KNOWLEDGE AND REASONING IN MATHEMATICAL PEDAGOGY: EXAMINING WHAT PROSPECTIVE TEACHERS BRING TO TEACHER EDUCATION

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ABSTRACT

KNOWLEDGE AND REASONING IN MATHEMATICAL PEDAGOGY: EXAMINING WHAT PROSPECTIVE TEACHERS BRING TO TEACHER EDUCATION

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This study focuses on the knowledge and beliefs about mathematics and the teaching of mathematics held by prospective teachers when they enter teacher education. It offers frameworks for thinking about the role and relationship of different kinds of knowledge in teaching mathematics. Grounded in a vision of a particular mathematical pedagogy, these frameworks are constructed around teacher knowledge, beliefs, and dispositions in the domains of subject matter, teaching and learning, students, and context.

A two-part interview was designed to explore prospective teachers' ideas and thinking about mathematics and the teaching and learning of mathematics. Nineteen prospective teachers were interviewed. The study's results include description and appraisal of the participants' understandings and ways of thinking about the subject matter, teaching and learning, the teacher's role, and students. These results suggest categories useful for examining what prospective teachers bring with them to their professional preparation to teach mathematics. Focal subject matter knowledge categories included the explicitness and connectedness of substantive understandings, as well as ideas about the justification and nature of mathematical knowledge and activity. With respect to teaching and learning, central categories were the teacher's role, how learning occurs and, with respect to learners, ideas about the sources of success and failure in mathematics, knowledge of and dispositions to learn about students. The analysis also highlighted the importance of clarifying terms such as "concept" and "explanation" which may be used in both professional and commonsense ways.

Teaching mathematics is more than a sum of different components of knowledge; teachers' understandings and beliefs in one domain interact with their understandings and beliefs in others to shape their pedagogical reasoning. Therefore, a second theoretical framework was developed to appraise teachers' instructional representations of mathematics. Examining teachers' pedagogical reasoning in terms of the tacit or explicit warrants they use to justify their representations of the subject brings the components of teacher knowledge back together. The framework of warrants also offers another perspective for examining what prospective teachers bring, focusing on the patterns of thinking and justification that they assume as they enter teacher education.

To

Richard, Sarah, and my parents

ACKNOWLEDGMENTS

The time to bind this tome has arrived. The work is not finished; no work that so fully absorbs and transforms one can truly end. And yet, with the binding of these pages comes a pause in the bubbling and churning of the past months. As I stop sculpting this text, I have a moment to breathe deeply. I find myself thinking back over the past months and years to the many people who have contributed significantly to my growth.

I start with my family: Richard, my special friend and husband, with whom I have now spent half my life, has supported my intensity in ways both big and small. Consistently supportive, he has nurtured a family life which grounds me and helps me to maintain perspective. I am grateful for how well he understands me and helps me understand myself, and for the room and freedom I have always had to grow. All that I owe to him I cannot even put into words.

For Sarah, almost nine, this dissertation has filled too big a proportion of her young life. Although she was, for a while only seeing her mother at the oddest hours, she has been cheerful and understanding at every turn. She also helped directly with my work. On her own initiative, she ran a phone-answering service to protect my writing time ("I'm sorry but my mother can't come to the phone right now"). She also tried the card sort task, provided child-like handwriting for samples used in the interviews, and checked the references (the responsibility for which is nonetheless mine).

Going a little further back, I owe much of my discipline and intellectual desire to my parents, Ina and Jerry, who consistently fostered in me dispositions such as loving to read, to learn, to write, and, above all, to argue. Their expectations of and belief in me have made mere possibilities attainable. My brother, Michael, finishing his doctoral dissertation (on heterogeneous reaction and diffusion), gave me the critical push I needed to finish, just as I drooped in mid-February. A mathematician himself, he talked through substantive issues with me — such as what proof means to him in his work. The pride that my parents-in-law, Norma and Ivan, have shown for me has also meant a great deal as I cut a course for my life quite different from anything they were used to.

I turn next to my teachers. Fortunate to have had a committee who, chosen for their distinct and distinctive intellectual interests and strengths, always worked together to help me learn, I have been pushed by each of them to think and to reach well beyond where I was when I began. Their prodding and challenging has been matched only by their support and their caring. Perry Lanier, the chair of my committee, Sharon Feiman-Nemser and Magdalene Lampert, who directed this dissertation, Margret Buchmann and Robert Floden — each of them, in ways both gentle and tough, has contributed in different and important ways to my development. Their influence is readily apparent within these covers — and in me.

Perry Lanier, with his gentle but insistent pedagogy, has helped me move from a fifth grade teacher who was inexplicitly worried about her mathematics teaching to a researcher and teacher educator who explicitly worries and works to improve it — while bringing the teacher in me along. His encouragement of my own study of mathematics helped me to stretch and redefine my interests, as well as to negotiate the foreign culture of the university mathematics classroom.

From Sharon Feiman-Nemser, with whom I have worked closely since my first term in graduate school, I have learned to teach and learned about learning to teach. She has fostered in me new ways of looking that have helped me begin to understand some of the most central questions about teaching, about being and becoming a teacher. Her thoughtful responses to my writing have, in addition to helping me clarify what I know, taught me immeasurably about writing and about helping others learn to think and to write.

Magdalene Lampert, as elementary school teacher-researcher-teacher educator, has helped me learn to integrate those same pieces in myself — in the dissertation as well as in my ongoing work. With Maggie, I can move from Lakatos to third grade to number theory and have a coherent conversation. Her understanding of and full absorption in the multiple dimensions of my work is helping me to develop both power and grace.

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Finally, from the beginning of this study, Robert Floden kept asking me hard questions — like what this study was really <u>about</u>. He read drafts and typed notes back to me with more questions. Bob's questions have made a big difference in the long run, for I've gotten better at answering his and better at asking my own.

As these teachers have also become my friends, so my friends have taught me too. They have pushed and encouraged me at critical points. I must begin with Suzanne Wilson, with whom I shared, amid laughter and tears, the labor of conceiving and giving birth to a dissertation. As a team, bonded by our commitment to <u>Finish</u>, we made fun what should have been a "shared ordeal" and learned a lot from one another along the way. Drinking strong, strong coffee at 3 o'clock in the morning, we have tried to figure out exactly the right word for a new idea, puzzled about the consequences of talking about "pedagogical reasoning" versus "pedagogical thinking," talked through mental outlines, challenged one another's thinking — as well as figured out how to get long footnotes to print all on the same page. With dissertations so similar and yet so different, our work grew stronger as we sharpened our ideas against the other's subject area, design and focus. We sometimes even talked about things beyond our dissertations — about our lives.

Kathy Roth and Saundra Dunn have lived through graduate school with me. Kathy, whose dissertation three years ago filled me with awe, has taken the time to talk, to listen, and to encourage me whenever I needed it. Realizing the value of coffee in a sleepless world, her gift of a coffee grinder was a major contribution to the dissertation. Saundra Dunn, patient with my immersion, understood when I spent hour after hour writing, even when it was really time to play. She even understood when I couldn't show up for my own birthday party this spring.

Bill McDiarmid, with whom I first wrote about representations in teaching, has shared my struggles about teacher knowledge. Our joint work in developing the NCRTE questionnaire, interviews, and observation guide helped me become clearer about my own work; our many conversations about what teachers need to know helped me see the different categories — as well as the problems <u>with</u> categories.

Many other friends have supported and encouraged me; I cannot mention them all here. Jessie Fry, Pam Schram, Ruhama Evan, Susan Irwin, Wendy Esmailka, and Jim Mosenthal are among those who have made a real difference.

I want to thank David Cohen for enticing and encouraging me to participate in the National Center for Research on Teacher Education and for believing I could do it; I am also grateful to Mary Kennedy and the NCRTE for the intellectual, personal, and financial support I have had throughout this process. NCRTE staff members — Marilyn Augustine, Sharon Jourdan, Lucy Sanchez, and Rose Snitgen — have helped in many ways also, and always with perpetual good humor and patience.

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CHAPTER 1

INTRODUCTION

> The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. . . Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it. . . .The reason for this failure of the science to live up to its great reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. (Whitehead, 1911 / 1948, pp. 1-2)

Alfred North Whitehead opens his book, <u>An Introduction to Mathematics</u> (1911 / 1948) by lamenting the trivial encounters that most people have with mathematics. In this thin but dense volume, he sets out to correct the "pedantry" of traditional mathematics teaching by helping his readers to develop an appreciation for the power of mathematics as a system of human thought. In a similar spirit, Michael Guillen (1983), a contemporary research mathematician and physicist, argues that those who are alienated from mathematics are "deprived of being able to consider for [themselves] the manifold and unique ways in which mathematics bears on our far-flung human concerns, including questions about God" (p. 2). Passionately, Guillen writes about his desire to ignite readers with his excitement about mathematics, to persuade them that mathematics is a form of human imagination and vision accessible to all.

Despite the good intentions and the passions of mathematics lovers like Whitehead and Guillen, the school mathematics experience of most Americans is and has been uninspiring at best and mentally and emotionally crushing at worst. "Mathematical presentations, whether in books or in the classroom, are often perceived as authoritarian" (Davis & Hersh, 1981, p. 282). Ironically, the most logical of the human disciplines of knowledge is transformed through a misrepresentative pedagogy into a body of precepts and facts to be remembered "because the teacher said so." Despite its power, rich traditions, and beauty, mathematics is too often unknown, misunderstood, and rendered inaccessible. The consequences of traditional mathematics teaching have been documented; they include lack of meaningful understanding, susceptibility to "mathematical puffery and nonsense" (Schoenfeld, in press b), low and uneven participation, personal dread.

As an experienced mathematics teacher and lateblooming mathematics amateur, I, too, worry about our culture's mathematical disenfranchisement and disenchantment. Because this dissertation grows out of my concern for improving the character and quality of mathematics teaching, I begin this first chapter by exploring ***mathematical pedagogy***[1] **—** a vision of mathematics teaching and learning quite different from what students encounter in most American classrooms.

Mathematical pedagogy, so named because disciplinary mathematics is at its core, affords students the opportunity to develop both understanding and power in mathematics. It is an approach to teaching math that emphasizes making sense and having control in situations that involve quantitative and spatial reasoning. In order to give the reader a vision of mathematical pedagogy, this chapter begins with a story from a third grade class. Examining what is going on in this class takes us on a short trek into the philosophy of mathematics and the inherent relationship between epistemology and pedagogy. To highlight the distinctive features of a mathematical pedagogy, I then compare this view of mathematics teaching with two other dominant views of mathematics teaching.

The chapter continues with a brief discussion of four factors that make a mathematical pedagogy difficult to enact. While the goals of mathematical pedagogy provoke little disagreement, this ideal is rarely realized in American elementary and secondary classrooms. No single cause adequately explains why American mathematics teaching is stung with inertia, despite frequent and varied efforts to change it.

Together, these two pieces — the vision of mathematical pedagogy and the problems of realizing it — set the stage for the focus and goals of this study, which I present in the third section of the chapter.

Mathematical Pedagogy

Rooted in mathematics itself, the goal of a mathematical pedagogy is to help students develop mathematical power and to become active participants in mathematics as a system of human thought. To do this, pupils must learn to make sense of and use concepts and procedures that others have invented — the body of accumulated knowledge in the discipline — but they also must have experience "doing" mathematics, developing and pursuing mathematical hunches themselves, inventing mathematics, and learning to make mathematical arguments for their ideas (see Romberg, 1983). Good mathematics teaching, according to this view, should eventually result in meaningful understandings of concepts and procedures, as well as in understandings *about* mathematics: what it means to "do" mathematics and how one establishes the validity of answers, for instance. I embrace this vision of mathematical pedagogy in reaction to the mindless mathematics activity that prevails in so many schools and results in so little meaningful student learning (Erlwanger, 1975; Wheeler, 1980).

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[1] I use this term stipulatively to refer to a particular view of mathematics teaching. The label was first suggested to me by Magdalene Lampert who argued that it appropriately reflected the primary influence of the discipline of mathematics on this approach to teaching.

Mathematical Pedagogy in Action

To illustrate concretely what a mathematical pedagogy might look like,[2] I use an example from a third grade classroom in which students are learning about perimeter and area and the relationship between these two measures.[3] On the day that I will describe, students were presenting their solutions to the following problem which the teacher had posed the day before:

> Suppose you had 64 meters of fence with which you were going to build a pen for your large dog, Bozo. What are some different pens you can make if you use all the fence? Which is the pen with the most play space for Bozo? Which pen allows him the least play space?

The students have worked on this problem in a variety of ways, using tools and strategies which they have acquired: graph paper for drawing alternative dogpens, rulers for drawing constructions of dogpens on plain paper,[4] making tables of the possible combinations of dimensions that total 64 and calculations of area using L x W (length times width). Some students have worked together on the problem; others have worked alone.

The teacher opens discussion by asking who would like to present a solution to the problem of finding the most play space for Bozo. A girl comes up to the board and carefully draws a 18 x 14-inch rectangular pen on the board. Turning to face the class, she announces that this is the largest pen that can be built with 64 meters of fence. The teacher asks her to explain. The girl points at her drawing and says that she knows she has used just 64 meters of fence because 18 + 18 = 36, and 14 + 14 = 28, and 36 + 28 = 64. "And how much play space does your dog have?" probes the teacher. "Two hundred fifty-two square meters," she replies, after glancing down at her wrinkled paper. "How did you get that?" asks a small boy sitting on the side of the class. Carefully she explains:

> 18 x 14 you can do in 18 x 10 and 18 x 4 and then put them back together. 18 x 10 is 180 [she writes this on the board]. 18 x 4 I have to do in parts, too. 10 x 4 is 40, and 8 x 4 is. . .32, so 18 x 4 is 72. Now — 180 + 72 is 252!

She smiles. Another girl raises her hand and announces that she has made a dogpen with a still larger play space than the one on the board. The teacher invites her to show the class. The girl quickly draws a square 16 x 16 and, breathlessly, explains that 16 + 16 + 16 + 16 is 64, and 16 x 16 is 256, which is more than 252. Several children nod, as they, too, have found this dogpen.

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[2] No single way of enacting a mathematical pedagogy is implied through the use of this example. Depending on a host of factors — the particular grade level, topic, time of day or year, for instance — teachers may do different things in the service of these goals.

[3] I draw this example from my own teaching. Other outstanding written examples exist — see, for example: Lampert, 1985b, 1986, in press a; Schoenfeld, 1985, in press b.

[4] There is a difference between using graph paper and using rulers to draw dogpens: graph paper is a simpler tool, allowing more direct and concrete exploration of the problem. With graph paper, the students can "see" the area of the dogpen without any formulaic knowledge — i.e., they can just count the number of squares inside the pen. With drawings on plain paper, they will probably have to decide on a scale for their drawings (because they will not want drawings where the dimensions total 64 inches or even 64 centimeters). They will also have to use some sort of formula for calculating the area of the dogpens they construct.

The teacher looks around and asks, "Is this the dogpen with the most play space? Raise your hand if you think it is." All but one raise their hands. The teacher looks at the boy who didn't raise his hand. "You don't think this is the largest, Jeremy?" she asks. He frowns, "I'm just not sure. How do we know that it's the largest one?" Another child turns to him and says, "No one found one with more play space." Jeremy shakes his head, saying that just because they didn't find one with a larger play space doesn't mean there isn't one.

The teacher is listening to all this. She asks the class how they could investigate Jeremy's question. Someone says that they could go back to looking for another dogpen with a larger play space.[5] The teacher notices that most of the pupils are now skeptical that the square allows the largest possible play space and she suggests that anyone who is interested should explore it further later. The class moves on to look at solutions for the smallest dogpen.

Later the teacher refers to this next part of the lesson as "the most exciting thing that happened." It starts off as before: One of the boys draws a 30 x 2 rectangle, proves that he has used exactly 64 meters of fence, and announces that the area is only 60 square meters — "very skinny." As before, another challenges with a 31 x 1 rectangular pen. Play space: 31 square meters! The difference catches everyone off guard and the teacher leads a brief discussion about why the area of the 30 x 2 dogpen is so much greater than the 31 x 1 pen. Everyone seems satisfied that this is the smallest, when a small girl named Julia raises her hand. "This may be crazy," she says hesitantly, "but I was thinking — now Bozo wouldn't like this much, but — what if we made a dog pen with 32 meters along one side and 32 along the other and that would use 64 meters of fence, and if we push the two sides together as close as we can, there would be almost no play space inside."[6] The other pupils nod in amazement. The teacher smiles and asks the class what conclusion they draw from this. Someone says this means that there isn't really a smallest play space, "because you couldn't really get it so there was no space at all between the two sides but you could get very, very little." The teacher asks what others think. Another child says that she sees now that she was assuming that she should use whole numbers only for the lengths of the sides of the pen, and she shouldn't have assumed that. "If it was just whole numbers, though," asks the teacher, "would your answer be different?" The child says

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[5] In fact, the largest dogpen would be a circular one. A circle with a circumference of 64 meters would have an area of 325 square meters. ($64 = 2\pi r$ [formula for circumference], so, $r = 10.2$ meters and $\pi r^2 = 325$ square meters.) With calculus, one can prove that, for a given perimeter, a circle will always have the largest area.

The teacher knows that her students do not have the knowledge they need to calculate the perimeter or area of a circle, nor do they have tools that would allow them to invent a procedure to do that. Still, she knows, from other experience with eight-year-olds, that someone may intuitively figure out that a circle with a given perimeter will have a greater "inside" than will a square. The uncertainty about whether this is the largest dogpen would still remain, however, part of what she wants to represent to her students. In any case, what the third graders decide to accept as a solution, given what they understand now, will later be refuted as they acquire more knowledge. This pedagogical feature represents one view of the growth of mathematical knowledge (see Lakatos, 1976).

[6] The child is intuitively articulating the mathematical idea of taking a limit. Without formal procedures, she is showing that the smaller you make the widths of the rectangle, the smaller the area will become: as the widths approach 0 meters in length, the area will approach 0 as well, thus the smallest area possible within the constraints of the problem. The child's proof exemplifies Hawkins's (1972/1974) description of how young children's mathematical explorations take them from time to time "near mathematically sacred ground" (p. 119).

yes, that then the solution would be 31 x 1 meters. Several others make comments both about the largest and smallest play space parts of the problem.

The teacher remarks that she is glad to see that they are becoming more skeptical when they think they have solved a problem like this one. "One of the important questions in mathematics," she says, "is how do you know when you have solved a problem? How do you know you have found the most sensible or the correct solution? You have to try to prove it to yourself and to others." She points out that, with the largest play space problem, they are all unsure now whether the square pen is the largest, but with the smallest play space, Julia's proof has persuaded them (for the moment, at least) that there isn't really a smallest play space. The teacher reminds everyone to explore the largest play space problem some more and then sends them out to recess.

