distribution satisfying a function f by the solutions of

$$\rho_i = f\left(N \sum_j P_{ji}\rho_j\right) . \tag{8}$$

In the case of scale-free networks, Eq. (8) becomes

$$\rho_i = \left(\sum_j P_{ji}\rho_j\right)^{-\gamma} , \tag{9}$$

where constant factors like normalization or network size have been absorbed by the elements of P, noting that Eq. (4) is zero degree homogeneous in P. For the case of a random network, degree distribution has an exponential tail that can be approximated by the solutions of

$$\rho_i = A \exp\left(-\eta \sum_j P_{ji}\rho_j\right) \tag{10}$$

as parameters. Here η characterizes the rate of exponential decay and A is a normalization factor.

4. Guild influence

4.1. Scale-free networks

To capture the influence exerted by the topology on the dynamical process, we define *guild influence* I_i as the equilibrium fractional abundance that a state will achieve assuming that it was initially occupying, only and entirely, the ith guild $(\rho_\alpha(t=0) = \rho_i)$,

$$I_i = \lim_{t\to\infty} \frac{\rho_\alpha(t)}{\rho_i} .$$

This quantity represents the average number of copies that an agent of a given guild creates during the entire process, or an effective infection rate proper of the guild. This is a formalization of what Wu et al. [27] and Sucheki et al. [28] have done.

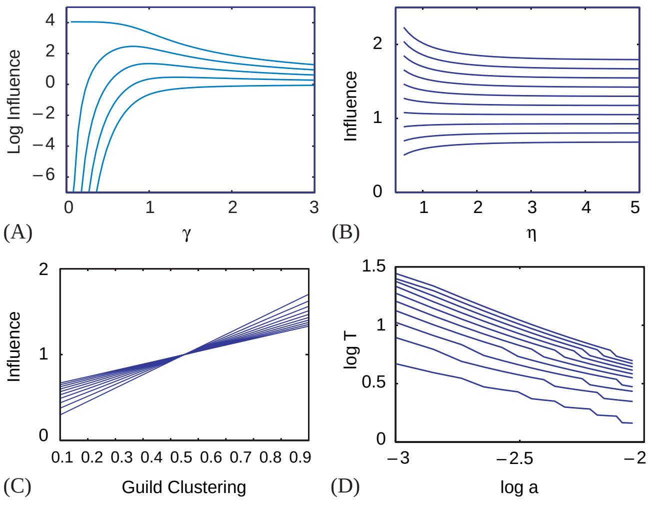

Fig. 3. (A) Guild influence against scale-free exponent γ for a 5 guild network calculated through numerical integration of Eq. (4). From top to bottom, guild sizes were chosen as 9×10^{-1}, 9×10^{-2}, 9×10^{-3}, 9×10^{-4}, and 9×10^{-5}. (B) Influence structure of a random network against the inverse of its characteristic length η, for a 10 guild network. From top to bottom guild sizes were chosen as 0.026, 0.032, 0.041, 0.053, 0.068, 0.088, 0.114, 0.147, 0.189 and 0.245. (C) Influence for a ten guild network in which each guild had a particular clustering coefficient. Different lines represent linearly spaced variations of the constant a in the 0.01–0.09 interval from lowest to highest slope respectively. The constant b was taken as 0.1 and 10 guilds were used. (D) Characteristic times against inter-guild connectivity for values of b chosen linearly spaced from top to bottom in the 0.1–0.01 interval.

Influence measures the affect of the group of agents with a certain degree, generalizing the bias depicted in the two state systems studied by them.

We studied guild influence on a scale-free network by choosing guild sizes and finding acquaintanceships probabilities for a given scale-free exponent according to Eq. (9). Fig. 3(A) shows that the influence of the highest degree guild increases with decreasing γ, until it reaches a steady value which is equal to the ratio between the system size and the number of agents in the highest degree guild. This is a consequence of the increasing fame of the highest degree guild that occurs for small values of γ. In the large exponent limit the influence of all guilds approach unity, meaning that there are no high *out* degree agents exerting an important influence on the system.

4.1.1. Exponential networks

An example involving a exponential network approximated by 10 guilds is shown in Fig. 3(B), the procedure used to determine the parameters being analogous to the one performed for the scale-free network. The figure shows that in a random network high degree guilds exert a mild influence compared to that occurring on its scale-free counterpart. This is a consequence of the reduced population of high degree guilds.

4.2. Clustering

The topological influence of clustering was studied by choosing acquaintanceship probabilities as

$$P_{ij} = a + \delta_{ij}(b \cdot i - a) \quad \text{with } b > a$$

and equal guild sizes for all the network, where δ_{ij} is the Kronecker delta. This model represents a random network in which agents of the ith guild have a probability a of being acquainted to agents in other guilds. As usual, we shall call this connectivity. On the contrary, the probability $b \cdot i$ to acquaint someone from its own guild together with the condition $b > a$ defines clustered groups with an internal connectivity that increases linearly throughout the network. This is similar to communities in the sense introduced by Newman and Girvan [7,25] and are also similar to the networks studied by Watts [26] at the end of last century.

In a random network, clustering is equal to the probability that two agents are acquainted, motivating us to define guild clustering as the clustering coefficient of a guild in the absence of the rest of the network. We studied the dependence of influence on guild clustering (Fig. 3(C)) finding that there is a linear dependence with a slope that decreases with a, meaning that the reinforcement mechanism is more effective in sparsely connected networks than in strongly connected ones. Either way, the effect continues to remain mild compared to its scale-free counterpart. On the other hand, the characteristic time T required for the system to converge increases with decreasing a as $T \sim a^{-\kappa(b)}$ with $\mathrm{d}\kappa/\mathrm{d}b < 0$ (see Fig. 3(D)).

