5 Write functions as compositions.

Decomposing Functions

When you form a composite function, you "compose" two functions to form a new function. It is also possible to reverse this process. That is, you can "decompose" a given function and express it as a composition of two functions. Although there is more than one way to do this, there is often a "natural" selection that comes to mind first. For example, consider the function h defined by

$$h(x) = (3x^2 - 4x + 1)^5.$$

The function h takes $3x^2 - 4x + 1$ and raises it to the power 5. A natural way to write h as a composition of two functions is to raise the function $g(x) = 3x^2 - 4x + 1$ to the power 5. Thus, if we let

$$f(x) = x^5 \text{ and } g(x) = 3x^2 - 4x + 1, \text{ then}$$
$$(f \circ g)(x) = f(g(x)) = f(3x^2 - 4x + 1) = (3x^2 - 4x + 1)^5.$$

Study Tip

Suppose the form of function h is $h(x) = (\text{algebraic expression})^{\text{power}}$. Function h can be expressed as a composition, $f \circ g$, using

$f(x) = x^{\text{power}}$

$g(x) = \text{algebraic expression}.$

EXAMPLE 6 Writing a Function as a Composition

Express $h(x)$ as a composition of two functions:

$$h(x) = \sqrt[3]{x^2 + 1}.$$

Solution The function h takes $x^2 + 1$ and takes its cube root. A natural way to write h as a composition of two functions is to take the cube root of the function $g(x) = x^2 + 1$. Thus, we let

$$f(x) = \sqrt[3]{x} \text{ and } g(x) = x^2 + 1.$$

We can check this composition by finding $(f \circ g)(x)$. This should give the original function, namely $h(x) = \sqrt[3]{x^2 + 1}$.

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 1) = \sqrt[3]{x^2 + 1} = h(x)$$

Check Point 6 Express $h(x)$ as a composition of two functions:

$$h(x) = \sqrt{x^2 + 5}.$$

Exercise Set 2.6

Practice Exercises

In Exercises 1–30, find the domain of each function.

1. $f(x) = 3(x - 4)$
2. $f(x) = 2(x + 5)$
3. $g(x) = \dfrac{3}{x - 4}$
4. $g(x) = \dfrac{2}{x + 5}$
5. $f(x) = x^2 - 2x - 15$
6. $f(x) = x^2 + x - 12$
7. $g(x) = \dfrac{3}{x^2 - 2x - 15}$
8. $g(x) = \dfrac{2}{x^2 + x - 12}$
9. $f(x) = \dfrac{1}{x + 7} + \dfrac{3}{x - 9}$
10. $f(x) = \dfrac{1}{x + 8} + \dfrac{3}{x - 10}$
11. $g(x) = \dfrac{1}{x^2 + 1} - \dfrac{1}{x^2 - 1}$
12. $g(x) = \dfrac{1}{x^2 + 4} - \dfrac{1}{x^2 - 4}$
13. $h(x) = \dfrac{4}{\frac{3}{x} - 1}$
14. $h(x) = \dfrac{5}{\frac{4}{x} - 1}$
15. $f(x) = \dfrac{1}{\frac{4}{x - 1} - 2}$
16. $f(x) = \dfrac{1}{\frac{4}{x - 2} - 3}$
17. $f(x) = \sqrt{x - 3}$
18. $f(x) = \sqrt{x + 2}$

19. $g(x) = \dfrac{1}{\sqrt{x-3}}$

20. $g(x) = \dfrac{1}{\sqrt{x+2}}$

21. $g(x) = \sqrt{5x+35}$

22. $g(x) = \sqrt{7x-70}$

23. $f(x) = \sqrt{24-2x}$

24. $f(x) = \sqrt{84-6x}$

25. $h(x) = \sqrt{x-2} + \sqrt{x+3}$

26. $h(x) = \sqrt{x-3} + \sqrt{x+4}$

27. $g(x) = \dfrac{\sqrt{x-2}}{x-5}$

28. $g(x) = \dfrac{\sqrt{x-3}}{x-6}$

29. $f(x) = \dfrac{2x+7}{x^3-5x^2-4x+20}$

30. $f(x) = \dfrac{7x+2}{x^3-2x^2-9x+18}$

In Exercises 31–48, find $f+g, f-g, fg$, and $\frac{f}{g}$. Determine the domain for each function.