What was going on in this lesson? With their eye on the problem — i.e., what is the largest and smallest area that can be enclosed by a given perimeter — the students were employing mathematical concepts and procedures in the service of its solution. Figuring out and manipulating various possible combinations of dimensions that can be made out of a given length of fence engaged the pupils in adding and multiplying as well as using place value and the distributive property. Examining the areas that result from those different combinations afforded them an opportunity to consider what the measurement concepts of area and perimeter are each about. The process of presenting and justifying solutions was fertile ground for inquiry and arguments as children searched for reasonable solutions that their peers would accept. The challenges they presented to one another revealed the nature and power of proof — how can one persuade others in the mathematical community that one's solution is reasonable or even the best? The idea that mathematics entails puzzles and uncertainties and that mathematical thinking involves questions as much as answers was represented vividly through the problem of knowing whether one has indeed found the largest area. In this classroom students are helped to acquire the skills and understanding needed to judge the validity of their own ideas and results — to be "independent learners" or to be "empowered" in ways that are specific to mathematics.

From Third Grade to the Philosophy of Mathematics

Imre Lakatos's Proofs and Refutations (1976) grounds this particular mathematical pedagogy in a conception of mathematical knowledge and activity.[7] Just as this mathematical pedagogy represents an attack on authoritarian and dogmatist norms of mathematics teaching, the Lakatos argument attacks the idealized picture of formal mathematics. Describing the classic "deductivist style" as starting with mystifying axioms and definitions and followed by slickly proved theorems, Lakatos argues that it presents a misleading picture of mathematics as an "ever-increasing set of eternal, immutable truths" surrounded with an authoritarian air. He is sharp in his criticism:

> Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of

[7] Other mathematical pedagogies might be developed to intentionally represent mainstream views of the nature of mathematical knowledge and activity. The mathematical pedagogy developed in this dissertation, however, is based on a fallibilist epistemology. Taking this view of mathematics as a pedagogical foundation can be justified based on its convergence with constructivist theories of learning as well as with the revisionism that is an inevitable function of both the process of learning and the organization of the school curriculum.

> the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility. (p. 142)

And Lakatos makes a direct link to pedagogy:

> It has not yet been sufficiently realised that the present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought. (p. 143)

While the formalists distinguish the "logic of discovery" from the "logic of justification," for Lakatos, these are inseparable in the doing of mathematics and in clarifying the nature of mathematical truth. Mathematicians develop hunches and conjectures, the pursuit of which entails seeking supporting arguments and counterexamples: "Global" counterexamples destroy a conjecture, while "local" ones help to revise and strengthen it. In other words, the process of figuring something out in mathematics fuses discovery and justification.

Lakatos's Proofs and Refutations portrays this fusion. Based on the development of the Euler-Descartes formula about the relationship between the number of vertices, edges, and faces of regular polyhedra, the book shows how a mathematical idea grows through challenge and revision. A classroom dialogue in which the teacher and students pursue the Euler conjecture together is the presentational form. Proofs and Refutations is a brilliant trifold: the history of the specific conjecture, an argument about the nature of mathematical knowledge and activity, and a corollary model of mathematical pedagogy. Each stands alone as a goal for his work; yet the power of the book derives from their conjoining. The teacher "proves" the conjecture, only to be immediately challenged by several students. Over time the class recreates the history of this famous conjecture, as alternative speculations are tried out, objections raised, and the conjecture elaborated and revised.

Greig (1988) speculates that Lakatos chose this form, in part, in order to model his vision of ideal mathematics teaching in which students pursue big mathematical ideas, simultaneously acquiring knowledge of and about mathematics. "Mathematical activity," Lakatos writes, "produces mathematics" (p. 146). Corollary, then, is student engagement in mathematical activity in order to acquire mathematical knowledge. In this activity, proof is conceived not as "mechanical procedure which carries truth from assumptions to conclusions" (Davis & Hersh, 1981, p. 347) but as a form of "thought experiment" (Lakatos, 1976, p. 9) in which the effort to explain advances one's understanding of the phenomenon. This view of mathematical knowledge and activity aligns pedagogical and disciplinary structures in ways that formalist mathematics precludes (see Schwab, 1961/1978, pp. 241-242; also Kline, 1970).

Comparing Mathematical Pedagogy to Other Views of Math Teaching

In mathematical pedagogy, the goals are different from those of the "ordinary" math class or of a more conceptual approach to mathematics teaching (e.g., Good, Grouws, & Ebmeier, 1983).[8] In this ideal, learning what mathematics is and how one engages in it are goals

[8] This contrast is not intended to map the field of alternative approaches to the teaching of mathematics.

coequal and interconnected with acquiring the "stuff" — concepts and procedures — of mathematics.

What *is* mathematics? The kind of math teaching that I am describing differs radically from "ordinary" mathematics teaching on what counts as "mathematics" — what students are supposed to learn, what matters about learning mathematics, what it means to know and to do mathematics, and where the authority for truth lies.

The ordinary math class makes mathematics synonymous with computation. Students learn algorithms for computational procedures; calculational speed and accuracy are valued goals and meaning is hardly ever the focus. Problem solving generally means word problems which are little more than symbols dressed up in words and have little to do either with real life or with mathematics (Lave, 1988). By default the book has epistemic authority: Teachers explain assignments to pupils by saying, "This is what they want you to do here," and the right answers are found in the answer key. In this kind of teaching, knowing mathematics means remembering definitions, rules, and formulae, and doing mathematics is portrayed as a straightforward step-by-step process. Learning to "play by the rules" often entails a "suspension of sense-making" in school mathematics (Davis, 1983, 1986; Lave, 1988; Schoenfeld, in press b, p. 9; Stodolsky, 1985). Math is primarily a "tool" subject, one that has applications to everyday life both by ordinary people who must balance their checkbooks and by experts who in some mysterious way use mathematics to construct bridges or send rockets into space.

In more conceptual mathematics teaching, mathematics is seen as "a body of logically consistent, closely related ideas" (Good, Grouws, and Ebmeier, 1983) and knowing mathematics entails understanding those relationships. For example, when students learn about fractions, they should be led to integrate that new knowledge — e.g., calculating with fractions and ordering fractions — with their previous knowledge about natural numbers. The goal is for students to remember what they have learned, but the assumption is that they will remember more effectively if they understand what they are being taught rather than simply learning by rote. Therefore, more emphasis is placed on meaning than in the ordinary math class. Knowledge of mathematical procedures entails knowing when to use them and understanding why they work. For example, in learning to subtract with regrouping, students should know when regrouping is necessary and when it is not and should be able to explain the steps using a model such as base 10 blocks or popsicle sticks bundled in tens. Ideally, doing mathematics is portrayed as smooth and straightforward, not frustrating or uncertain — if one has this kind of meaningful understanding of the content. Epistemic authority lies less with the book than with the teacher who dispenses concepts and evaluates the correctness of students' answers.

Mathematical pedagogy, however, is founded on yet another view of mathematics. On one hand, just as in conceptual mathematics teaching, "meaningful understanding" is emphasized. Students are helped to acquire knowledge of concepts and procedures, the relationships among them, and why they work. The goals are different, however. For example, learning computational skills is valued as much for what students can learn about numbers, numeration, and operations with numbers than as an end in itself. On the other hand, mathematical pedagogy also explicitly emphasizes not only the substance of mathematics but also its nature and epistemology (see Davis, 1967). Just as central as understanding mathematical concepts and procedures is understanding what it means to <u>do</u> mathematics, being able to validate one's own answers, having opportunities to engage in mathematical argument, and seeing value in mathematics beyond its utility in familiar everyday settings. Lampert (in press a) discusses how the substantive and epistemological dimensions of mathematical knowledge go hand-in-hand in this view of mathematics. She explains that, in her classroom, she tries to

> shift the locus of authority in the classroom — <u>away from</u> the teacher as a judge and the textbook as a standard for judgment and <u>toward</u> the teacher and students as inquirers who have the power to use mathematical tools to decide whether an answer or a procedure is reasonable. (p. 1)

But, she adds, students can do this only if they have meaningful control of the ideas:

> Students will not reason in mathematically appropriate ways about objects that have no meaning to them; in order for them to learn to reason about assertions involving such abstract symbols and operations as .000056 and $a^2 + b^3$, they need to connect these symbols and operations to a domain in which they are competent to "make sense." (p. 2)

For example, fifth graders could understand that if .000056 represented an imaginary amount of money, you would need about 1000 times that amount just to equal one nickel. Using money, a familiar domain, could enable eleven-year-olds to make sense of this incredibly tiny quantity and to argue about whether .000056 is more or less than .00003.

<u>Views of teaching, learning, learners, and context</u>. In addition to what counts as knowledge of mathematics, each of these kinds of teaching — ordinary, conceptual, and mathematical pedagogy — also implies certain embedded assumptions about the teaching and learning of mathematics: about pupils, teachers, and the context of classrooms.

The ordinary math class is based on the assumption that mathematics is only learned through repeated practice and drill, that "knowing" math means <u>remembering</u> procedures and concepts. The teacher's role is to show pupils how to do the procedures and to give them tricks, mnemonics, and shortcuts that make it easier for the pupils to keep track of everything. Consider, for example, the old rules of thumb that rattle around in our heads (e.g., "invert and multiply" or "to multiply by 10, add a zero"). Pupils are expected to absorb and retain what they have been shown, a commonsense sensory view of learning.

Conceptual mathematics teaching assumes that the teacher should do more than tell students how to do procedures and should, when possible, also emphasize the <u>meaning</u> of those procedures by showing how they work. The teacher plays an active role: leading, showing, directing, and structuring class time. The students play roles within this structure, answering teachers' questions, completing assignments, pursuing explorations set out by the teacher. Use of manipulatives is valued. This approach to teaching assumes that pupils will learn more if teachers are clear and direct about the content and spend more time developing student understanding and less on massive drill and practice. Still, practice remains an important component of this approach.

Mathematical pedagogy assumes that students must be actively involved in constructing their own understandings, in discovering and inventing mathematics. The basis for this emerges directly from a largely constructivist epistemology of the discipline. Mathematical pedagogy also takes a group orientation to classroom learning: The model is not of a teacher facilitating the learning of individual students. Instead, this approach uses the classroom as a mathematical community; learning involves collaboration among individuals. Although this emphasis on the group overlaps the current vogue for cooperative learning (e.g., Slavin, 1978), the warrant for this orientation in mathematical pedagogy derives from a view of mathematics as a disciplinary community.

Perhaps most significant in the classroom context is the teacher's role in guiding the direction, balance, and rhythm of classroom discourse by deciding which points the group should pursue, which questions to play down, which issues to table for the moment. The teacher in mathematical pedagogy has a critical role to play in facilitating students' learning. The teacher introduces a variety of representational systems which can be used to reason about mathematics, models mathematical thinking and activity, and asks questions that push students to examine and articulate their ideas.

Finally, practice also takes on an entirely different meaning in this approach than in e2ither of the other approaches. Here students engage in the practice of mathematics, learning what it means to do mathematics (Collins, Brown, & Newmann, 1987; Lampert, 1986, in press a; Lave, 1987; Romberg, 1983; Schoenfeld, 1985): making conjectures, attempting to prove them to other members of the community, revising and elaborating ideas.[9]

Accomplishing Change in the Teaching of Mathematics

Mathematical pedagogy is far from common. Despite serious efforts to reform math teaching over the past 30 years, mathematics continues to be taught much as it always been with, at best, little pockets of change. Each of us probably knows some teachers who manage to engage pupils in solving genuine mathematical problems or who approach algebra conceptually or whose conceptions of what mathematics is worth teaching include topics like probability or number theory. Rarer are teachers for whom the logic of mathematical discovery and the logic of justification are coequal and focal dimensions of mathematical knowledge.

Most pupils spend their time in ordinary mathematics classrooms — classrooms where mathematics is no more than a set of arbitrary rules and procedures to be memorized. In their book, The Mathematical Experience, Davis and Hersh (1981) characterize a typical mathematics

[9] In this section I have contrasted mathematical pedagogy with two other contemporary approaches to math teaching. However, one might inquire how mathematical pedagogy compares with the trends in mathematics education in the 1950s and 1960s, when the emphasis also appeared to be disciplined-based. Lakatos's (1976) points about fusing the logic of discovery with the logic of justification and the concern for the nature of mathematical truth suggest some ways in which mathematical pedagogy differs from *discovery teaching*: the latter emphasized process, and in which justification was much less explicitly (if at all) the focus. Mathematical pedagogy is also different from the disciplinary-based "*new mathematics*" of the same period that focused on "the" unifying concepts and "structures" of the discipline, and, by default, represented a formalist view of mathematics (see Bruner, 1970; Fey, 1978; Shulman, 1970).

class:

> The program is fairly clearcut. We have problems to solve, or a method of calculation to explain, or a theorem to prove. The main work will be done in writing, usually on the blackboard. If the problems are solved, the theorems proved, or the calculations completed, then the teacher and the class know they have completed the daily task. (p. 3)

When students don't "get it," their confusions are addressed by repeating the steps in "excruciatingly fine detail," more slowly, and sometimes even louder (Davis & Hersh, p. 279).[10]

This pattern is old and dominant. Why is mathematical pedagogy, as an ideal of teaching and learning mathematics, so rarely realized? No single cause can account for the failure of past reform efforts to change the face of mathematics teaching in American classrooms, and, yet, the patterns of the "ordinary" math class have dominated and continue to prevail. To set a context for my own work, below I discuss four factors that contribute to this inertia:

***Factor #1*: Culturally embedded views of knowledge and of teaching**
***Factor #2*: The organization of schools and the conditions of teaching**
***Factor #3*: Poor curriculum materials**
***Factor #4*: Inappropriate mathematics teacher education**

***Factor #1: Culturally embedded views of knowledge and of teaching*.** Ordinary mathematics teaching reflects a culturally grounded epistemology — what Jackson (1986) calls the mimetic tradition. Knowledge is fixed; teachers give knowledge to pupils who store and remember it. This tradition is firmly embedded in Western culture. Cohen (in press) writes that, as much as 300 years ago,

> most teaching proceeded as though learning was a passive process of assimilation. Students were expected to follow their teachers' directions rigorously. To study was to imitate: to copy a passage, to repeat a teacher's words, or to memorize some sentences, dates, or numbers. Students may have posed questions in formal discourse, and perhaps even embroidered the answers. But school learning seems to have been a matter of imitative assimilation. (p. 19)

Cohen (in press) argues that innovative approaches to teaching and learning embody assumptions about knowledge and about teaching and learning that are "a radical departure from inherited ideas and practices" in this culture. Mathematical pedagogy is such a case. Reflecting a view of knowledge as "emergent, uncertain, and subject to revision — a human creation rather than a human reception" (Cohen, in press, p. 15), mathematical pedagogy places the teacher in the role of guide and the student as inventor and practitioner of mathematics. Such ideas fly in the face of centuries-old intellectual traditions, not just in mathematics but in

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[10] These patterns are reminiscent of what Putnam (1987) has called "curriculum scripts": A loosely ordered but well-defined set of skills and concepts that students are expected to learn, along with the activities and strategies for teaching this material. . . . Although student input is important, the curriculum script is the major determinant of the agenda. (p.17)

all disciplines. These traditions live outside the institution of schools as well, in the everyday occasions for informal teaching — parenting, for example. A view of knowledge as fixed, of teaching as transmission, and of teachers as authorities runs very deep.

Mathematical pedagogy is also at odds with the individualistic American intellectual tradition in which working independently and figuring things out for oneself is valued. Indeed, modal school practices — testing, remedial programs, individualized self-paced instruction — reflect this tradition. We speak of "individual differences" and different personal "learning styles." Psychology, the science most often presumed to be foundational to teaching, reflects this same bias, thus further reinforcing the traditional focus on the individual. Ordinary mathematics classrooms reflect this in the norms of quiet individual work and of interpretations of collaboration as cheating. Jackson (1968, p. 16) describes this dominant feature of classroom life, that students must ignore those who are around them and "learn how to be alone in a crowd":

> In elementary classrooms students are frequently assigned seatwork on which they are expected to focus their individual energies. During these seatwork periods talking and other forms of communication between students are discouraged, if not openly forbidden. The general admonition in such situations is to do your own work and leave others alone.

In addition, right answers are the currency of the classroom economy, earning privilege and reward. Just as the views of knowledge and of teaching discussed above are embedded in the surrounding culture, so too is this intellectual competitiveness.

Individualism, however, is not the image of the academic community of discourse (e.g. Kuhn, 1962). Scientists disagree and hammer out new understandings in the pursuit of their work and in solving novel puzzles. Mathematical pedagogy, too, implies a learning community (Schwab, 1976), a group collaborating in the enjoyable pursuit of mathematical understanding. Students are expected to share ideas, to work with others, and to see the collective as the intellectual unit and arbiter. While right answers may not be the issue, intellectual progress and strength are a resource to everyone, not just to some; each person's contributions can help ethe community.

Factor #2: The organization of schools and the conditions of teaching. Institutional and organizational factors in schools also present obstacles to change. Schools are charged with multiple and competing goals, not the least of which is the goal of fostering individual excellence and advancement, while simultaneously ensuring equity and access. The rhetoric of "individualism" dominates for many teachers; yet, they know that they are expected to meet individual needs and to guarantee equity. The push to standardization and to avoid risks or experiments is great, in light of these pressures (Cohen & Neufeld, 1981; Goodlad, 1984). Conservative administrators and school boards preoccupied with test scores put pressure on teachers to emphasize "basic skills" — computation and memorization of "facts." Teachers are generally responsible for a curriculum that is both traditional and warranted by its very traditions.

The structure of the school day means that teachers are isolated from one another and have little time or support for learning or trying out innovations. Time is segmented into 54-

minute blocks, content must be covered, pupils must be prepared not only for tests, but also for the next level. Moreover, working with groups of 30 children makes experimenting with pedagogy risky for teachers who must also maintain order and routines. Elementary teachers must teach many other subjects in addition to mathematics; secondary teachers must teach many more students.

Teachers often do not have time to plan and organize rich experiences for pupils, nor can they afford the looseness of more exploratory curricula. They feel pressure to make sure that pupils master required content. For example, the time required for students to "get inside" a topic like measurement may seem to be at odds with ensuring that students also get to everything else. The pull toward neat, algorithmic curriculum is very strong. Teaching measurement by giving out formulae — L x W = some number of square units, and L x W x H = some number of cubic units — may seem much more efficient than hauling out containers and blocks and rulers and having students explore the different ways of answering questions of "how big" or "how much." That this results in sixth graders who think you measure water with rulers goes almost unnoticed.

