Calculus - 5.6 Uninhibited and Inhibited Growth and Decay Models

- Solve Differential Equations involving uninhibted growth and decay
- Solve differential equations involving inhibited growth and decay
- Apply: Newton's Law of Cooling

Uninhibited Growth and Decay models:

Things that grow or decay proportionally to itself, with no limits, is known as "uninhibited" growth/decay.

Examples: Radioactive decay, bacteria growth, population growth, etc...

We saw a model in Algebra 2 which depicted this kind of growth/decay, but the originator of this model is a differential equation.

There are situations in science, nature, and economics such as radioactive decay, population growth, and interest paid on an investment, in which a quantity A varies with time t in such a way that the rate of change of A with respect to t is proportional to A itself. These situations can be modeled by the differential equation

$$\boxed{\frac{dA}{dt} = kA} \qquad (1)$$

where $k \neq 0$ is a real number.

- If $k > 0$ and $\frac{dA}{dt} = kA$, the rate of change of A with respect to t is positive, and the amount A is increasing. (Then $\frac{dA}{dt}$ is a **growth model**.)
- If $k < 0$ and $\frac{dA}{dt} = kA$, the rate of change of A with respect to t is negative, and the amount A is decreasing. (Then $\frac{dA}{dt}$ is a **decay model**.)

Suppose that the initial amount A_0 of the substance is known, giving us the boundary condition, or initial condition, $A = A(0) = A_0$ when $t = 0$.

The solution to this differential equation gives us the function...

growth/decay factor

$$\boxed{A = A_0 e^{kt}}$$

time

Total Amount (after time t)

Initial Amount

Uninhibited Growth Example:

Assume that a colony of bacteria grows at a rate proportional to the number of bacteria present. If the number of bacteria doubles in 5 hours (h), how long will it take for the number of bacteria to triple?

$A_0 = 1$; @ $t = 5$ $A = 2$

What is t when $A(t) = 3$

$A = A_0 e^{kt}$

$2 = e^{k(5)}$

$\log_e 2 = 5k$

$k = \frac{1}{5} \ln 2$

$3 = e^{\frac{1}{5} \ln 2 \cdot t}$

$\frac{5 \cdot \ln 3}{\ln 2} = \cancel{\frac{1}{5}} \cancel{\ln 2} \cdot t$

$t = 5 \frac{\ln 3}{\ln 2} \approx 7.92$ hrs

Uninhibited Decay Example:

The skull of an animal found in an archaeology dig contains about 20% of the original amount of carbon-14. If the half-life of carbon-14 is 5730 years, how long ago did the animal die?

What is t when $A(t) = .2$

$$\frac{1}{2} = 1e^{k(5730)}$$

$$\ln\frac{1}{2} = 5730k \qquad k = -\frac{\ln 2}{5730}$$

$$\ln 2^{-1}$$

$$-\ln 2$$

$$0.2 = e^{-\frac{\ln 2}{5730}\cdot t}$$

$$-\ln 5 = -\frac{\cancel{\ln 2}}{\cancel{5730}}t \cdot -\frac{\cancel{5730}}{\cancel{\ln 2}}$$

$$\cdot -\frac{5730}{\ln 2}$$

$$t = 5730 \cdot \frac{\ln 5}{\ln 2} \approx 13305 \text{ yrs}$$

Inhibited Growth/Decay:

Excerpt:

In uninhibited growth or decay, the amount y either grows without bound or decays to zero. In many situations there is an upper value M that the amount y cannot exceed (inhibited growth), or there is a lower value M that y cannot go below (inhibited decay). For inhibited growth or decay models the rate of change of y with respect to time t satisfies a different differential equation.

A popular example of inhibited growth/decay is Newton's Law of Cooling, which is the idea that an object placed in an environment will have it's temperature change over time based on the temperature of the surrounding environment.

Differential Equation for Newton's Law of Cooling:

$$\boxed{\frac{du}{dt} = k[u(t) - T]}$$

Solution to the equation above:

rate

$$\boxed{u(t) = (u_0 - T)e^{kt} + T}$$

time

initial temp of object

Temp of object after time t

Temp of environment object is placed into

Example with inhibited decay:

An object is heated to 90 degrees Celsius and allowed to cool in a room with a constant ambient temperature of 20 degrees Celsius. If after 10 minutes the temperature of the object is 60 degrees Celsius, what will its temperature be after 20 minutes?

$u_0 = 90 \qquad T = 20$

at $t = 10$, $u(t) = 60$

$60 = (90 - 20)e^{k(10)} + 20$

$40 = 70e^{10k}$

$\frac{4}{7} = e^{10k} \qquad \ln\frac{4}{7} = 10k$

$k = \frac{1}{10}\ln\frac{4}{7}$

$u(20) = 70e^{\frac{1}{10}\ln(4/7)\cdot 20} + 20$

$u(20) \approx 42.86$

Suggested Homework:

1-7 Odd & AP Practice Problems