31. $f(x) = 2x+3, g(x) = x-1$

32. $f(x) = 3x-4, g(x) = x+2$

33. $f(x) = x-5, g(x) = 3x^2$

34. $f(x) = x-6, g(x) = 5x^2$

35. $f(x) = 2x^2-x-3, g(x) = x+1$

36. $f(x) = 6x^2-x-1, g(x) = x-1$

37. $f(x) = 3-x^2, g(x) = x^2+2x-15$

38. $f(x) = 5-x^2, g(x) = x^2+4x-12$

39. $f(x) = \sqrt{x}, g(x) = x-4$

40. $f(x) = \sqrt{x}, g(x) = x-5$

41. $f(x) = 2+\dfrac{1}{x}, g(x) = \dfrac{1}{x}$

42. $f(x) = 6-\dfrac{1}{x}, g(x) = \dfrac{1}{x}$

43. $f(x) = \dfrac{5x+1}{x^2-9}, g(x) = \dfrac{4x-2}{x^2-9}$

44. $f(x) = \dfrac{3x+1}{x^2-25}, g(x) = \dfrac{2x-4}{x^2-25}$

45. $f(x) = \sqrt{x+4}, g(x) = \sqrt{x-1}$

46. $f(x) = \sqrt{x+6}, g(x) = \sqrt{x-3}$

47. $f(x) = \sqrt{x-2}, g(x) = \sqrt{2-x}$

48. $f(x) = \sqrt{x-5}, g(x) = \sqrt{5-x}$

In Exercises 49–64, find

a. $(f \circ g)(x)$; **b.** $(g \circ f)(x)$; **c.** $(f \circ g)(2)$.

49. $f(x) = 2x, g(x) = x+7$

50. $f(x) = 3x, g(x) = x-5$

51. $f(x) = x+4, g(x) = 2x+1$

52. $f(x) = 5x+2, g(x) = 3x-4$

53. $f(x) = 4x-3, g(x) = 5x^2-2$

54. $f(x) = 7x+1, g(x) = 2x^2-9$

55. $f(x) = x^2+2, g(x) = x^2-2$

56. $f(x) = x^2+1, g(x) = x^2-3$

57. $f(x) = 4-x, g(x) = 2x^2+x+5$

58. $f(x) = 5x-2, g(x) = -x^2+4x-1$

59. $f(x) = \sqrt{x}, g(x) = x-1$

60. $f(x) = \sqrt{x}, g(x) = x+2$

61. $f(x) = 2x-3, g(x) = \dfrac{x+3}{2}$

62. $f(x) = 6x-3, g(x) = \dfrac{x+3}{6}$

63. $f(x) = \dfrac{1}{x}, g(x) = \dfrac{1}{x}$

64. $f(x) = \dfrac{2}{x}, g(x) = \dfrac{2}{x}$

In Exercises 65–72, find

a. $(f \circ g)(x)$; **b.** *the domain of* $f \circ g$.

65. $f(x) = \dfrac{2}{x+3}, g(x) = \dfrac{1}{x}$

66. $f(x) = \dfrac{5}{x+4}, g(x) = \dfrac{1}{x}$

67. $f(x) = \dfrac{x}{x+1}, g(x) = \dfrac{4}{x}$

68. $f(x) = \dfrac{x}{x+5}, g(x) = \dfrac{6}{x}$

69. $f(x) = \sqrt{x}, g(x) = x-2$

70. $f(x) = \sqrt{x}, g(x) = x-3$

71. $f(x) = x^2+4, g(x) = \sqrt{1-x}$

72. $f(x) = x^2+1, g(x) = \sqrt{2-x}$

In Exercises 73–80, express the given function h as a composition of two functions f and g so that $h(x) = (f \circ g)(x)$.

73. $h(x) = (3x-1)^4$

74. $h(x) = (2x-5)^3$

75. $h(x) = \sqrt[3]{x^2-9}$

76. $h(x) = \sqrt{5x^2+3}$

77. $h(x) = |2x-5|$

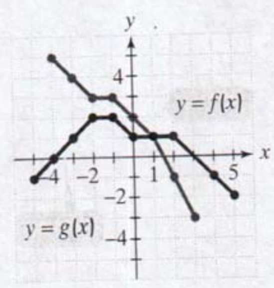
78. $h(x) = |3x-4|$

79. $h(x) = \dfrac{1}{2x-3}$

80. $h(x) = \dfrac{1}{4x+5}$

Practice Plus

Use the graphs of f and g to solve Exercises 81–88.



81. Find $(f+g)(-3)$.

82. Find $(g-f)(-2)$.

83. Find $(fg)(2)$.

84. Find $\left(\dfrac{g}{f}\right)(3)$.

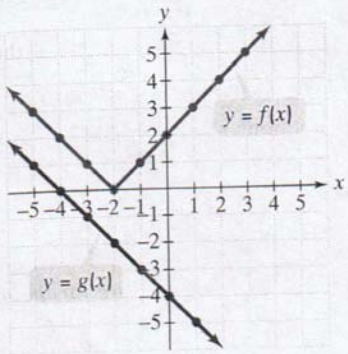
85. Find the domain of $f+g$.