All these features of school organization and of the conditions of teaching are part of the context within which reforms must operate (Cuban, 1984; Sarason, 1971). Sarason (1971) argues that the massive failure of the new math was due, in large measure, to the reformers' failure to take a sufficiently broad perspective of the "regularities" of the school setting and the culture of the school:

> It becomes clear that introducing a new curriculum should involve one in more than its development and delivery. It should confront one with problems that stem from the fact that the school is, in a social and professional sense, highly structured and differentiated — a fact that is related to attitudes, conceptions, and regularities of all who are in the setting . . . . Any attempt to change a curriculum independent of changing some characteristic institutional feature runs the risk of partial or complete failure. (pp. 35-36)

Without taking the wider context into account, change, if it occurs at all, is likely to be superficial — changing textbooks but not mode of instruction, for instance (Sarason, 1971).

These conditions of schooling have direct consequences for efforts to effect change in the teaching of mathematics — indeed, for the concept of "implementation" itself (Farrar, Descantis, & Cohen, 1980; Wildavsky & Giadomenico, 1979). Sarason would argue that reformers who do not consider what schools are like will operate under the misguided impression that changes can be simply "put into place." Quite the contrary: Innovations are interpreted and adapted by teachers; both the intent, the enactment, and the effect of an innovation changes in the translation (Berman & McLaughlin, 1975; McLaughlin & Marsh, 1978; Sarason, 1971). Expecting that reforms can be instituted faithfully from the top down is a fantasy that ignores the loose connections between official authority and actual practice. American teachers, in fact, have considerable elbow room at the classroom level, and typically "[arbitrate] between their own priorities and the implied priorities of external policies" (Schwille, Porter, Floden, Freeman, Knappen, Kuhs, & Schmidt, 1983, p. 387).

Factor #3: Poor curriculum materials. Moving from the least tangible to the most concrete, another factor that contributes to the continuity and conservatism of mathematics teaching is the nature of the materials from which teachers teach. Typical mathematics curriculum materials — textbooks, activity cards, kits — tend to emphasize calculation skills and a "here's how to do it" approach to mathematics. Stodolsky's (1988) analysis of elementary math textbooks suggests that concepts and procedures are often inadequately developed, with just one or two examples given, and an emphasis on "hints and reminders" to students about what to do. Even when teachers attend workshops or read about the importance of focusing on problem-solving and teaching procedures with conceptual understanding, their textbooks provide them with little guidance or support for acting on these recommendations. For example, area and perimeter, the content of the example above, are presented in terms of the formulas — L x W and 2 x L + 2 x W — with, perhaps, some pictures to illustrate. Practice is provided in calculating the area and perimeter of some rectangles, with reminders to state the answers to area problems in terms of "square units." Multiplication by ten is explained in terms of "adding a zero," algebra texts claim that vertical lines have "no slope," and rectangles are represented as figures with two long and two shorter sides. Geometry is scant, measurement more procedural than conceptual, and probability investigations relegated to the little "Time Out" boxes on a few random pages. Although recent editions of popular textbooks include new pages on "problem solving," they are often additions to rather than changes in the textbook's representation of mathematics. They tend, furthermore, to package mathematical problem solving as procedural knowledge or as a topic in its own right. Kline (1977) blasts school mathematics texts for their dogmatic presentations and failure to let students in on the struggles of mathematical discovery and activity.

Factor #4: Inappropriate mathematics teacher education. That these three factors affect the problem of "no change" in the teaching of mathematics is unquestionable. A fourth factor, one focused on what teachers know and believe, is equally critical. The consistency of the ordinary math class is in itself a barrier to change. Prospective teachers probably learn something from being inundated with that kind of teaching over a ten or twelve-year period — and what they learn may well include a rule-bound knowledge of mathematics. If we want schools to change, teacher education has to find out how to break into this cycle.

Critics observe that preservice teacher education typically has a weak effect on teachers' knowledge and beliefs and that whatever prospective teachers learn at the university tends to be "washed out" once they get to schools. In fact, it is rather unsurprising that a handful of university courses often fails to substantially alter the knowledge and assumptions which prospective teachers have had "washed in" through years of firsthand observations of teachers. In the case of mathematics teaching, they have already clocked over 2,000 hours in a specialized "apprenticeship of observation" (Lortie, 1975, p. 61) which has instilled not only traditional images of teaching and learning but has also shaped their understandings of mathematics. As this is the mathematics they will teach, what they have learned about the subject matter in elementary and high school turns out to be a significant component of their preparation for teaching. Thinking that they already know a lot about teaching based on their experiences in schools and on their good common sense, prospective teachers may not be disposed to inquire or to learn about teaching mathematics.

Furthermore, and equally serious, what we know about what students learn in ordinary mathematics classes suggests that prospective teachers are unlikely to know math in the ways that they will need to in order to teach. The weakness of teacher education tends to be attributed either to its truncated structure or its misfocused curriculum. Efforts to reform teacher education assume that what is taught to prospective teachers either is not optimally organized or is not enough to break with traditional modes of teaching. In response, many people have ready answers about how preservice teacher education should be changed — proposing to alter content, duration, requirements, or structure (see, for example, Holmes Group, 1986; Prakash, 1986; Carnegie Task Force, 1986). With respect to the content of teacher education, consider the preparation of mathematics teachers. Disagreements abound about what knowledge is most important to cover: Piagetian theory or a variety of perspectives on human learning? A survey of the history of mathematics or how to teach specific topics like place value? A review of basic arithmetic and algebra or exposure to nontraditional topics such as probability? Classroom management or more mathematics courses? Some advocate that elementary teachers should specialize (e.g., Elliott, 1985); others recommend the abolition of the undergraduate major in education (Holmes Group, 1986). Still alive and well are debates about the relative importance of generic skills of teaching and subject matter knowledge (e.g., Guyton & Farokhi, 1987; others).[11]

Discussions about the ideal curriculum for teacher education, however, are premature (Lampert, in press b). They are premature because increasing the impact of teacher education on the way mathematics is taught depends on a reconsideration of our assumptions about teacher learning, not just a revision of what teacher educators deliver. While what is taught is unquestionably a critical issue for teacher education, the fact that formal teacher education is often a weak intervention is attributable in part to its failure to acknowledge that "the mind of the education student is not a blank awaiting inscription" (Lortie, 1975, p. 66) as well as its failure to confront effectively the inappropriate or insufficient understandings that prospective teachers bring. Questions of what to teach properly depend both on what learners need to know and on what the learners already know.

The first three factors discussed in this chapter — cultural views of knowledge and of learning, organizational features of schools and the conditions of teaching, poor curriculum materials — all help to explain the persistence of ordinary mathematics teaching. These factors are all useful explanatory perspectives. I have chosen to focus on the last factor, contending that to ignore teachers — what they know and do as well as how they learn — dooms efforts to change the teaching of mathematics.

While widespread change in mathematics teaching is unlikely to occur by attempting only to change individuals, disciplinary-based mathematics teaching is likewise impossible without appropriately educated teachers.

[11] Assumptions about this relationship are reflected in current initiatives to certify college graduates with majors in mathematics through alternative routes to certification — in New Jersey and California, for instance.

What Do Prospective Mathematics Teachers Bring With Them to Their Professional Education?

What prospective teachers bring to teacher education programs is a critical influence on what they actually learn there. We do not know enough about what they bring nor, in preparing them to teach mathematics, what we should especially pay attention to about what they bring. These issues are necessarily prior to reconsiderations of the structure or curriculum of mathematics teacher education.

This point would seem to be little more than an ordinary and self-evident caveat. After all, there is mounting evidence from cognitive science that children's prior knowledge powerfully affects the way they make sense of new ideas. This research has been influential in the curricula of both school mathematics and teacher education. Mathematics educators who develop innovative mathematics curriculum materials for classrooms design them to take into account what we know about how children learn and the kinds of misconceptions they are likely to have. Some teacher educators talk about "constructivism" and student misconceptions in their courses. To make the same point about teacher learning should be trivial — it should seem obvious that teachers, like their pupils, may have ideas and understandings that influence their learning.

Ironically, however, this perspective on human learning rarely influences what university educators do with their students. University classes, whether in mathematics, history, or educational psychology, are typically "delivered," even when they are about the problems with delivering knowledge to children in schools. As one teacher remarked wryly, she had been in courses where she got lectures on constructivism. All too often the assumption in higher education is that teaching equals learning, that those who are capable enough and who want to learn the material will and will make sense of it correctly.[12] And this assumption is made by the very persons who are raising these issues about lower education.[13]

In order for teacher education to help prepare teachers to approach mathematics teaching from the perspective of mathematical pedagogy, we need to understand more about (1) the students of mathematics teacher education — prospective teachers, and (2) what they learn from different approaches to professional preparation. Brown and Cooney (1982) argue that

> If . . . it is important to know how children learn mathematical knowledge in order to articulate effective instructional programs, then we are faced with the inescapable conclusion that it is important to know how teachers learn to teach mathematics in order to design effective teacher education programs. . .We have as a profession appreciated the naivete of assuming that it is possible to design a teacher-proof curriculum . . .but have we advanced beyond our initial naivete to realize that the teachers we train are far more than passive conduits through

[12] Ausubel (1968) distinguishes between the mode (reception versus discovery) and the nature (rote versus meaningful) of the learning. Still, it is hazardous to assume that meaningful learning will occur through a steady diet of the reception mode.

[13] There are exceptions to this, of course. Feiman (1979) writes about a course she designed to challenge and extend prospective teachers' ideas about pedagogy and about children. Other examples of this kind of thoughtful college-level pedagogy include Ball (1988); Feiman-Nemser, McDiarmid, Melnick, and Parker (1988).

which intended curricula become learned curricula? (p. 17)

Both foci of inquiry — what teachers bring and what they learn — are important to the improvement of mathematics teacher preparation, but this dissertation focuses on the first aspect: knowledge about what future math teachers bring to their formal preparation to teach mathematics. This work should also contribute conceptually to investigations of what teachers learn from different kinds of programs or experiences by offering a theoretical framework of what things to investigate and follow track about teachers' learning over time through a professional sequence.

My overarching research question is:

What do prospective elementary and secondary teachers bring with them to teacher education that is likely to affect their learning to teach mathematics?

The study addresses this question both conceptually and empirically. Below I describe the research design and discuss some related work that provides theoretical foundations for this dissertation.

Teachers as Learners in Teacher Education

What does it mean for a learner to "bring" something to a situation? A metaphor, it conveys the idea that learners do not arrive empty-handed (or empty-headed), but that they come instead with ideas, understandings, ways of thinking, inclinations, and habits.

Recent research on student learning in a variety of domains suggests that pupils' prior knowledge and beliefs — what they bring — shapes the ways in which they make sense of new ideas (see, for example, Anderson, 1984; Confrey, 1987; Davis, 1983; diSessa, 1982; Posner, Strike, Hewson, and Gertzog, 1982; Roth, 1985; Schoenfeld, 1983). Confrey (1987), tracing the recent history of this perspective on learning, describes it as reflecting "a basic rejection of a *tabula rasa* approach to learning" and, as being founded instead on the assumption that "existing knowledge serves as both a filter and a catalyst to the acquisition of new ideas" (p. 7).

Although teacher educators talk about this constructivist perspective on learning in their courses, it rarely influences what they do with their students. The professional preparation of teachers instead seems often to be based on the assumption that preservice teachers simply lack the knowledge they need and that the purpose of professional preparation is to fill them up with or give them the necessary knowledge and skills (Zarinnia, Lamon, & Romberg, in press). Debates about teacher education, when they occur, center on what knowledge is of most worth, on methods, or on the sequence and structure of professional education (e.g., Holmes Group, 1986; Sarason, Davidson, & Blatt, 1962; Smith, 1980).

Feiman-Nemser (1983) argues that teacher educators "tend to underestimate the pervasive effects" of school and culture on prospective teachers. She suggests that the ideas they bring to teacher education are rarely challenged; that, instead, these ideas are generally ignored and, as a result, teachers tend to maintain their preconceptions: "Formal training does not mark a separation between the perceptions of naive laypersons and the informed judgments of professionals" (p. 153). Because teaching is a familiar cultural activity, the notion that it requires specialized knowledge or perspectives is not commonplace (Buchmann, 1987b;

Jackson, 1986). Still, the "teaching knowledge" derived from everyday experience with teachers and classrooms is at sometimes insufficient and at other times misleading as preparation for teaching. It is a central and difficult task of teacher education to help prospective teachers to move from commonsense to pedagogical ways of thinking (Feiman-Nemser & Buchmann, in press).

Although evidence exists that formal teacher education is a weak intervention compared to the potency of teacher candidates' prior experiences, there has been little research on what teacher candidates do or do not learn from teacher education. The Knowledge Use in Learning to Teach project, conducted by Sharon Feiman-Nemser and her colleagues at the Institute for Research on Teaching, is one exception. Between 1982 and 1984, researchers (myself included) followed seven prospective elementary teachers through two years of undergraduate teacher education, observing and documenting courses and field experiences and interviewing the teacher candidates about what they were learning. We focused on how they made sense of their courses and field experiences, and traced changes in their knowledge, skills, and dispositions over the two years of their professional program. Several analyses revealed how powerfully prospective teachers' preconceptions could shape what they learned from their teacher education programs.

One example from this work shows the way in which Janice, one of the teacher candidates, made sense of Jean Anyon's (1981) "Social Class and School Knowledge," an article that analyzes and critiques the variation in curriculum across schools serving different socioeconomic populations. The article describes four urban and suburban schools and the differences in the social studies curricula among these four schools. Janice's instructor wanted the teacher candidates to confront and question the inequities in the distribution and packaging of knowledge by socioeconomic class. But Janice, a young woman from a rural background whose firsthand experience with migrant workers had helped to shape her views of children's differential academic needs, understood Anyon in prescriptive rather than critical terms. She interpreted the author as showing what pupils of different social classes <u>needed</u>:

> It just made me think that, maybe, some things maybe aren't important. Certain things should be stressed in certain schools, depending on where they're located. (p. 247)

Feiman-Nemser and Buchmann (1986) argue that, in general, because the "pull of prior beliefs is [so] strong"

> very little normatively correct learning can be trusted to come about without instruction that takes the preconceptions of future teachers into account . . . In learning to teach, neither firsthand experience nor university instruction can be left to work themselves out by themselves. Without guidance, conventional beliefs are likely to be maintained and new information or puzzling experiences absorbed into old frameworks. (p. 255)

The Knowledge Use in Learning to Teach study also suggests some of the ways in which prospective teachers' knowledge of subject matter interacts with their understandings and assumptions about teaching and learning and about pupils to shape what they learn from their courses as well as what they do as they begin teaching. For example, urged to avoid textbooks and to create their own instructional materials instead, the prospective elementary teachers we were studying designed their own substantively thin or even incorrect curriculum, often misrepresenting or even sidestepping the content. They, however, did not see that but felt proud that they were doing something of their "own" (Ball & Feiman-Nemser, in press).

In addition to the fact that little research has been conducted on what teacher candidates learn from their professional education, most of it has also been generic — that is, not focused on learning to teach in particular subject areas. An exception is the Knowledge Growth in a Profession project, conducted by Lee Shulman and his colleagues at Stanford University between 1984 and 1987. Researchers in this project studied how beginning secondary school teachers began learning to transform their own understandings of academic subjects in order to teach them. Interesting subject matter differences emerge from this work, suggesting the need to examine the process of learning to teach in specific subject areas.

For example, secondary social studies teachers come from majoring in a range of social science disciplines, such as anthropology, sociology, history, and political science, and then are assigned to teach particular social studies courses, often outside of their area of specialization. Wilson and Wineburg (in press) show how beginning teachers' "disciplinary lenses" shaped their teaching of social studies. For example, a former political science major, Fred viewed history as dry knowledge of facts and dates, while Jane, a history major, saw facts as "part of history woven together by themes and questions, and . . . embedded in a context that lends meaning and perspective" (p. 5). Wilson and Wineburg write:

> What is interesting in our findings is the way in which our teachers' undergraduate training influenced their teaching. The curriculum they were given and the courses they subsequently taught were shaped by what they did and didn't know. (p. 14)

The study's findings about beginning mathematics teachers were quite different. Conceptual understanding of the content was a bigger issue than it was with English or social studies majors. While undergraduate mathematics majors study similar mathematics topics, they do not necessarily seem to develop the same deep understandings of either the substance or the nature of their subject as do majors in other disciplines. Some of the beginning math teachers in the study struggled to teach high school content because their own understanding was weak. At times, they made mistakes or misrepresented the content, and some were unable to provide "why" explanations. Steinberg, Haymore, & Marks (1985) suggest that their results "contrast with the popular belief that any person who goes through calculus masters the high school content very well" (p. 22). They argue that difficulties in teaching high school mathematics may well stem from problems in teachers' own understanding of the subject matter.

If research on what teacher candidates learn from teacher education is scant, investigations of what prospective teachers bring to teacher education is non-existent. Virtually no research has been done to investigate what prospective elementary teachers know and believe relative to the teaching of mathematics before they enter formal programs of teacher education. This study represents a foray into that domain.

Examining What Prospective Mathematics Teachers Bring

Prospective mathematics teachers bring lots of ideas with them to teacher education. They may believe in assigning plenty of homework or they may have a repertoire of bulletin board displays. Reading may be a passion and penmanship a weakness. Obviously not all of what teacher candidates bring with them is equally related to the knowledge and beliefs most critical to this approach to learning to teach mathematics from the perspective of mathematical pedagogy. A framework for looking has helped to focus my inquiry, exploring the knowledge and beliefs most critical to this approach to teaching mathematics. The appropriate framework for investigating what teacher candidates know and believe thus derives from the logical requirements of this approach to teaching. What does it take, and where are entering teacher candidates in relation to that standard? My speculative framework, described below and represented graphically in Figure 1.1, has four components: *subject matter knowledge, knowledge about teaching and learning, knowledge about students as learners and knowledge about context.*[14]

Subject matter knowledge. Because the goal of mathematical pedagogy is to help pupils become active participants in mathematics as a system of human thought, teachers must themselves be *grounded in*

[14] Different views of teaching will include the same four domains, but lead logically to different specifics within those domains about what teachers may need to know, to be able to do, and to care about. For example, the teacher in the ordinary math class must know a variety of mnemonic devices, be able to state the steps of mathematical procedures parsimoniously and clearly, and should have facile and rapid calculation skills. Although its focus is on learning mathematical pedagogy, the results of this study should also be of interest to those whose view of mathematics teaching is closer to the ordinary or the conceptual modes.