5. Equilibrium distribution

Using the formalism recently introduced we can also find how states are distributed when the network reaches equilibria. In order to do this, first we notice that Eq.(4) vanishes when

$$\frac{\Upsilon_{i\alpha}}{\Upsilon_i} = \frac{\rho_{i\alpha}}{\rho_i} \ . \tag{11}$$

If we multiply Eq. (4) by P_{ji} and add over all i we will find a rate equation for $\Upsilon_{j\alpha}$. This must vanish in equilibria implying that

$$\Upsilon_j \sum_i \frac{\Upsilon_{i\alpha}}{\Upsilon_i} = \sum_i \Upsilon_{j\alpha} \ . \tag{12}$$

The term on the right-hand side does not depend on i so the summation just yields the number of guilds we have chosen to approximate the system. Considering a system with m guilds leads us to conclude that (12) is the same as

$$\frac{1}{m} \sum_i \frac{\Upsilon_{i\alpha}}{\Upsilon_i} = \frac{\Upsilon_{j\alpha}}{\Upsilon_j} \ . \tag{13}$$

The left-hand side of this equality does not depend on j, so when the system reaches the steady state, the ratio between $\Upsilon_{j\alpha}/\Upsilon_j$ must be the same for every guild. Combining this with condition (11) we conclude that upon equilibria the system reaches a configuration in which the ratio between the number of agents in a certain state over the total number of agents is independent of the degree, thus each guild has exactly the same distribution of states as the entire network.

6. Network structure

The formalism presented above can be used to understand some topological characteristics of the studied network. Recently, two groups have presented works indicating that a simple dynamical process like the one studied in this paper can be solved, in the case in which two states are involved, by the use of conserved quantities or martingales [27,28]. The first of them proposes that a weighted average over degree should be conserved in a dynamical process of this kind. In our formalism, we average the fraction of agents in each state weighted by their relative degree as

$$\sum_{i\alpha} \Upsilon_{i\alpha} \, . \tag{14}$$

We can show that this is a conserved quantity by taking Eq. (4), multiply it by P_{ji} and adding it over i, j and α to get

$$\partial_t \Upsilon = \sum_{ij\mu\alpha} \left(\frac{P_{ji}\Upsilon_{i\alpha}\rho_{i\mu}}{\Upsilon_i} - \frac{\rho_{i\alpha}\Upsilon_{i\mu}P_{ji}}{\Upsilon_i} \right) .$$

Performing the summation over α on the first term and over μ in the second term and cancelling the Υ_i we get

$$\partial_t \Upsilon = \sum_{ij\mu} P_{ji}\rho_{i\mu} - \sum ij\alpha P_{ji}\rho_{i\alpha} \, ,$$

which is clearly equal to zero. This quantity is naturally conserved if we consider that $\Upsilon_{i\alpha}$ represents the fraction of agents in the α state that are known by someone on the i'th guild. Thus Υ_i represents the total numbers of agents known by the i'th guild, and Υ therefore represents the fraction of agents known by all the guilds combined, or in the other words the number of non-intersecting sub-graphs of the network that are able to reach all the nodes of it.

In a two state system in which each one of them is represented by a spin up or down we can show that the degree averaged magnetization does not depend on time by replacing the fraction of agents in a certain state by the fraction of agents in that state times its spin.

$$\rho_{i\alpha} \to \rho_{i\alpha}\sigma_\alpha \, .$$

If we span $\Upsilon_{i\alpha}$ we can show that the change on this quantity can be constructed by using our formalism as

$$\partial_t \sum_{ij\alpha} P_{ij}\rho_{j\alpha}\sigma_\alpha = \sum_{ij\mu\alpha k}\left(\frac{P_{ji}P_{ik}\rho_{k\alpha}\sigma_\alpha\rho_{i\mu}\sigma_\mu}{\Upsilon_i} - \frac{\rho_{i\alpha}P_{ik}\rho_{k\mu}P_{ji}}{\Upsilon_i}\right),$$

where the two added terms are strictly equal if we consider that μ and α are mute variables. This agrees with the results found by Wu et al. [27] and can be extended to an arbitrary number of spins, which in this case are nothing more than labels, and we believe its *names* should not enter the calculations.

7. Conclusion

Using a topologically considerate mean-field approach we were able to characterize the simple dynamics of a node updating mechanism on a complex network with an arbitrary degree distribution.

The main idea behind this technique was to group nodes with similar topological characteristics into aggregates we called guilds. We characterized the reproductive capability of guilds by a quantity we called influence and showed that in the case of a scale-free network the system bias dramatically towards the states pertaining to the highest degree guilds as the characteristic exponent decrease, while on an exponential network this effect is comparatively mild.

It was also shown that the convergence time scale as a function of connectivity, with a characteristic exponent close to 1, when we considered a network formed by mildly connected clusters. This is a consequence of the reinforcement mechanism introduced by the node updating rule. We were also able to show that clustered groups tend to be more influential than un-clustered ones. This is due to the fact that they are harder to invade, allowing them to act over the system for a longer period of time.

The final configuration of the system was unravelled, showing that it arrives to a stationary distribution in which the fraction of nodes with a certain degree that are in a certain state, over the total number of nodes with that degree, is independent of the particular degree and it therefore mimics the network.

In the last section we discussed a purely topological quantity, which is naturally conserved and represents the number of subgraphs present in the network with the ability to hold it together. This suggests a direct link with network resilience and will be present in future discussions. We also showed that the degree weighted average of states over the system remains constant, regardless of the value and number of them in agreement with previous results.

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