86. Find the domain of $\dfrac{f}{g}$.

87. Graph $f+g$.

88. Graph $f-g$.

In Exercises 89–92, use the graphs of f and g to evaluate each composite function.



89. $(f \circ g)(-1)$

90. $(f \circ g)(1)$

91. $(g \circ f)(0)$

92. $(g \circ f)(-1)$

In Exercises 93–94, find all values of x satisfying the given conditions.

93. $f(x) = 2x - 5$, $g(x) = x^2 - 3x + 8$, and $(f \circ g)(x) = 7$.

94. $f(x) = 1 - 2x$, $g(x) = 3x^2 + x - 1$, and $(f \circ g)(x) = -5$.

Application Exercises

We opened the section with functions that model the numbers of births and deaths in the United States from 2000 through 2005:

$B(x) = 7.4x^2 - 15x + 4046$ $\quad$ $D(x) = -3.5x^2 + 20x + 2405$.

Number of births, $B(x)$, in thousands, x years after 2000

Number of deaths, $D(x)$, in thousands, x years after 2000

Use these functions to solve Exercises 95–96.

95. a. Write a function that models the change in U.S. population for each year from 2000 through 2005.

b. Use the function from part (a) to find the change in U.S. population in 2003.

c. Does the result in part (b) overestimate or underestimate the actual population change in 2003 obtained from the data in **Figure 2.49** on page 270? By how much?

96. a. Write a function that models the total number of births and deaths in the United States for each year from 2000 through 2005.

b. Use the function from part (a) to find the total number of births and deaths in the United States in 2005.

c. Does the result in part (b) overestimate or underestimate the actual number of total births and deaths in 2005 obtained from the data in **Figure 2.49** on page 270? By how much?

97. A company that sells radios has yearly fixed costs of \$600,000. It costs the company \$45 to produce each radio. Each radio will sell for \$65. The company's costs and revenue are modeled by the following functions, where x represents the number of radios produced and sold:

$C(x) = 600{,}000 + 45x$ — This function models the company's costs.

$R(x) = 65x$. — This function models the company's revenue.

Find and interpret $(R - C)(20{,}000)$, $(R - C)(30{,}000)$, and $(R - C)(40{,}000)$.

98. A department store has two locations in a city. From 2004 through 2008, the profits for each of the store's two branches are modeled by the functions $f(x) = -0.44x + 13.62$ and $g(x) = 0.51x + 11.14$. In each model, x represents the number of years after 2004, and f and g represent the profit, in millions of dollars.

a. What is the slope of f? Describe what this means.

b. What is the slope of g? Describe what this means.

c. Find $f + g$. What is the slope of this function? What does this mean?

99. The regular price of a computer is x dollars. Let $f(x) = x - 400$ and $g(x) = 0.75x$.

a. Describe what the functions f and g model in terms of the price of the computer.

b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the computer.

c. Repeat part (b) for $(g \circ f)(x)$.

d. Which composite function models the greater discount on the computer, $f \circ g$ or $g \circ f$? Explain.

100. The regular price of a pair of jeans is x dollars. Let $f(x) = x - 5$ and $g(x) = 0.6x$.

a. Describe what functions f and g model in terms of the price of the jeans.

b. Find $(f \circ g)(x)$ and describe what this models in terms of the price of the jeans.

c. Repeat part (b) for $(g \circ f)(x)$.

d. Which composite function models the greater discount on the jeans, $f \circ g$ or $g \circ f$? Explain.

Writing in Mathematics

101. If a function is defined by an equation, explain how to find its domain.

102. If equations for f and g are given, explain how to find $f - g$.

103. If equations for two functions are given, explain how to obtain the quotient function and its domain.

104. Describe a procedure for finding $(f \circ g)(x)$. What is the name of this function?

105. Describe the values of x that must be excluded from the domain of $(f \circ g)(x)$.

Technology Exercises

106. Graph $y_1 = x^2 - 2x$, $y_2 = x$, and $y_3 = y_1 \div y_2$ in the same $[-10, 10, 1]$ by $[-10, 10, 1]$ viewing rectangle. Then use the $\boxed{\text{TRACE}}$ feature to trace along y_3. What happens at $x = 0$? Explain why this occurs.

107. Graph $y_1 = \sqrt{2 - x}$, $y_2 = \sqrt{x}$, and $y_3 = \sqrt{2 - y_2}$ in the same $[-4, 4, 1]$ by $[0, 2, 1]$ viewing rectangle. If y_1 represents f and y_2 represents g, use the graph of y_3 to find the domain of $f \circ g$. Then verify your observation algebraically.

Critical Thinking Exercises

Make Sense? *In Exercises 108–111, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

108. I used a function to model data from 1980 through 2005. The independent variable in my model represented the number of years after 1980, so the function's domain was $\{x \mid x = 0, 1, 2, 3, \ldots, 25\}$.