SUBJECT MATTER KNOWLEDGE

knowledge about mathematics as a discipline

substantive knowledge of mathematics concepts and procedures

appreciation of and propensity toward mathematics

TEACHING AND LEARNING MATHEMATICS

goal: using, engaging in, and appreciating mathematics

learning as evolving, collective sense-making

teaching as representing the discipline
in ways that engage and help students
learn and appreciate mathematics

teacher as guide

STUDENTS AS LEARNERS OF MATHEMATICS

students as capable knowers and doers of mathematics

what students of particular ages find interesting or difficult

what students of particular ages can do, can understand

CONTEXT

disposition to attend to cultural and social factors

classroom as a learning community

Figure 1.1
Mathematical pedagogy: A theoretical framework for
essential teacher knowledge, beliefs, and dispositions

The subject matter. What does "grounded" mean? As the term suggests, foundations are central to this conception of the knowledge requirements of teaching. In other words, teachers need to **understand about mathematics —** where the knowledge comes from and how it is justified, what it means to do mathematics, what the connections are between mathematics and other domains. Understanding about mathematics places rules and procedures in an appropriate relationship to the ideas they were invented to model or simplify, and shapes what it means to "explain" or to justify mathematical knowledge.

Teachers also need *substantive knowledge of mathematics* **—** of particular concepts and procedures. This knowledge, in order to be helpful in teaching, must be explicit and conceptual. Even to guide students in constructing their own understandings, teachers must understand where students are headed and which paths may be more or less fruitful to pursue. They must understand the procedures and concepts in ways that enable them to select and construct fruitful tasks and activities for groups of pupils that will help them both to understand and to do mathematics.

Furthermore, teachers need the capacity to be flexible in interpreting and appraising pupils' ideas (Buchmann, 1984), and to help them to extend and formalize intuitions.[15] While everyday experience generates informal mathematical theories which pupils can be helped to articulate mathematically, there are also times when intuition and common sense mislead or conflict with disciplinary understanding. Teachers need to be sensitive to the difference and to be able to create situations which challenge common sense beliefs when appropriate (Davis, 1967).[16]

Knowledge about teaching and learning. Four essential ideas about teaching and learning undergird mathematical pedagogy. First is its **core goal —** to help students develop their capacities to use, engage in, and appreciate mathematics with competence and confidence.[17] This goal, elaborated above, is foundational to mathematical pedagogy; what distinguishes it from other approaches to mathematics teaching rests in what is meant by each of the terms: use, engage in, appreciate, mathematics. Prospective teachers need to know about and understand it but also to be committed to it as a reasonable and elegant system of human thinking.

Second, mathematical pedagogy is predicated on a conception of ***learning as sense-making***. In other words, it assumes that learning is the result of an interaction between what the learner knows and what she or he encounters. Learning does not arise from "filling up" the learner with information nor simply organically through maturation or development (Smith, 1975). Rather, students learn as they are confronted and challenged with ideas and situations that are novel or nonroutine in some way. In using their current knowledge and skills to

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[15] In teaching students how mathematical ideas are recorded, for example, teachers help pupils learn mathematical language to express ideas they already have. Many young children understand the concept of multiplication as repeated addition of sets, for instance, and yet do not know how to represent multiplication symbolically; others may intuitively work with proportional relationships but not be able to express their ideas in mathematical terms or symbols.

[16] For instance, many people assume there is a direct relationship between perimeter and area. It would seem that as you increase the length of fencing, the area of a dog pen would automatically increase. While this "feels" right, it is not.

[17] The emphasis on appreciating mathematics is one especially advocated by Perry Lanier.

wrestle with these problems, students reinforce, extend, and revise what they know (Anderson, 1984).[18]

The teacher's role as ***guide*** follows logically from this view of learning for the teacher is a central source of experiences for students. Teachers define mathematical behavior through the questions they pose, the tasks they set, and the directions in which they steer classroom discourse; in other words, through whatever they do. This role is significantly different from a more traditional role of teacher as dispenser of knowledge.

This function of guiding implies the third key dimension of teaching and learning: that teaching entails ***representing the discipline in ways that engage and help pupils learn and enjoy mathematics***. As a central source of students' mathematical experience, the teacher is inevitably responsible for portraying and communicating what it means to do mathematics as well as what particular ideas are about or mean. If teachers are to help students learn mathematics, the ways in which they represent the discipline must connect with those students and facilitate their learning. An inescapable aspect of teaching, representation is a conscious and focal activity for teachers in mathematical pedagogy.

<u>Knowledge about students as learners of mathematics</u>. Mathematical pedagogy entails involving students in authentic mathematical activity; as such, teachers must view all students as ***capable of understanding and doing mathematics***. This perspective differs from the dominant cultural view which divides the population into the "haves" and the "have-nots" or "math types" and "English types," assuming a "soft" versus "hard" disciplinary and mental structure.

Representing mathematics in ways that are both appropriate for and helpful to students means considering ***what students of particular ages find interesting or difficult*** as well as ***what students of particular ages can do and understand***. Some forms or modes of representation may be less useful in helping younger students while older students may make better connections with others. For instance, measuring and comparing areas of floor in the classroom may be an engaging and helpful activity for fifth graders learning about area and perimeter; the resulting numbers and required calculations would be impossible for most first graders who would do better with square tiles and drawings of different areas.

Furthermore, the teacher's judgment of what bears emphasis or extension should depend, in part, on the students. Teachers, therefore, must not only have general knowledge about students but must also be *disposed* to find out about their particular students. For example, teachers need to know that while most six-year-olds struggle with <u>writing</u> simple inequalities correctly, they probably <u>understand</u> division. Eleventh graders are likely to find both theory and applications of statistics and probability interesting.

<u>Knowledge of context</u>. In the face of the diversity of American society, teachers cannot presume that the cultural and social context in which they work is the same or even similar to that with which they are familiar. In order to be able to engage students appropriately and helpfully in mathematics to generate or select representations which help particular students

[18] Of course, some learning does not entail this kind of reorganization but requires instead repeated practice and memorization — learning basic multiplication "facts," for example. Such knowledge can nevertheless be acquired in context, in the service of doing mathematics, thus conferring meaning upon even rote learning.

learn, teachers need to understand the context of their students' lives. However, general knowledge is only minimally helpful in this domain because teachers cannot know in advance about all the kinds of students they will teach, the kinds of cultural and social contexts in which they may teach. Teachers must be ***disposed to attend to social and cultural factors*** (McDiarmid, Kleinfeld, & Parrett, 1987) — that is, both to actively seek insight and information in their particular settings as well as to consider how that knowledge, once acquired, can or should bear on their mathematics teaching.

As one goal of mathematical pedagogy is to use the classroom to represent the growth of knowledge in the mathematical community, teachers need to view the ***classroom as a learning community***, not as a collection of individuals, each learning at his or her own rate (Schwab, 1976).

The purpose of this study is to provide a way of examining and appraising what prospective mathematics teachers bring with them to their professional preparation as well as to contribute to knowledge about what they do bring. The work can help to clarify concepts and to challenge assumptions embedded in debates about teaching, teacher education, and teacher learning. How does what prospective teachers believe about mathematics as a subject, about how to teach it, and about how students learn it fit with mathematical pedagogy? What do prospective teachers believe about themselves in relation to mathematics and mathematics teaching? The point of asking the questions this way is not simply to highlight what prospective teachers <u>lack</u>, but hopefully also to identify things they <u>do</u> bring that may contribute to or interfere with learning to teach mathematics.

This study cannot, however, answer questions about what teacher educators should <u>do</u> to prepare teachers who can teach mathematics from the perspective of mathematical pedagogy. While crucial to debates about effective and appropriate teacher education, knowledge about prospective teachers' ideas and understandings cannot lead directly to obvious implications for action in teacher education and curriculum. On the one hand, very little is known about how beliefs and attitudes are acquired, shaped, or changed. Moreover, we should not assume that all preconceptions that seem inappropriate will interfere with what teacher educators try to teach. Some ideas that prospective teachers bring with them may simply be exchanged for ones that are pedagogically more appropriate (requiring "informational" rather than "conceptual" change). Some may be thin, or partial, and in need of development.

On the other hand, we also do not know enough about the continuum of teacher learning — about when, where, and how teachers may best learn certain things. If prospective teachers do not know much about students, this is hardly surprising or necessarily troublesome. Still, is this something that formal teacher education can or should address? On another count, while it may be all too clear that prospective teachers' understandings of subject matter tend to be critically inadequate for teaching mathematics, less clear is what could help them develop deeper and more appropriate subject matter knowledge. Do teachers really learn their subjects best by teaching them once they are in the classroom? What about the effects of increasing liberal arts requirements for prospective teachers?

Although this study cannot answer such questions, its findings and analyses raise questions and provide knowledge that can contribute to the discussion. I will return to some key issues about mathematics teacher preparation and research on mathematics teaching and teacher learning in the panel discussion in Chapter 7.

CHAPTER 2

METHODS OF DATA COLLECTION AND ANALYSIS

Overview

The empirical component of this study drew on interviews with prospective elementary and secondary teachers. I interviewed 19 teacher education students — sophomores and juniors — at the point at which they were about to enter their first education course. My goal was to learn about the knowledge and beliefs of these 19 individuals as well as to develop a theoretical framework for assessing what teacher candidates bring to their formal preparation to teach mathematics. Based on the view of teaching — mathematical pedagogy — advanced in Chapter 1, I articulated an initial framework for the knowledge and beliefs needed in order to teach mathematics. Using a process of what Bogdan and Biklen (1982) call "modified analytic induction," my intent was to use the data I was collecting to revise and reformulate this preliminary framework.

Toward that end, I set out to gather information about the teacher candidates that would help me learn what they knew and believed, as well as how they thought and felt, about mathematics, about the teaching and learning of mathematics, and about students as learners of math. I developed interview questions and tasks to examine what the teacher candidates knew and believed. These data were to provide grist for my analytic mill. On one hand, I would compare these teacher candidates' understandings with the framework I had derived from the view of teaching and would describe and appraise what these 19 teacher candidates were bringing to their formal preparation to teach mathematics. On the other hand, I would also continually assess the adequacy of that framework for capturing what the teacher candidates were bringing.

The interviews included questions about their own past experiences with mathematics in school, about how they thought that people get to be good at math and what they thought that meant, and about what they thought they needed to learn in preparing to teach mathematics. I also presented them with scenarios constructed out of common tasks of teaching math — for example, deciding what to do in response to a student's question, helping a group of children learn a particular topic or idea, and analyzing curricular materials. These tasks were embedded in particular mathematical topics and ideas.

The interview analysis focused on the prospective teachers' subject matter knowledge, as well as on their beliefs about mathematics, about teaching and learning math, and about learners. I also examined how these ideas and understandings <u>interacted</u> in the prospective teachers' thinking about teaching.[19]

In this chapter, I describe the sample of students who were interviewed, the data collected, and the procedures used for collecting and analyzing those data. I conclude with one

[19] The analysis of these interactions represents the major modification of the initial theoretical framework, and is the focus of Chapter 6

case from the intellectual history of this study, a case that illustrates the evolution of ideas that developed out of the dialectic between the empirical and theoretical work of this dissertation.[20]

The Sample

Selection Procedures

Because I wanted to interview prospective teachers who had no formal teacher preparation, I drew my sample from two introductory education courses within the first week of the academic term. For elementary candidates, I went to Teacher Education 101, Exploring Teaching, a course most elementary majors take before applying to a teacher education program at Michigan State University. Secondary candidates typically do not take this course; their first education course is Teacher Education 200, educational psychology.

I spoke to several sections of these two courses, explaining the study and asking for volunteers to be interviewed. I emphasized my interest in interviewing a variety of people: people who enjoyed mathematics and were looking forward to teaching it, people who had not enjoyed their experiences with math in school, people who had taken a lot of math courses, and people who had not taken any math since seventh grade. I gave a number of examples such as these because I wanted the sample to vary and was concerned that only those who felt confident about mathematics might volunteer. This strategy seemed effective, as the students who did volunteer varied significantly in mathematics background, success in math courses, and feelings about math.

All volunteers completed a three-page information form which helped me to select the sample. The form included questions about their major, level, grade point average, other degrees earned, and mathematics courses taken in college. One item focused on their general feelings about mathematics.

Many more elementary teacher candidates volunteered than I needed. Prospective secondary teachers were more scarce, not because they did not volunteer, but because they were simply unavailable: Fewer of them were enrolled in the introductory courses. For example, during the first term of the study, only 8 mathematics majors were enrolled across all 5 sections of educational psychology. Partly due to this scarcity, I decided to include two of the mathematics minors who had volunteered to participate in the project; these students identified themselves as "planning to teach math."

Demographic Characteristics

I conducted interviews at the beginning of two consecutive terms. Each term I selected 5 elementary and 5 secondary teacher candidates. The former were majoring in elementary education or child development and had no disciplinary specialization. The latter were either mathematics majors or minors.[21] As the study was not designed to produce generalizable

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[20] Further detail about data collection and analysis is included in Chapters 3, 4, and 5, in the context of discussing particular foci of the study.

[21] The reason for the difference between the two samples is that all elementary teacher candidates are actually preparing to teach math, while among secondary candidates, only math majors and minors are preparing to teach math.

findings about prospective teachers, the sample did not need to be representative of the teacher education student population at Michigan State. However, since the purpose of the interviews was to learn the range and diversity of what prospective teachers bring to teacher education that might affect their learning to teach mathematics, the sample should be varied.[22] A few key criteria were used to build variation into the sample: gender, academic history in college mathematics,[23] and stated feelings about math (based on the item posed on the volunteer form).[24]

All but one of the volunteers chosen agreed to participate in the study. One prospective secondary teacher dropped out during the second round of interviews but was not replaced. The final sample therefore consisted of 10 elementary and 9 secondary teacher candidates.

Table 2.1 shows the gender composition of the sample.[25] One student was black, one was Asian; the others were Caucasian. Fourteen of the students were between 19 and 21 years old; two were in their thirties. All 19 students were from Michigan. Over half the students Table 2.1 came from mid-sized suburban high schools, seven were from very small towns or rural areas, and two from urban high schools.[26, 27]

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[22] Bogdan and Biklen (1982, p. 67) explain the rationale for the *purposeful sampling*: "This research procedure insures that a variety of types of subjects are included, but it does not tell you how many, nor in what proportion the types appear in the population. . . . You choose particular subjects to include because they are believed to facilitate the expansion of the developing theory."

[23] The experience of taking college mathematics may be related to the question of what prospective teachers "bring." For instance, a student who has completed the calculus sequence has spent many more hours in mathematics classes than one who has taken no college level mathematics. While the decision to enroll or not in mathematics courses may itself indicate differences in attitudes, interests, or sense of self among the informants, taking mathematics may also <u>affect</u> such differences.

[24] There are other variables potentially connected to what prospective teachers bring with them to mathematics teacher education — high school mathematics experience, age, or whether they attended a community college prior to coming to the university, for instance. The sample used in this study was not large enough to justify stratifying it along many dimensions. I did collect additional academic data about the teacher candidates which, in analyzing interview responses, may suggest possible connections worth systematic exploration in the future.

[25] I tried, without success, to oversample for males in the elementary sample. Only about 4 males were present in class across all the sections of Teacher Education 101, however, and of those, only one volunteered.

[26] The gender and ethnic composition of the sample generally reflects the composition of both the elementary and secondary math student populations in this college, although not by my express design. That is, I did not set out to obtain a representative sample.

[27] While I am not claiming to make general statements about what prospective teachers know or believe based on a small number of interviews with teacher education students at Michigan State University, the demographic data on this sample of prospective teachers indicate that they are very similar to the population of teacher education students in terms of characteristics such as ethnicity, gender, social class, and age.

Table 2.1
Gender Composition of Sample

	Male	Female
Elementary n=10	1	9
Secondary n=9	5	4
Total	6	13

Data Collection

I used interviews and academic records to learn about each prospective teacher in the sample. In this section I describe the interview, strategies used in conducting it, and academic records examined.

The Interview

The primary purpose of the interviews was to learn more about what these different entering teacher education students were bringing with them to their formal preparation to teach mathematics. In particular, I wanted to learn about their understanding of mathematics and their ideas about teaching and learning math. The interview was designed based on my preliminary framework, discussed in Chapter 1, for exploring what prospective teachers bring, and included the following: knowledge of and about mathematics, ideas about the teaching and learning of mathematics, about students, and about self in relation to mathematics. Context figured too, although less prominently.

The interview was a tool for learning about what prospective teachers bring to teacher education. But, at the same time, the interview was itself also my working hypothesis, shaped by my ideas of what was critical to explore. As a net, what did it capture about prospective teachers' ideas and ways of thinking and feeling related to mathematics and the teaching and learning of mathematics? My challenge was to be sufficiently focused that I might get beneath the surface of these teacher candidates' ideas, yet open enough to enable me to do more than verify my initial assumptions.[28]

Because the interview protocol played a dual role in the study, reflecting as well contributing to the theoretical framework, I collected data over two terms. I interviewed 10 students during the first term and 9 during the second. Between terms I revised the questions and tasks before I interviewed the second group of students, focusing some questions, altering some, and adding others, based on what I was learning from the first set of interviews. This strategy helped me to use the interviewing to advance the theoretical framework for assessing what prospective mathematics teachers bring to teacher education.

Structure of the interview. Because it was long and demanding for participants, the interview was conducted in two sessions, each lasting about two hours. The first interview (sections A and B) explored the prospective teacher's personal history and ideas about mathematics, teaching and learning math, and about self. The tasks and questions in the second session (section C) were grounded in scenarios of classroom teaching. Below I explain each of these parts of the interview. The complete interview protocol can be found in Appendix B.

Section A. I designed Section A partly to establish rapport, beginning with questions which would demonstrate my interest in the people I was interviewing and which would be less intense than the subsequent tasks and questions. I began by asking them why they were thinking about becoming a teacher and followed with questions focused on the teacher candidates' experiences in school. The "stuff" of their "apprenticeship of observation," these experiences might be a significant component of, or influence on, what the prospective teachers were bringing to teacher education.

I asked the students to talk about their elementary, high school, and college math courses. These questions helped me understand how they understood what they had been taught, what kind of perspective they had of the substance of their courses, the way they organized that knowledge, and the articulateness with which they could talk about it. I designed these questions to provide an alternate avenue (in addition to the second part of the interview) for exploring what prospective teachers "bring" with respect to knowledge of and about mathematics. I was also trying to learn their ideas about good (or bad) teaching, how people learn mathematics, and their feelings about the subject. For example, I asked prospective teachers what they remembered about math from high school — what they took, what they remember learning, and what stood out to them. I especially tried to find out what they thought their classes were about. If they commented that they had had a particularly good (or bad) teacher, I probed to learn why they considered that person so good (or bad) at teaching math.

Section B. The questions in section B focused on people's ideas about mathematics as a subject and as a discipline, about teaching and learning mathematics, about pupils, about self, and about learning to teach mathematics. I developed questions to probe the teacher candidates' beliefs, both explicit notions and underlying assumptions. For instance, I asked them to think of a person whom they would consider to be "good at math" and then probed their notions of what it means to be "good at math" and how people get to be that way.

Card sort task. Besides answering questions, teacher candidates also completed a card sort task. I gave them a set of 35 cards with statements about math and the teaching and learning of math and asked them to create three piles: those with which they agreed, those with which they disagreed, and those about which they were not sure. "Not sure" included those

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[28] Bogdan and Biklen (1982, p. 66) describe the discipline of beginning from an initial conceptualization of the phenomenon under study and modifying it in light of new cases that do not fit that initial formulation. Analysis proceeds by "interrogating" the data. Data do not speak but, in seeking supporting and disconfirming evidence for developing ideas, the data may signal that there is too little support for an idea, or that the idea, "though supportable," is not particularly cogent" (Schatzman & Strauss, 1973, p. 120). In this way, the researcher's working hypotheses undergo a process of continual testing and revision.

statements they did not understand as well as those about which they vacillated. After the prospective teachers sorted the statements, I asked them to go through their piles of cards and talk about the statements about which they were either unsure or about which they had strong feelings.

The teacher candidates' comments about the statements opened another window on their ideas and were especially useful for learning how they understood certain terms or phrases in the statements — for example, "concrete," "problem solving," or "argument." This task differed from the other more open-ended questions. Instead of explaining what they thought in their own terms — their own language and category schemes — the prospective teachers reacted to given statements. This task afforded me the opportunity to extend my analyses of teacher candidates' responses to the open questions. The card sort statements also raised issues that an individual person might not have addressed: the nature of mathematical knowledge or the process of learning mathematics, for example.

I changed the card sort statements after the first term of interviews. Based on issues that emerged during those interviews, and drawing on the prospective teachers' language, I added some statements, dropped some, and revised others. For example, I was struck by the number of people who used mechanical terms (e.g., drumming, pushing, drilling) to describe learning math. In order to explore these metaphors, I added a new statement drawn directly from the first interviews: "To learn mathematics, many basic principles must be drilled into the learner's head." Although many statements were altered for the second term of interviews, I asked the second term participants to respond to the original statements on a brief written questionnaire, so that I could aggregate data across all 19 students.

Section C: Scenarios. A major assumption of the study's conceptual framework is that, in carrying out the activities of teaching, teachers' understanding of subject matter interacts with their knowledge and beliefs about teaching, learning, and the contexts of schooling. Therefore, in order to learn about prospective teachers' understanding of mathematics and the role that understanding plays in their thinking about teaching and learning math, I designed questions based on common tasks of teaching. These tasks were ones that all teachers, whatever their view of teaching, perform, although the way in which they deal with these tasks varies as a function of their view of teaching. The tasks include responding to unanticipated student questions or novel ideas, examining students' written work, evaluating curriculum materials, and planning approaches to teaching. Each question is cast in the form of a scenario. After describing the scenario to the teacher candidates, I asked what they would do or say if this situation came up in their own teaching and why.

I wove the questions with particular mathematical concepts, procedures, and ideas: rectangles and squares, perimeter and area, place value, subtraction with regrouping, multiplication, division, fractions, zero and infinity, proportion, variables and solving equations, theory and proof, slope and graphing. Four criteria guided the selection of content for the scenarios. I sought topics that:

(1) are taught throughout the K-12 curriculum,

(2) are central to mathematics,

(3) are often difficult for pupils to learn,

(4) can be taught algorithmically or conceptually.

The first two criteria helped to ensure that the content sampled was significant mathematically and also related to what prospective teachers were likely to teach. The third criterion focused my attention on content that teachers ought to understand especially well and yet which they themselves may not have learned thoroughly. I thought that the fourth criterion would lead to topics that might reveal different kinds of mathematical understanding as well as different approaches to teaching among the prospective teachers interviewed.[29]

Each question was cast in the form of a scenario, or sketch of a teaching situation. After I described the scenario, I asked the prospective teachers what they would do or say if this situation came up in their own teaching, and why. One longer scenario, on planning and teaching mathematics, engaged the prospective teachers in several related tasks of teaching: appraising textbook material, planning, evaluating student written work, and responding to student questions. I designed the scenarios in order to explore three things:

1. what the prospective teachers seemed to know and believe — about mathematics and the teaching and learning of mathematics in classrooms;

2. what ideas and commitments the prospective teachers would draw on in responding to each scenario;

3. how prospective teachers weave together different kinds of knowledge and commitments in making interpretations, judgments, choices, and in justifying those.

To illustrate how I wove each question with subject matter, as well as issues about teaching, learning, and learners, in this section I discuss one of these scenarios in detail. I have chosen Question C3, a scenario in which some eighth graders are having a problem with place value in multiplying large numbers correctly; I asked prospective teachers what they would do if these were their students. First I will explain what the scenario entails in terms of the mathematical content and issues about teaching and learning, learners, and context.[30] Then I will show how the exercise and the responses it elicits helped me to make inferences about what the prospective teachers knew and how they thought about mathematics and about teaching and learning mathematics in classrooms. The reader should read the complete question (see Figure 2.1) before proceeding.

Interview scenario C3: Conceptual analysis. First I explain the ***subject matter content*** of the question: multiplication with regrouping. The algorithm for multiplying large numbers is derived from the process of decomposing numbers into "expanded form" and multiplying the number in parts. That the procedure has a substantive logic, as opposed to being an arbitrary series of steps, is in itself a fundamental component of the subject matter knowledge. To understand this procedure, one must understand decimal numerals as representations of

[29] In Chapter 3, I provide rationales for the specific mathematics content on which I have focused.
[30] These domains correspond to the framework outlined in Chapter 1.

numbers in terms of hundreds, tens, and ones. Understanding the procedure also entails knowledge of the distributive property of multiplication over addition: knowing that one can decompose numbers, multiply their parts separately, and then add the results of the three products.

Often, people have learned to carry out this multiplication algorithm without writing down the zeros, so that their computation looks like this:

$$\begin{array}{r} \mathbf{123} \\ \underline{\mathbf{x\ \ 645}} \\ \mathbf{615} \\ \mathbf{492} \\ \underline{\mathbf{\ \ 738}} \end{array}$$

Question C3: <u>Responding to Student Difficulties: Place Value</u>

Some eighth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

$$\begin{array}{r} 123 \\ \underline{x\ 645} \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \underline{x\ 645} \\ 615 \\ 492 \\ \underline{\ \ 738} \\ 1845 \end{array}$$

instead of this:

$$\begin{array}{r} 123 \\ \underline{x\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 79335 \end{array}$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching eighth grade and you noticed that several of your students were doing this?

Figure 2.1.
Question C3 of the interview

Consequently, many people can perform the multiplication without understanding why it works or seeing the relationship of the base ten place value system to this algorithm. This procedure, which depends conceptually on place value and the distributive property, is therefore a useful site for examining how people understand place value and the decimal numeration system.

What the prospective teachers thought there was for the errant pupils in the scenario to know gave clues to their understanding both of and about mathematics. Did they think that these pupils needed to be told to "line their numbers up"? Or would the pupils need to understand that they are actually multiplying by 40, not by 4, in the second step? I developed probes to help me explore how the prospective teachers understood place value and numeration. For example, if a teacher candidate said, "I'd show them to put zeros in," I probed by responding, "What if some student asks, `How can we just add zeros like that — it changes the numbers, like from 492 to 4,920!!'" This probe helped me learn more about the prospective teachers' understanding of place value and about the role of the number 0 in our numeration system.

If the prospective teachers said they would "explain" the procedure, I was interested in what they considered to be an explanation. Going over the steps of the procedure is very different from discussing the underlying place value principles. What counts as "explanation" is one component of knowledge about mathematics — about the nature and justification of particular ideas.

The scenario was also designed to capture aspects of the teacher candidates' knowledge and beliefs about ***teaching and learning and the teacher's role***. For example, if prospective teachers would tell pupils to "line up the numbers" or "put down zeros as placeholders," it might be because they believe that this is the way that students will learn the material best. Perhaps they think pupils should acquire algorithms first, and conceptual understanding later. Several other issues about teaching and learning that might influence what teacher candidates would do in this scenario were woven into the scenario as well.

For example, teachers' responses to this scenario are influenced by their ideas about how pupils learn mathematical procedures most effectively, which in turn are shaped by their ideas about what there is to learn. Do pupils need drill on this procedure so that they know it "cold"? Do they need a clear explanation of the procedure, accompanied by mnemonics to help them remember, followed by practice? Or perhaps they need an opportunity to explore why the procedure works.

Prospective teachers' assumptions about what teachers should do when kids have apparently failed to learn — i.e., what some call "remediation" — could also influence what they say about this scenario. One view that prospective teachers are likely to hold is that teachers should "go over" material — more slowly and systematically — when students haven't learned.

The task depends also on ***knowledge and assumptions about students.*** Prospective teachers' ideas about children of this age could play a role in their thinking about what to do. For example, if prospective teachers know that this topic was probably first taught to these pupils in fourth or fifth grade, that could affect their decisions about what to do. Their notions about how to work with 13-year-olds may also play a role. Worried about students' feelings and about their relations with students, for example, teacher candidates may be reluctant to tell students that they have made an error — a consideration that could color their response to this situation in yet another way.

In real teaching, this situation would demand some consideration of the classroom ***context***. For example, the scenario states that "several students" are having difficulty with the multiplication algorithm. Prospective teachers might not take the rest of the class into account, nor wonder whether other pupils were having similar confusions. My hunch was that most prospective teachers would respond in terms of one student.[31]

Although the issues vary from one scenario to another, the way in which I have opened up this particular question (C3) exemplifies the way in which I built the analytic categories into the scenarios. These tasks were experimental from a methodological perspective. Two reasons underlie my experimentation with this form. First, in order to learn about what prospective teachers understand within the domains of the framework, I wanted to ask them questions which would probe what they knew and believed about a number of separate dimensions — mathematics, for example. At the same time, in order to learn how they weave together different kinds of knowledge and belief, questions or situations which would invite the teacher candidates to draw from a number of areas were necessary. The goals of being both analytic and integrative are in tension with each other at times (I discuss this more in the context of individual analyses). Still, as an interview form, the scenario seemed to hold promise as a means of learning how prospective teachers reason about teaching.[32]

Interview Procedures

I interviewed each prospective teacher twice within a two-week period. Each session lasted approximately $1\frac{1}{2}$ - 2 hours.

Getting inside the questions. Probing was critical in order to learn why people said the things they did — what they were drawing on, what they used to justify their decisions, whether they had any alternatives, and what they meant by the things they said. I used some standard probes. For example, whenever a person described something he or she would do with a student, I always asked (as neutrally as possible) why. I probed to find out if the person had other options in his or her repertoire — i.e., alternative interpretations, courses of action — and if so, what the choice depended on. I also probed terms and phrases. Although both mathematical vocabulary and the language of teaching and learning are in the public domain, people frequently mean different things by the same terms. In an effort to get as much information as possible about certain topics or ideas, I asked, "Could you tell me more about that?" and also sometimes just refrained from saying anything — silence frequently proved to be an effective stimulus.

Sometimes the prospective teachers asked me questions. For instance, people wanted to

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[31] I knew, however, that most of the experienced teachers whom I had interviewed with this question referred to the rest of the class. For example, some said that they would want to know how many pupils were making this mistake in order to decide whether to review it with everyone or not.

[32] Scenarios, or structured exercises, are being developed and tried by researchers in a number of current studies (e.g., the National Center for Research on Teacher Education at Michigan State University; the Teacher Assessment Project at Stanford University; Wilson [1988]; and the Connecticut Beginning Teacher Assessment Project.)

know if their mathematics was "right." When this happened, I tried to deflect their query and use it instead as an opportunity to learn more about their thinking by asking, "What is it about this that feels confusing to you?" That probe was often both effective and helped me learn more about the person's ideas about math. Sometimes the teacher candidates asked what I meant by a term in a question or on a card. I responded by asking what it meant to them, or by saying that people seem to have different meanings for that term and that I was interested in their interpretation of it. Overall, my general principle was to get the teacher education students to help me understand their ideas, feelings, and ways of thinking.

Recording the interviews. The interview sessions were audiotaped and videotaped; technical equipment was set up to be as unobtrusive as possible. Although some of the teacher education students expressed anxiety about being recorded, I became so relaxed about it that I could usually put people at ease.

The interviews were tape-recorded so that I could obtain verbatim transcriptions. In analyzing the data I wanted to examine the metaphors and terms people used, as well as their capacity to articulate mathematical ideas. I assumed that people's inflection and tone would carry meaning, too. I also wanted a record of my role: how I probed and how I influenced the pace or direction of the interview, for instance.

The videotape allowed me to record non-verbal information which I thought would play a role in interpreting interview responses. For example, I expected that the prospective teachers' non-verbal responses to questions (e.g., posture, facial expression) would offer clues to their attitudes toward mathematics and conceptions of themselves as knowers of mathematics. Moreover, videotape would allow me to examine further how I may have affected the interviews (e.g., smiling, nodding, shifts in position).[33]

The interviews were transcribed and edited to be as faithful as possible to people's exact words, inflection, and tone. The transcripts averaged over 50 pages single-spaced.

Academic Records

Student academic records were an additional source of data about the prospective teachers. I obtained the following information for each study participant:

a) High school: final grade point average and rank in graduating class;[34]

b) College matriculation: college entrance examination scores (SAT or ACT), university placement test scores (English and mathematics);

c) College: up-to-date college transcript (showing courses taken and grades received).

I gathered this information for two reasons. On one hand, I wanted to describe certain academic characteristics of the people I interviewed. I also wanted to be able to respond to critics who might claim that my results were due to the poor academic caliber of the students

[33] In fact, I have done little analysis of the face-to-face interaction as recorded on these videotapes. I discuss reasons for this in the data analysis section of this chapter.

[34] High school transcripts were also requested but were unavailable as the university does not retain these on file. In the interview I did ask people what they took in high school, but not what their graders were.

whom I interviewed. The academic data show that this was not the case. On the other hand, I also wanted to consider possible connections or patterns (e.g., between level of mathematical course-taking and particular ideas about mathematics). As policymakers argue for increased course requirements for teachers, it seems important to ask questions about the differences among students who have been successful (i.e., received good grades) in college math, those who have been unsuccessful, and those who have not enrolled in math courses at all. While this study was not designed to make such comparisons, I thought that it might shed light on some important issues worth further exploration.

Data Analysis

I collected data over two terms, with two groups of prospective elementary and secondary teachers. This two-term data collection cycle allowed me to begin analyzing what I was learning from the interviews and, based on this preliminary analysis, sharpen the focus of my probes during the interviews and modify some of the interview tasks in relatively minor ways. The changes included adding probes, altering the setting up of tasks, and revising card sort statements. I also collected academic records data. In this section I discuss the methods used to analyze the data.

Interviews

The interview transcripts comprise an enormous data set; infinite possibilities exist for analysis. Some research questions would suggest analyzing the data by individual, leading to the construction of cases around particular issues or themes. Since the purpose of my study was to describe what prospective teachers bring to teacher education and to articulate a theoretical framework that would help appraise what they bring, I focused on analytic comparisons and contrasts across individuals. To do this, I analyzed the interview transcripts from three perspectives: by person, by question, and by analytic theme.

Reduction and analysis by person. Data analysis began with editing the transcripts. I listened to each interview and edited the transcript to correspond, adding emphasis, laughs, correcting transcription where necessary. As I listened to the taped interviews, I created an elaborated index for each transcript. The index included the person's responses to each question, other significant comments, and the page numbers of particular responses. Creating these indexes was helpful in making access to the transcripts easier and also in beginning to notice themes across individuals. I wrote notes to myself: questions, observations, and ideas for analysis.

The videotapes were used mainly as a means of revisiting the interviews and augmenting the index for each person. Additional features of each interview were noted, including both people's nonverbal reactions and their comments. I did not, however, do a careful interactional analysis.[35]

Reduction and analysis by question. Another step in the process of reducing and

[35] Undoubtedly more could be learned from careful analysis of the interactions in these interviews. Still, this kind of analysis is beyond the scope or primary focus of the present study.

analyzing data was to read the transcripts by interview question. Drawing from careful substantive analyses of each question (such as the one I discussed above), I created a set of projected response categories for each one. As I read, I modified these categories to better accommodate the kinds of responses people gave. Consider, for example, question C6:

Suppose a student asks you what 7 divided by 0 is. How would you respond?. . . . Why?

In this question, along the dimension of substantive mathematical knowledge, I was looking for evidence of people's understanding of the number 0, the concepts of division, infinity, and "undefined." I anticipated three basic categories of response in relation to the mathematics of the question: that people might focus on meaning, a rule, or might not know.[36] I constructed a table with these three columns and began reading and categorizing people's responses to the question. I soon found that this scheme was inadequate to account for the data. Among the people who focused on a rule, almost half of them gave an incorrect rule — e.g., "Anything divided by 0 is 0, no matter what the number is." In contrast, some people gave a correct rule — stating that division by zero was undefined or that "you can't do it." The difference between giving a correct and an incorrect rule is significant in analyzing people's understanding of mathematics and so I modified the rule category to include subcategories for correct and incorrect rules. I returned to categorizing people's responses, entering their names and key phrases in the appropriate columns.

I constructed summary analytic tables for each question. Most questions were cross-analyzed on several dimensions: subject matter understanding, ideas about teaching, learning and the teacher's role, and feelings or attitudes about mathematics, pupils, or self. As I proceeded, I modified and elaborated these dimensions which were part of my initial theoretical framework for examining what prospective teachers bring to teacher education.

To analyze the teacher candidates' responses to the card sort task, I constructed a matrix of items by people. Using a grid, I entered each person's sort: agree, disagree, not sure. This matrix allowed me to look for patterns, such as statements classified in the same way by everyone, statements where there seemed to be a split between elementary and secondary majors, and statements which revealed considerable variation.

Analysis by theme. One more level of analysis was thematic, using themes that I brought to the study, such as subject matter understanding or ideas about teaching, but modified and elaborated by the person- and question-level analyses and further reading.

The basic thematic questions were:

1. How did the prospective teachers understand mathematics?

2. How did the prospective teachers feel about mathematics and about themselves in relation to mathematics?

[36] For example, they might focus on what division means and show how dividing by 0 does not make sense. They might state a rule — "You can't divide by 0" — or they might simply not remember.

3. What ideas did the prospective teachers have about teaching, learning, and the teacher's role with respect to teaching mathematics?

4. What did the prospective teachers know and assume about students as learners of mathematics?

5. What did the prospective teachers seem to consider in deciding what to do in particular mathematics teaching situations?

As the themes form the core of Chapters 3 - 6, I discuss my analysis of these questions in context.

Academic Records

I analyzed the students' academic records after I had analyzed the interview data. Questions and patterns suggested by the interview analysis helped me decide which data were most relevant to analyze in investigating what prospective teachers bring to learning to teach mathematics. For example, because many of the secondary teacher candidates were unable to correctly answer or articulately explain basic mathematical concepts, I wanted to know how successful they had been in their mathematics courses as reflected in final course grades. Because so many elementary and secondary teacher candidates had difficulty with the mathematics content of the interview, I also decided to examine how these students compared to all college-bound seniors by looking at rank in high school graduating class and college entrance examination scores. I used the following questions, based on analyzing the interview transcripts, to guide my examination of the students' academic records:

1. In general, how were these students doing in college?

2. How did these students compare with other students in high school — in general and in mathematics?

3. What mathematics have these students taken in high school and college?

4. How successful have they been in college mathematics courses?

5. On all these dimensions, how do the elementary and secondary majors compare?

First I read each student's file. Each file consisted of a complete current Michigan State University transcript, college entrance examination scores, high school grade point average and final class standing (for students who entered MSU directly from high school only), previous college transcripts (for students who transferred to MSU from another college), and MSU placement test scores. I calculated each student's cumulative grade point average overall and in mathematics.

I constructed a large summary table in which I collected and displayed the selected data by individual student: age, number of credits completed overall and in math, grade point average overall and in mathematics, mathematics courses taken and grades received in each, college entrance examination scores (scaled scores and percentile ranks), MSU placement scores in arithmetic and mathematics, and, where available, high school grade point average and rank.

Next, working from this summary table, I constructed smaller working tables to reduce and aggregate information about the 19 students. I decided which tables to make based on questions raised by my analysis of the interviews. For example, I developed tables to compare the number of college credits completed in mathematics and the number of years of high school math taken by the elementary and secondary teacher candidates. Other tables included: grade point averages overall and in mathematics by intervals, means and standard deviations of grade point averages overall and in mathematics of the elementary and secondary teacher groups, type of high school attended (size, location, public/private), and mathematics college entrance examination scores by interval. These tables enabled me to examine patterns within the data (Miles & Huberman, 1984). A summary of my analysis of the prospective teachers' academic records is presented in Appendix A.

The Evolution of Themes:
The Case of "Subject Matter Knowledge"

The process of analysis was a spiralling one, for the themes were both the frames of the analysis and the results of the study. In this section I will illustrate this with a brief discussion of the evolution of the theme of **subject matter understanding**. At the start of the study, I divided subject matter knowledge into categories, distinguishing between what I called knowledge <u>of</u> and <u>about</u> mathematics. By knowledge <u>of</u> mathematics, I meant an understanding of the **substance —** the topics, concepts, procedures — of the subject; I claimed that teachers needed "flexible" understanding of these. I also included Shulman's (1986) concept of "pedagogical content knowledge" — ways of representing mathematics so that others can understand — which, I claimed, included "both what teachers need to know and how they need to know it" (Ball, 1986, p. 8). By knowledge <u>about</u> mathematics, I meant ideas about what mathematics is — where it comes from, what it is good for, and how right answers are established. I also had a separate category: "attitudes toward mathematics and sense of self in relation to mathematics."

While none of these ideas was entirely off the mark, neither was any one adequately conceptualized. For example, I had no operational means of identifying "flexible" understanding. I also did not anticipate the interpretive issues entailed in analyzing teacher candidates' "right" answers. For example, when prospective teachers said that division by zero was "undefined" or that "you can't divide by 0," they were correct. Yet saying that division by zero is undefined represents a qualitatively different kind of understanding than saying that there is no number that can be multiplied by 0 to yield 7. The latter explains <u>why</u> it is undefined — that it violates the <u>definition</u> of division — while the former is no more than the restatement of a rule. In a similar vein, while "how right answers are justified" makes sense as a dimension of knowledge **about** mathematics, I didn't fully appreciate how that understanding plays out in teachers' <u>explanations</u> until I had read and re-read teacher candidates' interviews. Moreover, my conception of subject matter knowledge did not deal with the connections between knowledge <u>of</u> and <u>about</u> mathematics — <u>as</u>, for example, in the case of division by zero. In this example, knowing that division by zero is undefined and knowing <u>why</u> go hand in hand to comprise conceptual understanding.

I substantially revised the initial conceptualizations of "subject matter knowledge" over the course of the study, both through direct analysis of interview data, reading, writing, and other conceptual work in which I was engaged. For instance, I had asserted from the start that teachers need to understand concepts, not just get right answers. An uncontroversial claim, it proved thorny in analyzing what the prospective teachers knew. In some cases they seemed to know more than they could say. This pushed me to consider what it means to say someone "knows" something in mathematics, as well as how to infer what people understand. Several of the prospective teachers could "do" the math presented in the interview questions and their comments suggested that they knew more about what they were doing than just the steps of the algorithm. Yet they were vague or awkward in explaining their understandings. One teacher candidate exclaimed at one point, "I know what I'm saying, I know what I'm thinking, I just don't know if I can explain it!" I saw that I needed to articulate more clearly what I meant by "knowledge of and about mathematics" in order to describe the understandings that prospective teachers bring to teacher education. I worked with the interview data, looking more closely at instances where people did not articulate concepts. I also read what others have written about knowledge in order to conceptualize what seemed to be "submerged," or perhaps tacit, understandings of concepts. Is tacit knowledge sufficient for teachers? How is it different from explicit but algorithmic knowledge?

Another dimension of subject matter knowledge, knowledge <u>about</u> mathematics, was also elaborated during the study. One problem was the term "knowledge." The prospective teachers knew things about mathematics — for example, that unsolved problems exist, or that examples are not proofs. They also held beliefs about math that are <u>not</u> true — that problems have one definite answer, or that doing proofs involves correctly reproducing a series of memorized steps. When they "explained" things to hypothetical students in the interviews, their explanations were often simply restatements of rules or definitions — a weak form of explanation. Moreover, their ideas about math were part of the larger web of personal understanding of mathematics which included their feelings about themselves in relation to the subject. For example, in trying to determine the reasonableness of novel student ideas, several prospective teachers were very uncertain. Their uncertainty stemmed from two interacting notions: on one hand, that math is arbitrary and filled with "weirdo exceptions," and, on the other, that they felt they were personally not good at math.

Furthermore, as I analyzed the ways in which prospective teachers represented their understandings of the mathematical content in the questions, the problem of conceptualizing subject matter knowledge for teaching increased for me. I read literature about the kind of subject matter knowledge that teachers need and worked with the interview data in relation to what I was reading. The conceptual framework in my original proposal included "pedagogical content knowledge" (Shulman, 1986) within subject matter knowledge, but left subject matter knowledge separate from other domains of knowledge and belief — e.g., about teaching and learning, about learners, or about the context). The interviews made me realize that this was too static; it did not account for the ways in which teachers' ideas about students shape how teachers represent the subject, or how the particular subject matter may influence a teacher's beliefs about good teaching. It seemed to me that the interaction among these different domains was critical in thinking about teaching mathematics, and that I needed to focus both on what prospective teachers know and on how their ideas interact. I drew a lot of Venn diagrams, trying to represent the relationships in teaching between people's personal understandings of mathematics and their ideas about teaching and learning. Some of my efforts seemed to portray these different domains of knowledge in overlapping, but static, boxes. Where was pedagogical thinking (Feiman-Nemser & Buchmann, 1986) in such a model? Other attempts I made seemed to discount any accumulation of pedagogical knowledge, and implied that teachers generate everything themselves, from scratch, each time they teach someone something. In studying the interviews, I tried to develop a better way of talking about "subject matter knowledge," a way that could more adequately capture what I observed as the prospective teachers worked with mathematics in the interview.

Developing a clearer conceptualization of subject matter knowledge and subject matter knowledge for teaching was a spiralling process which drew on both the data and on outside literature. I give this case from the intellectual history of my dissertation research as an illustration of the close interplay between my initial ideas, the data I collected, and the conceptual work. More of this story will emerge throughout the next four chapters of this dissertation.

In Chapters 3 through 6, I describe and analyze the knowledge, beliefs, and assumptions of the prospective teachers whom I interviewed. The discussion includes both conceptual and empirical results in the domains of focus. In Chapter 6, I present a theoretical framework for integrating these domains into a model of pedagogical thinking in the teaching of mathematics, a model that can be used to describe and appraise the knowledge and reasoning of beginning as well as experienced teachers. Thus, Chapters 3 through 6 comprise the results of this research, providing findings about these prospective teachers, as well as advancing a theoretical framework.

CHAPTER 3

PROSPECTIVE TEACHERS' SUBSTANTIVE KNOWLEDGE OF MATHEMATICS

> I am really worried about teaching something to kids I may not know. Like long division — I can do it — but I don't know if I could really teach it because I don't know if I really know it or know how to word it. (Cathy, elementary teacher candidate)

> Teaching the material is no problem. I have had so much math now — I feel very relaxed about algebra and geometry. (Mark, prospective secondary mathematics teacher)

> I'm not scared that kids will ask me, you know, a computational question that I cannot solve, I'm more worried about answering conceptual questions. Right now, my biggest fear — and I'm going to have to confront this on the 3rd of February — is what I am going to do if they ask me some kind of question like, "Why are there negative numbers?" (Cindy, prospective secondary mathematics teacher)

Cathy, Mark, and Cindy differ in what they think they need to know in order to teach mathematics, as well as in their assessments of their own preparedness in mathematics. While Mark has confidence in the sufficiency of his mathematics knowledge, both Cindy and Cathy suspect that they may come up short when they try to teach. From a broader perspective, these three teacher candidates represent alternative points of view about the subject matter preparation of teachers. Cathy's view — that she understands the mathematics herself, but needs to learn to teach it — is the basis for traditional formal preservice teacher education. Mark expresses a view that undergirds many of the current proposals to reform teacher education: that people who have majored in mathematics are steeped in the subject matter and have thus acquired the subject matter knowledge needed to teach. Cindy's fear that, although she can do the mathematics, she may not have the kind of understanding she will need in order to help students learn, is a fear shared by a few and is the basis of this study.

The mathematics knowledge that prospective teachers bring to teacher education is the focus of the next two chapters of this dissertation. Despite the fact that subject matter knowledge is logically central to teaching (Buchmann, 1984), the subject matter knowledge of prospective teachers rarely figures prominently in preparing teachers. Although the constraints on formal preservice teacher education are often offered as the explanation for the fact that the subject matter preparation of teachers is left to precollege and "liberal arts" college mathematics classes, I argue that this typical arrangement reflects implicit and questionable assumptions about the understandings of mathematics that prospective teachers bring to teacher education, about what they need to know in order to teach mathematics, and where and when that can best be learned.

First, I explain what I mean by "subject matter knowledge" for teaching mathematics, given mathematical pedagogy as the goal (see Chapter 1). Next, I present and discuss the results of my interviews with prospective elementary and secondary teacher education students that reveal the nature of the mathematical understandings that they bring with them to teacher education. Finally, I show how these results point to the need to reconsider assumptions about what prospective teachers need to know and how they can learn that, assumptions that underlie current teacher education practice as well as proposals to reform teacher preparation.

Subject Matter Knowledge for Teaching Mathematics

Although most researchers have moved away from the earlier use of course lists or credits earned as a proxy for teachers' knowledge, how they currently conceptualize and study "subject matter" varies. Some researchers examine teachers' ***conceptions of or beliefs about mathematics*** (e.g., Blaire, 1981; Ernest, 1988; Ferrini-Mundy, 1986; Kuhs, 1980; Lerman, 1983; Thompson, 1984). These researchers use a variety of methods to identify teachers' conceptions, including interviews, questionnaires, and inferences based on teachers' practices. These studies generally highlight the influence of teachers' assumptions about mathematics on their teaching of the subject.

Other researchers focus on teachers' ***understanding of mathematical concepts and procedures*** (e.g., Ball & McDiarmid, in press; Evan, in progress; Leinhardt & Smith, 1985; Owens, 1987; Schram, in progress; Steinberg, Haymore, & Marks, 1985). Using interviews and structured tasks, they explore how teachers think about their mathematical knowledge and how they understand (or misunderstand) specific ideas. What counts, according to these researchers, is the way teachers organize the field and how they understand and think about concepts (as opposed to just whether they can give "right" answers).

Faced with these different ways of thinking about subject matter knowledge, I needed to make explicit the definition of subject matter knowledge that shaped my approaches to collecting and analyzing data about these prospective teachers' knowledge of mathematics. I had to consider what I meant by "knowing" mathematics. Does knowing mathematics for teaching mean being able do it oneself? Does it mean being able to explain it to someone else? Is subject matter knowledge a question of "knowledge structures" — that is, a function of the richness of the connections among mathematical concepts and principles? What is the relationship among "attitudes," "conceptions," and "knowledge" of mathematics? In this section I describe the conception of subject matter knowledge that undergirds this study.[37]

Mathematical Understanding: Interweaving Ideas *Of* and *About* the Subject

My premise is that understanding mathematics involves a mélange of knowledge, beliefs, and feelings about the subject. Central is propositional and procedural knowledge of mathematics — that is, understandings of particular topics (e.g., fractions and trigonometry),

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[37] Elsewhere I and my colleagues have provided conceptual and empirical justification for this definition of subject matter knowledge for teaching mathematics. The justification is founded on showing how teachers' understandings of mathematics interacts with other ideas about teaching and learning, about students, and about the contexts of teaching, to shape what they do as teachers. The argument, for example, underscores the centrality of knowledge about as well as of mathematics in teaching. See Ball, in press; McDiarmid, Ball, and Anderson, in press.

procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity), and the relationships among these topics, procedures, and concepts (Davis, 1986; Hiebert & Lefevre, 1986; Skemp, 1978). This substantive knowledge of mathematics is what is most easily recognized by others as "subject matter knowledge."

Another critical dimension, however, is knowledge about mathematics. This includes understandings about the nature of knowledge in the discipline — where it comes from, how it changes, and how truth is established. Knowledge about mathematics also includes what it means to "know" and "do" mathematics, the relative centrality of different ideas, as well as what is arbitrary or conventional versus what is necessary or logical, and a sense of the philosophical debates within the discipline.[38]

A subject which, in mainstream American culture, elicits, alternately, awe, anxiety, and intense dislike, mathematics is a domain of feelings as well as knowledge (Bassarear, 1986; Brandau, 1985; Buerk, 1982; McLeod, 1986). Understanding mathematics is colored by one's emotional responses to the subject and one's inclinations and sense of self in relation to it.

The three preceding paragraphs sweep broadly over a vast conceptual issue: what "understanding mathematics" means and includes. The discussion has barely sketched the domains and boundaries. How would one comprehensively map the essential substantive knowledge of mathematics? And what is entailed in "understanding" any of it? The first question — about a map — is one I did not attempt to address in this work. I chose instead to sample purposefully specific mathematical topics[39] and to explore how the teacher candidates understood the particular underlying concepts and procedures. The second question — about what understanding mathematics for teaching means and includes — was, however, a significant question for this study. Below I discuss three issues of critical concern in exploring prospective teachers' knowledge of and about mathematics: (1) the fact that prospective teachers' "knowledge" is not necessarily true, (2) the difference between tacit ways of knowing and explicit conceptual knowledge in doing mathematics, and (3) connectedness of knowledge as a crucial dimension of mathematical understanding.

Talking Carefully About "Knowledge"

Describing prospective teachers' understandings of mathematics as "knowledge" is problematic, for they of course "know" things that are wrong. Some of the prospective teachers whom I interviewed, for instance, thought that $7 \div 0 = 0$, that squares were not rectangles, and that doing mathematics meant adding, subtracting, multiplying, and dividing. Although false from a disciplinary point of view, and therefore not "knowledge" (Scheffler, 1965), these ideas were what they "knew."

Using the term "understanding" (instead of "knowledge") might be one acceptable way of dealing with the fact that people's "knowledge" is not always true. However, with the goal of preparing teachers to teach mathematics, teacher educators have a dual focus: We are interested in understanding not only in what the ideas mean to them as knowers, but also in

[38] Schwab (1961/1978) refers to these kinds of understandings as knowledge of the *substantive* and *syntactic* structures of a discipline — in this case, mathematics.

[39] I explain why I chose each of the topics later in this chapter.

appraising their understanding in light of standards of disciplinary knowledge. To avoid the problem of referring to incorrect notions as "knowledge," I considered talking about their understandings, right or wrong, as "ideas" (a term I use in referring to the teacher candidates' knowledge about teaching, learning, and learners). This term applied better to issues about the nature of mathematics than it did to specific mathematical topics, however. It seemed odd to talk about specific substance — that $1\frac{3}{4} \div \frac{1}{2} = 3\frac{1}{2}$, for instance — as an "idea."

Knowledge is also not an all or nothing matter (Nickerson, 1985). What does one say about the knowledge of a person who says that "you can't divide by zero"? This <u>is</u> true, but of interest here is also <u>how</u> she understands it — as an arbitrary "fact" or as a logical consequent of other mathematical ideas and principles. She may, by way of explanation, say that "it's just one of those things you have to remember," "zero can't do anything to a number," "it's undefined." Or she may prove her assertion by comparing division by 0 to division by 2 or by using the inverse relationship of multiplication and division. In each case, she would reveal significantly different things about her understanding of division by zero as well as her understanding of mathematics more broadly. These differences matter in teaching mathematics from the perspective of mathematical pedagogy in which the goals include helping students understand mathematics as a discipline (see Chapter 1).

In this research, what matters are the qualitative dimensions of prospective teachers' knowledge — what they know and how they think about it. Still, the truth value of their ideas is equally critical, for teachers are responsible for helping their pupils access disciplinary knowledge. Consequently, "knowledge," "understanding," "belief," and "idea" are all used in my discussions, albeit cautiously and with qualification.

Tacit Versus Explicit Ways of Knowing[40]

Uncovering what people know in mathematics is a endeavor fraught with practical and conceptual difficulties. Not the least of these is the problem of inferring what people <u>know</u> from what they <u>do</u> or <u>say</u>. On one hand, observing what people do when they are presented with mathematical problems raises questions about the relationships between procedural and conceptual knowledge. Assuming that people have conceptual knowledge of procedures which they have learned to perform is a fallacy (Hatano, 1982). As one of the math majors reflected when he tried to explain the basis for the multiplication procedure, "I absolutely do it [multiplication] by the rote process — I would have to think about it." Not clear here was whether he had, at one time, understood the underlying principles — but had simply forgotten — or whether it was something he had never known, never considered — and would therefore have to figure out. Certainly many children and adults go through mathematical motions without ever understanding the underlying principles or meaning. How many people, for example, can say <u>why</u> "cross-multiply and divide" works?

Still, mathematical understanding may also be tacit. Successful mathematicians can unravel perplexing problems without being able to articulate all of what they know. Not unrelated to Schon's (1983) "knowing-in-action," the mathematicians' work reflects both tacit

[40] The distinction between tacit and explicit ways of knowing is not intended to suggest a dichotomy, but rather a qualitative dimension along which understanding varies.

understanding and intuitive and habituated actions (Noddings, 1985).

Experts in all domains, while able to perform skillfully, may not always be able to specify the components of or bases for their actions. Their activity nevertheless implies knowledge. Similarly, in everyday life, people understand things which they cannot articulate. For instance, a woman may find her way around the town she grew up in, identifying friends' homes and old hangouts, yet not be able to give directions to a visitor. A man may use colloquial French expressions in speaking French but be unable to explain their meaning to a fellow American.

Polanyi (1958) describes what he calls the "ineffable domain" — those things about which our tacit understanding far exceeds our capacity to articulate what we know. He argues further that "nothing we know can be said precisely, and so what I call `ineffable' may simply mean something I know and can describe even less precisely than usual, or even only very vaguely" (p. 88). It is unclear whether we would want to say that the woman understands her way around less well than someone who can give directions, or that the man understands French less well than someone who can translate. Clumsy attempts to articulate understanding may reflect an area in which, according to Polanyi (1958) the tacit predominates.

On the other hand, apparently clumsy talk may not be clumsy or inarticulate at all, but rather may reflect how the speaker actually understands what he or she is talking about. Orr (1987) argues that teachers often "fill in" the gaps in what their pupils say, assuming they know what the pupils "mean." She said that when her high school geometry students would talk about distances as locations and locations as distances, she thought these were careless mistakes or awkward wording. Suddenly it occurred to her that these nonstandard ways of talking might actually represent nonstandard understandings of the relationship between location and distance. She began asking some different questions of her students to try to elicit what they understood — asking them to construct diagrams showing where certain cities were located and the distances among them, for example. She discovered in case after case that her students' explanations were accurate reflections of how they were thinking.

Just like mathematicians, ordinary people do things in mathematics — and do them correctly — which they cannot, however, explain. This raises two points critical for research on teacher knowledge: one methodological and one theoretical.

What do teachers understand? Problems of inference. Analyzing teachers' knowledge is complicated by the extent to which they are able to talk or otherwise represent that knowledge. Did the less articulate prospective teachers whom I interviewed understand the concepts less deeply than those who were very explicit? Were convoluted explanations a sign of confusion or of limited capacity to put into words what they understood? This issue clearly presents methodological problems of inference in studies of teachers' subject matter knowledge. A second consideration, however, affords a way out of this methodological tangle.

What do teachers need to know? Tacit knowledge, whatever its role in independent mathematical activity, is inadequate for teaching. In order to help someone else understand and do mathematics, being able to "do it" oneself is not sufficient. A necessary level of knowledge for teaching involves being able to talk about mathematics, not just describing the steps for following an algorithm, but also about the judgments made and the meanings and reasons for certain relationships or procedures.

Explicit knowledge of mathematics entails more than saying the words of mathematical statements or formulae; rather, it must include language that goes beyond the surface mathematical representation. Explicit knowledge involves reasons and relationships: being able to explain why, as well as being able to relate particular ideas or procedures to others within mathematics.

Explaining is the teacher's trade. Explanations open up ideas and make them more accessible by clarifying terms or delving into details (Buchmann, 1987a). Explaining something in mathematics means clarifying an idea by unpacking underlying concepts as well as giving reasons that reveal its meaning and logic.

Being able to explain mathematics is essential knowledge even for teachers who do not teach mathematics in a "show and tell" mode (see Chapter 5). Facilitating students' construction of mathematical understanding, for instance, involves selecting fruitful tasks, asking good questions, judging which student ideas should be especially pursued. All of this demands explicit analytic knowledge, the same kind of understanding entailed in constructing direct explanations.

Connectedness in Mathematical Understanding

A third dimension of mathematical understanding critical for mathematical pedagogy is **connectedness** for, however explicitly articulated or carefully memorized, mathematical knowledge is not a collection of separate topics, nor a laundry list of rules and definitions. Addition is fundamentally connected to multiplication, algebra is a first cousin of arithmetic, and the measurement of irregular shapes is akin to integration in calculus.

In the school mathematics curriculum, however, these are delivered in compartments separated in time and meaning. Even content taught within the same grade or course is often disconnected. Rarely are students encouraged or helped to make connections among the ideas they encounter in school. In mathematical pedagogy, teachers aim to help students to connect mathematical ideas with one another as well as to other domains, to acquire both "breadth and depth" (Duckworth, 1979).

The standard school mathematics curriculum to which most prospective teachers have been subjected treats mathematics as a collection of discrete bits of procedural knowledge. Take division, a focal topic in this study, as an example. A typical textbook page introducing division of fractions, for example, says simply, "Dividing by a fraction is the same as multiplying by its reciprocal." No or little attention is given to the meaning of division with fractions and no connections are made between division with fractions and division with whole numbers (Fielker, 1986). (Dividing by a number, after all, always produces the same result as multiplying by its reciprocal.) Division with decimals is still another case — i.e., "move the decimal point over until you get a whole number in the divisor, then move the decimal point over the same number of places in the dividend." Division in algebra becomes yet more meaningless as pupils are told to "get rid of the denominator." And division by 0 is a special fluke — you just "can't do it."

Not only does this tendency to compartmentalize mathematical knowledge seriously misrepresent the logic and conceptual organization of the discipline to students, but it also substantially increases the cognitive load required to "do" or "know" mathematics. Each idea or procedure seems to be a separate case. Each requires a different rule, all of which must be individually memorized and accessed. "Knowing" mathematics is easily reduced to a senseless activity.

If teachers are to break away from this common approach to teaching and learning mathematics, they must have connected rather than compartmentalized knowledge of mathematics themselves.

* * *

The discussion of prospective teachers' understandings of mathematics is divided into two chapters. In this chapter, I discuss the substantive knowledge of mathematics of the 19 undergraduate teacher education students in this study. The chapter is organized around four mathematical topic areas: place value and numeration, division, the number zero, and slope and graphing. In each section, I discuss the rationale for exploring that particular area, explain the mathematics involved, describe my approach to exploring the teacher candidates' understanding of it, and present and appraise the results. Pursuant to the discussion above, the analysis focuses on three critical dimensions of the teacher candidates' knowledge of mathematics: the correctness, explicitness, and connectedness of their understanding. In Chapter 4, the discussion focuses on the teacher candidates' ideas **about** mathematics — exploring their understandings of theory and proof, about the nature of mathematics, as well as their feelings about mathematics and about themselves in relation to the subject.

Knowledge of Place Value and Numeration[41]

Numeration — how we record numbers — has been a central mathematical issue for thousands of years. Ancient peoples sought ways to efficiently represent quantities in writing, and the history of these searches provides fascinating testimony both to the inventiveness of the human mind as well as to the constructed and evolutionary nature of mathematical knowledge. The system we use is a multiplicative grouping system, grounded in base ten and positional notation, but many systems have preceded or coexisted with the current one (Bennett & Nelson, 1979). The Roman system, for example, familiar to most of us, is a simple grouping system which works additively and subtractively (for example III means 1 + 1 + 1, or 3, while IV means 5 - 1, or 4).

The base ten positional numeration system is part of adult working knowledge of numbers in our society. That is, adults read, write, and make sense of written numerals. They know that "56" does not mean 5 + 6 or 11. They know that "04" represents the same quantity as "4" but that "40" does not. This is a critical understanding for everyday as well as mathematical activity.

Children, however, do not automatically understand the numeration system in terms of the underlying grouping concepts. While they may be able at an early age to read numerals like

[41] This is the first of four topically organized sections examining the teacher candidates' mathematics knowledge. In order to help the reader appraise the prospective teachers' understandings, each section includes an explanation of the mathematics under discussion and investigation. The explanations are intended to highlight the underlying depth and complexities of the content (Duckworth, 1987), commonly invisible or taken for granted.

"74" or "714," they are nevertheless unlikely to see a difference between the "7" in each numeral. In fact, the way they learn arithmetic may be a hindrance rather than a help to understanding numeration and place value.

Some research suggests that place value is particularly difficult for children to learn. Elementary school students may write 365 as 300605, for example, which represents the way the number sounds rather than place value. Kamii (1985) argues that traditional math instruction actually forces young children to operate with numerals without understanding what they represent. We have all heard children performing addition calculations reciting, "5 plus 7 is 12, put down the 2, carry the 1" or doing a long division calculation such as 8945 divided by 43 by saying, "43 goes into 89 twice, put up the 2, 2 times 43 is 86" and so on. These "algorithm rhymes"[42] which pupils learn interfere with paying attention to the essence of the numeration system — that numerals have different values depending on their place. The "1" in the addition rhyme actually means 10. The "89" in the division chant actually means 8900 and that "2" represents the fact that there are 200 groups of 43 in 8945.

Since place value (and its root idea, grouping) is a fundamental mathematical idea and since pupils often find it difficult, it seemed a critical area of prospective teachers' knowledge to investigate. I embedded place value concepts in two different interview tasks: one a classroom scenario focused on student difficulties with the multiplication algorithm, the other a structured planning-teaching-assessment exercise on subtraction with regrouping ("borrowing"). Although place value is the underlying foundation of these conventional procedures, adults can perform the procedures competently without thinking about place value at all. I wanted to examine how well the prospective teachers' adult operational knowledge of numbers equipped them to help pupils make sense of the meaning of written numerals and the meanings of these operations with them.

The classroom scenario task in the interview read as follows:

===

Some eighth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ \\ 1845 \end{array}$$

[42] I borrow the term "algorithm rhyme" from Blake and Verhille (1985).

instead of this:

```
   123
 x 645
   615
  492
 738
 79335
```

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching eighth grade and you noticed that several of your students were doing this?

==

The algorithm for multiplying large numbers is derived from the process of decomposing numbers into "expanded form" and multiplying them in parts. To understand this, one must understand decimal numerals as representations of numbers in terms of hundreds, tens, and ones, i.e., that in the numeral 123, the 1 represents 1 hundred, the 2 represents 2 tens, and the 3 represents 3 ones. In this example, 123 x 645, first one multiplies 5 x 123:

```
  123
x   5
  615
```

then 40 x 123:

```
  123
x  40
 4920
```

and then 600 x 123:

```
  123
x 600
73800
```

In the final step, one adds the results of these three products:

```
  123
x 645
  615
 4920
73800
```

In effect, one is putting the results of the "parts" of the operation back together — i.e., 645 x 123 = (600 x 123) + (40 x 123) + (5 x 123).

Few people write their computation out this way. Most "shortcut" it by writing:

123

$$\begin{array}{r} \underline{\times\ \ 645} \\ 615 \\ 492\ \ \\ \underline{738\ \ \ \ } \end{array}$$

This shortcut, in effect, hides the conceptual base of the procedure. Furthermore, people can learn the shortcut without learning the conceptual base. This procedure, which depends conceptually on place value and the distributive property, is thus a strategic site for exploring prospective teachers' understandings of place value.

Correctness

The simple way to report what the interviews revealed is to say that only 5 of the 19 prospective teachers whom I interviewed talked explicitly about the concepts of place value and numeration that underlie the multiplication algorithm. The others gave answers that were ambiguous or focused exclusively on the procedure. For this topic, however, to stop at such a summary would be to miss much of what was significant about the teacher candidates' knowledge of place value: what they focused on and their ways of explaining what they were thinking.

Understanding the Role of Place Value in Multiplication

Some of the prospective teachers' responses to the scenario were relatively easy to interpret because they focused explicitly either on the role of place value in the algorithm or the steps of the procedure. For example, Allen, an elementary major, said that he would "have to explain about that not being 123 x <u>4</u>. That it's 123 x <u>40</u>." In contrast, Teri, a prospective elementary teacher, focused on the steps:

> I would show them how to line them up correctly. I would do what I still do, which is once I multiply out the first number and then I start to do the second line, put a zero there. That's how I was taught to do it and that's how I still, when I have big numbers to multiply, I do, because otherwise I'd get them too mixed up, probably. It helps to keep everything in line, like after the first line, you do one zero and then you do two zeroes to shift things over.

Allen's answer showed that he understood that "moving the numbers over" is not just a rule to remember, but reflects that 123 x 4 is 4,920, not 492. In contrast, Teri's understanding was wholly procedural: the numbers must be lined up and the zeroes help you to remember to "shift things over." There was no hint in her answer that she saw any meaningful basis for the procedure.

Others, however, talked in ways that were harder to interpret. One reason for this was the ambiguity in the use of the term "place," which held conceptual meaning in some cases and largely procedural reference in others.

The prospective teachers also varied in what they understood to be the focal content issue: some saw it as place value, some saw it as remembering to "line up and move over," others emphasized the role of zero.

<u>Place *value* or "places"?</u> Some prospective teachers' responses were difficult to interpret because they used conceptual language — e.g., "the tens place" — to describe procedures, or procedural language — e.g., "add a zero" — to (perhaps) refer to concepts. Rachel's response was an example of this ambiguity:

> You would take the last number and multiply it by all three of the top numbers and you put those underneath and then you start with the next one. You'd want to put it underneath the number that you are using. They aren't understanding that they need to be underneath of that instead of just down in one straight row.

This seemed like an answer focused on the rules of the multiplication algorithm. Rachel was talking only about where to put the numbers and what to do next. But then she said that the students would "probably need to know about places":

> You know, the hundreds, the thousands, you know, whatever. If they don't understand that there is a difference in placing, that could also lead to this if they don't remember. . . . They need to understand that there is a difference in the placing, too.

What did Rachel mean when she said "placing"? She may have been talking about where to write the numbers — where to place them — or she may have been talking about the difference in the value of a number depending on its placing. To probe how she understood "places," I asked her why this mattered. She replied:

> Because of the fact that you are working with such a large number, like your second and third numbers are not going to be ones. You move out of, your numbers get larger and larger and since you are working with such a large sum, you have to know how to work in the thousands, you know, to keep your numbers that way. I guess it all goes back to them understanding why the numbers should be underneath of what you are multiplying.

She added that she wasn't sure "how that affects the placing."

Rachel's response was not as clear as Allen's or Teri's. She seemed to focus on lining up the numbers correctly, but then she talked about "places," too. Was her reference to "placing" and "places" conceptual — i.e., addressing the values of different places within a numeral? Or was Rachel just talking about "placing" the numbers in the right place — so that they would be lined up correctly?

Zero as a "placeholder." Also ambiguous were the responses of several prospective teachers who talked about the importance of writing in zeros in the partial products. Janet, a secondary candidate and mathematics major, said she would get her students to focus on putting the numbers "in the right places" and would "encourage them to use zero as a placeholder," and Cindy, another math major, commented, "We were taught to put a 0 here, and a 0 there, to represent the places." Cathy, an elementary major, tried to explain the role of 0:

> I don't exactly know how to explain it, but something having to do with this first column. . .is the ones, and the next column is the tens, and maybe something like there's a 0, you know, the tens there's always one 0, and so you have — God, I don't know. Like to make it balance out for the tens you'd have to add the 0 and for the hundreds you'd have to add two zeros. Something to that effect. I don't

know.

In some of these cases, interpreting what the prospective teachers understood about place value was difficult. Although their answers focused on how to write the partial products, it is not clear what that meant to them. People could talk about the importance of zero as a "placeholder" and mean simply that using zeros helps one remember to get the numbers lined up correctly.

Some prospective teachers who talked about zeros did elaborate their answers explicitly and their responses reveal different kinds of thinking that can underlie answers focused on "putting down zeros." The responses of Pam and Allen, both elementary teacher candidates, illustrate such differences in thinking.

Pam said she would show pupils to "physically put a zero every time you moved down a line." She explained that "zero doesn't add anything more to the problem. It's just empty. But instead of having an empty space, you have something to fill in the space so that you can use it as a guideline."

Allen also said he would "make it mandatory that the zeros start showing up" on his pupils' papers. But he explained it differently. He said he would "have to explain about that not being 123 x 4. That's 123 x 40, which is a multiple of 10 — which has that 0 on the right side which is why the 0 has to be there."

Both Pam and Allen would have their pupils put the zeros down, yet their explanations revealed strikingly different understandings of the role of zero in our decimal positional numeration system. Pam saw the zero as useful for keeping the columns of numbers lined up but says that zero "adds nothing" to the number. Her statement suggested that she confused "adding zero" to a number (78 + 0 = 78) with the role of 0 in a numeral (e.g., 780). Allen knew that 123 is multiplied by 5, 40, and 600. He said the zeros "have to be there" because the products are "a multiple of 10 off." Still, his response did not show what he knew about the zeros in place value numeration. Was it a rule he had memorized — that multiples of 10 have one zero, multiples of 100 have two zeros, and so forth? Or did he know why putting a zero "on the right side" produces a number that is ten times the original? Understanding what prospective teachers know is complicated by these confusions and inadequacies of their language.[43]

Partial and inexplicit understanding. Those who mentioned "places" and "ones, tens, and hundreds" may have had a partial, fuzzy, or inexplicit understanding of the underlying concepts of place value. Some prospective teachers figured it out in the course of answering the question. Barb, a post-B.A. student with an undergraduate mathematics major, was one of these. She began her answer much as many others did, focusing on "moving over" from column to column:

> You start in the units column and you multiply that, and then you start in your tens column and so you have to start in your tens column of the next one and

[43] Although it was not a focal topic in this study, the number zero and the ways in which prospective teachers thought about it came through on a number of the tasks. Preliminary analyses suggested that misconceptions about the number zero were common among the prospective teachers in this study.

> you multiply 4 x 123 and then you move over into your hundreds column over here where you're taking 6 x 123.

Then she talked about how she was taught to "put the zeros there because it helped me line up my columns." As she pondered this aloud, she began to pull her own submerged knowledge to the surface:

> A lot of the time you say, "Well, put a zero here, put a zero there, and zero there, and you put a zero here, and a zero there," and you get into the <u>method</u> of it and you know that you put a zero here, but they don't really understand <u>why</u>. And I think it goes back to the units and tens and hundreds and all that. And that might be an easier way to take a look at it. `Cause you're going to take 5 times that, and you take 40, and then 600, and you can see where those zeros come from.

She still wasn't entirely clear about this, however, as is shown by her efforts to make explicit the reason for the zeroes. First, she said that "when you take the 4 or the 40, you're gonna want to start in, understand that you're working with tens now, so you want to move into the tens column" but then she stopped and said, "God, I don't know any other way that I'd be able to describe it than, I'd have to think about it." She paused and looked at the problem and suddenly realized that 123 x 40 "is going to be the same as this [492] with a 0 on it!" She talked to herself under her breath and then a few moments later looked up and said, "Wow, I haven't even thought about it that way before!. . . <u>that's</u> where those zeros come from, <u>oh</u>! <u>Wow</u>, okay."

Although Barb could multiply correctly, she did not explicitly understand the mathematical principles underlying the procedure. She was, however, able to put different pieces of knowledge together and figure it out as she talked. Others who lacked explicit understanding also seemed to realize that there was more to know than just procedures, but could not always uncover the deeper levels. Sandi, an elementary major, struggled and then gave up. Her answer seemed to focus on the rules of lining up the numbers:

> I would explain that every time you move over this isn't ones, this is tens, so it's ten more, so you have to have an extra ten there, you have to put the zero there to hold it in place. Does that make sense?

I asked if it made sense to <u>her</u>. She replied,

> Oh, I know what I'm saying, I know what I'm thinking, I just, I don't know if I can explain it. . . I guess it's because the stuff is so basic to me.

What Sandi could <u>say</u> was that "you have to put the zero there to hold it in place." Her explanation that in the tens place "it's ten <u>more</u>" misrepresents the fact that the value of the tens place is ten <u>times</u> more. Still, her comment that she knew what she was thinking but doesn't know if she can explain it is worth pondering. Sandi seemed to have part of the idea, that something about the value of the places mattered, but was unable to pull it together.

Understanding the Role of Place Value in Subtraction with Regrouping

To extend what I was learning about the explicitness of the teacher candidates' knowledge of place value, as well to investigate the degree to which that knowledge was connected across contexts, I explored their understanding of the role of place value in subtraction with regrouping (a procedure commonly known as "borrowing"[44]). I designed an exercise, longer than the rest, in which I asked the teacher candidates to examine a section from a second grade math book (Bolster, Cox, Gibb, et al., 1975). This section (two pages) dealt with subtracting two-digit numbers with regrouping. The task, as it was conceived and constructed, had several purposes. I asked the teacher candidates to appraise the section, to talk about what they perceived as its strengths and weaknesses, and then to describe how they might go about helping second graders to learn "this." I did not specify what "this" was because I wanted to see what they would focus on. I also asked them what they thought pupils would need to know before they could learn this, and what they would use as evidence that their pupils were "getting it." Finally they examined a second grade pupil's work on one of the pages, and were asked to talk about what they thought she understood and what they would do next with her.

Generally, compared to the previous scenario, the teacher candidates <u>did</u> seem to have explicit knowledge of the role of "tens and ones," or place value, in "borrowing." Everyone used "tens and ones" language, and most focused to some extent on the underlying place value concepts. Still, some were preoccupied with the "borrowing" metaphor, and most saw subtraction, not place value, as the central concept.[45]

<u>"Tens and ones."</u> Of the 9 elementary majors who did the task,[46] 7 focused explicitly on concepts of place value. These teacher candidates' responses showed that they were aware that "tens and ones" played some sort of role in teaching subtraction with regrouping (which they all referred to as "borrowing"). For some, this awareness of tens and ones was at the surface, readily accessible. For example, Teri described what she would say for:

$$\cancel{6}\,{}^{1}4$$

> I would say, you know, obviously these numbers, you can't subtract in your head. All right, you have to cross out one of the tens from the top. And put it

---

[44] Using the tern "borrowing" misrepresents the conceptual essence of the procedure. The traditional metaphor conveys the impression that one number gets something from another on a short-term basis. It completely ignores the place <u>value</u> aspect of the number. Regrouping is a truer representation of the procedure. to subtract

$$\begin{array}{r} 71 \\ \underline{-\ 39} \end{array}$$

the number 71 (7 tens and 1 one) is regrouped so that it is expressed as 6 tens and 11 ones. 71 is still the <u>total</u>. It is simply represented as

$$\cancel{7}\,{}^{1}1$$

instead. To say, on the other hand, that "the 1 borrows from the 7," while cute, does not call attention to the conceptual basis.

[45] Both are of course key, but place value is actually more central to learning the <u>procedure</u> (see note 4).

[46] Anne, the tenth elementary candidate, did the secondary structured exercise as she was a prospective middle school teacher.

> over in the ones column on the top, so you are able to subtract the two numbers. And then when you cross that tens number, change it, like subtract 1 from it. So you change, like if it was 64, change it to uh, you know, the 6 to a 5, and the 4 to a 14. And maybe I would show them, like 64, like maybe I would write 64 on the board. And then put that it equals 50 plus 14, so they see it is still the same amount.

Teri, in the midst of a procedural description ("change the 6 to a 5"), explicitly added an important piece of conceptual understanding: that 64 equals 50 plus 14 and so the crossing out has not changed the value of the number.

Rachel anticipated that pupils would be confused about "why it doesn't just become — if you take 1 away from the 6, why doesn't the 4 just become 5?"

$$\begin{array}{r} \cancel{6}\,{}^{1}4 \\ -\underline{3\;\;6} \end{array}$$

She thought about it and decided that "you would have to let them know the difference between the ones and the tens columns. . .they would need to understand that there is a difference, you know, where the number is placed." Then she said that pupils would have to know that "64 is like one number instead of just thinking of it like a 6 and a 4." A few moments later, she elaborated her insight:

> Oh, okay. I want them to understand that you know 60 — 6, 0, is a lot bigger than just a 6 and a 0 together. The total concept that — I'm not exactly sure how to explain it — is that 60 is how large it is, you know. The 6 means a lot more than the 4, that it has a (pause) I mean, not means a lot more, but that it contains a lot more — how do you explain that? That it is a larger number when it is in the tens than when it is in the ones, you know.

In her response, Rachel seemed to bring some submerged understanding of place value to the surface. She realized that the "where the number is placed" affects its value, and thus appeared to unpack what it means to understand "the difference between the ones and the tens places."

Cathy tried to do the same thing, but with less success. First she said she had "no idea" about how she would "get into" the "borrowing". I asked her why that seemed hard and she explained,

I mean maybe because I don't really exa — I mean, I don't, it's just like second nature to me and I don't know exactly why we do it.

> [looking at:
>
> $$\begin{array}{r} \cancel{6}\,{}^{1}4 \\ -\underline{3\;\;6} \end{array} \quad]$$
>
> You go over to the other column and you borrow 1 from the 6. So you lose 1 on that side and it becomes a 5 so — and like, I could say that and that you add 1 — the 4 becomes a tens. But why, I mean, how can you do that? I mean, I guess you are saying that the 64 becomes a 54 and why? You know?

Unlike Teri, Cathy did not seem to understand that the procedure transforms 64 into 50 + 14. And unlike Rachel, Cathy seemed unable to retrieve the underlying concepts in the course of the interview.

<u>**"Borrowing."**</u> Although the textbook page was labeled "subtraction with renaming," all the prospective teachers referred to the procedure as "borrowing." Rachel was the only one who noticed the difference, and it puzzled her:

> Somehow renaming just doesn't seem the same because with me with borrowing, it seems like I know what I am doing..I am <u>borrowing</u> from the number beside it, by taking numbers from that, but with <u>renaming</u> it is. . . I don't know, it doesn't have the same connotation. When you rename it, why would you have to pick the one beside it, you know?. . . I would like to understand better about the renaming. Why they call it "renaming." What concept is behind that. I have a feeling now that it is going to be a difficult thing for me to change my own mind from borrowing to renaming.

Rachel's puzzlement suggested that prospective teachers' talk about "tens and ones" could have been, at least in some cases, largely procedural — i.e., to give directions about how to carry out the steps of the algorithm: "go to the next place — the tens — and borrow a ten and move it over to the ones." For example, although Teri mentioned that 64 was the same as 5 tens and 14 ones, Cathy realized that she did not know how "64 becomes a 54."

For two of the teacher candidates, subtraction and the metaphor of "borrowing" were especially focal. Pam said she would start by making sure her pupils could do "basic adding" before they got to the "concept of taking something away." Then she would make up simple stories around "borrowing something from your neighbor." She would start out with single-digit numbers "just to make sure they understand the concept":

> A situation where your mother had 5 apples and she needed 7 apples so she would send the child over to get 2 more apples from a neighbor, that sort of concept. Now in this [looking at:

$$\begin{array}{r} \cancel{6}\,{}^{1}4 \\ -\underline{\;3\;\;6\;} \end{array}$$

> it would be possibly she had 4 apples and the neighbor had 6 and she need 1 more apple.

I asked Pam what she meant by "the concept" and she replied, "Oh, subtracting one number from another." She said she would "advance" from these simpler stories to ones with larger numbers — "tens and ones." Pam's attention seemed to be on subtraction and on the idea of "borrowing" in the real world, not on place value. Her distraction from the conceptual essence of the problem was typical of the prospective teachers, who often tended to misrepresent the content. Here Pam confuses <u>60</u> apples with <u>6</u> apples, and <u>10</u> apples with <u>1</u>, serious confusions.

Mei Ling, another prospective teacher, explained how she would help pupils understand the <u>need</u> for the procedure:

> I would say "Ok, you've got. . . for example. . . you've got 4 Cokes here and you've got 6 people. Everyone of these people, they want to drink one Coke. Do you have enough, is it going to work?" They'll say "no" and someone will say "Well, we'll split one down the middle." (laughs) So I say, "Okay, that's not going to work, but then you go back to the store. . . I would just show that relationship isn't going to work. I'd say, "4 minus 6 doesn't work, so you have to borrow".

Mei Ling's story of 4 Cokes and 6 people represents division as well as subtraction. Although she wanted pupils to see that 4 - 6 "doesn't work," pupils may, as she anticipated, suggest sharing, or dividing, the Cokes. In that case, the story may represent 4 divided by 6. Mei Ling would explain, however, that "that relationship doesn't work, so you have to borrow." (In fact, she wanted pupils to see that, short of Cokes, they would go back to the store and get more — not exactly "borrowing," either.) Mei Ling's story does not pick up the "borrowing" of tens either, the central idea underlying regrouping in our numeration system. Like Pam, Mei Ling focused instead on getting enough more — however many that takes. In regrouping, however, one always regroups by tens, regardless of the quantity one is short. The story could also have been interpreted as a representation involving negative numbers (i.e., we are short 2 Cokes, or 4 - 6 = -2), however, like many of the other prospective teachers, Mei Ling was trying to make the (false) point that a larger number cannot be subtracted from a smaller one.

Still, all the teacher candidates were more explicit about place value when talking about subtraction with regrouping than they were when they discussed the multiplication algorithm. With multiplication, for instance, Teri focused on "lining up the numbers" and "shifting things over" on each line. She did not seem to understand that the partial product written as 492 was really 4920 ("adding the zero just keeps everything in line"). Yet, in talking about subtraction with regrouping, Teri talked explicitly about 50 + 14 being "the same amount" as 64.

Explicitness and Connectedness in Knowledge About Place Value

Explicitness and meaning. Overall, the teacher candidates' knowledge of place value was inexplicit. Although they were able to perform the computations involved, they were less able to explain the underlying concepts.

In the multiplication question, many of the teacher candidates mentioned "places" and yet did not necessarily seem to focus on place value. They talked about making sure the "4" was "under the tens place" or about "placing the numbers correctly under the one you're multiplying by." "Place" seemed to be equivalent to placement, or getting the numbers in the correct columns. In the subtraction task, however, most of the prospective teachers did seem to be talking about place value — tens and ones. On one hand, this suggests that they did have some explicit understanding of the decimal numeration system.[47]

Their mention of "tens and ones," on the other hand, may have been more procedural than conceptual. The meaning of "borrow one from the tens" is ambiguous. It may mean, literally, take 1 away from the number in the tens place — i.e., cross out the 6 and make it a 5. Or it may mean take 1 ten away from 6 tens, leaving 5 tens. Cathy's confusion about how "64 becomes 54" is evidence of the possibility that a teacher candidate could talk in terms of ones and tens without engaging the concept of grouping (and regrouping) by tens, just as reference to places in the multiplication algorithm may not signal attention to place value.

The fact that the steps of the "borrowing rhyme" refer explicitly to tens and ones (e.g., "borrow one from the tens, move it to the ones") may help to explain why teacher candidates seemed to focus more on place value when they talked about subtraction with regrouping than they did when they were thinking about multiplication.

Fragmented understanding. Almost all the teacher candidates were more explicit about place value when talking about subtraction with regrouping than they were when they discussed the multiplication algorithm. The teacher candidates may have understood the role of "tens and ones," or place value, in "borrowing" and yet not have connected that understanding with the multiplication algorithm. Their knowledge of place value seemed to be compartmentalized in the specific context of subtraction with regrouping. It was apparently not readily usable in the other equally relevant context — multiplication.

Summary: The Prospective Teachers' Knowledge of Place Value

Place value is a concept central to mathematics and one that is critical to helping pupils learn about the meaning of written numerals and operations with them. Yet, for adults, it is knowledge that is submerged under the routines of everyday operations with numbers, and may be rarely, if ever, made explicit. For these prospective teachers, knowledge of place value was not instantly retrievable. Being asked why it was permissible to write 4,920 instead of 492 did not, for example, readily trigger a conceptual justification. Furthermore, the teacher candidates' ability to recognize and mention the role of place value in one context and not in another suggests that place value does not occupy a central structural place in their knowledge of mathematics. Analyzing prospective teachers' knowledge of place value highlights the dangers of assuming that they have explicit and connected understandings of basic mathematical ideas, even when they are able to operate with them.

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[47] Two alternative explanations exist for this difference. First, their own teachers may have talked about tens and ones and place value in teaching subtraction with regrouping, but not in teaching multiplication. Thus, the teacher candidates might have understood the role of place value in one context, but might never have considered it in the other.

Second, the textbook page in the subtraction exercise displayed numbers in columns labeled "tens" and "ones":

	tens	ones
	6	4
-	3	6

The multiplication question contained no such explicit cue. This alone might have led some prospective teachers to think about place value. Still, the book did not deal conceptually with place value, so the cue was limited.

Knowledge of Division

Why division? An overrated and overemphasized piece of the school mathematics curriculum, critics would argue. Mathematics educators despair at the fact that children spend the better part of fourth and fifth grade "in" division — being trained to do what a $5 machine can do faster and more accurately (Schwartz, 1987). They urge teachers to emphasize division less, especially long division, and to teach better mathematical content instead.

Still the concept of division is a central one in mathematics at all levels and it figures prominently throughout the K-12 curriculum. Furthermore, division is worthwhile content for what students can learn about rational and irrational numbers, about place value, about the connections among the four basic operations, as well as about the limits and power of relating mathematics to the real world. For these reasons, as well as the fact that students also often have difficulty learning it, division is a topic about which teachers should have explicit conceptual and connected knowledge.

In order to examine the connectedness of teacher candidates' knowledge of division, I chose three different mathematical contexts: division with fractions, division by zero, and division with algebraic equations. These contexts, because they are separated in time and meaning by the school curriculum, would not appear obviously connected to teacher candidates. Yet division is the key to each. The kinds of tasks I posed also invited the teacher candidates to display explicit conceptual understanding: asking them to explain and to generate representations.

In order to help the reader appraise the prospective teachers' knowledge in the section that follows, I first present a brief discussion of division and its meaning in each of the three contexts.

At its foundation, division has to do with forming groups. Two kinds of groupings are possible:

> (1) Forming groups of a certain size (e.g., taking a class of 28 students and forming groups of 4). The problem is how many groups of that size can be formed? This is sometimes referred to as the **measurement** model of division.
>
> (2) Forming a certain number of groups (e.g., taking a class of 28 students and forming 4 groups). The problem is to determine the size of each group. This model is sometimes referred to as the **partitive** model of division.

Consider a typical division statement with whole numbers, such as $7 \div 2$. What does this mean? It may represent one of two kinds of situations:

> (1) I have 7 slices of pizza. If I want to serve 2 slices per person, how many portions do I have? (Measurement interpretation — Answer: $3\frac{1}{2}$ portions)
>
> (2) I have 7 slices of pizza. I want to split the pizza equally between 2 people. How much pizza will each person get? (Partitive interpretation — Answer: $3\frac{1}{2}$ slices)

Notice that because these two situations represent two different meanings for division, the referent for $3\frac{1}{2}$ is different in the two cases. In the first situation, the answer is a number of portions (the size of the portion was already decided); in the second, the answer is a number of slices per portion (the question was how large each portion would be). In each case, multiplying the result ($3\frac{1}{2}$) by the number used to divide the original total yields that total ($3\frac{1}{2} \times 2 = 7$).

Dividing by fractions, the first of the three contexts in the prospective teacher interview, is not different conceptually from dividing by whole numbers. What about the specific topic at hand, division with fractions? Remembering to "invert and multiply" — that is, to invert the divisor and multiply it by the dividend — is one way of understanding division by fractions. A typical sixth grade textbook page introducing division of fractions says simply, "Dividing by a fraction is the same as multiplying by its reciprocal." No or little attention is given to the meaning of division with fractions and no connections are made between division with fractions and division with whole numbers. Each is treated as a special case.[48,49] Since division with fractions is most often taught algorithmically, it is a strategic site for examining the extent to which prospective teachers understand the meaning of division itself (Davis, 1983).

Division by zero, the second case of division explored in this study, is also no more complicated than division with other rational numbers. The result, however, is different. Extending the debt example, suppose you owe \$100 and want to know how long it will take you to repay your debt if you make payments of \$0 per week. What is $100 \div 0$? Obviously you will never repay your debt at that rate — you will be indebted forever. There is no solution to your indebtedness if you choose to repay in installments of \$0. Going one step further, there is no number of weeks that you can multiply by \$0 and come up with a total of \$100. Yet, in the

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[48] Dividing by fraction is not different conceptually from dividing by whole numbers. Suppose, for example, that you owe a friend \$100 and must repay the money, although without interest. You can explore how long it will take to repay this debt, giving different payment amounts. If you pay \$2 per week, it will take you 50 weeks. This can be formulated mathematically as $100 \div 2 = 50$. Now, consider how long it will take you if you repay at a rate of 50¢ per week. 50¢ = $\frac{1}{2}$ dollar, so this option can be expressed as $100 \div \frac{1}{2}$. If you understand division, and have a feel for amounts of money, you know that 200 weeks is a reasonable answer to this (even though 4 years to pay back \$100 will probably jeopardize your friendship). If you are uncertain about your calculations, you can verify them by multiplying the number of weeks by the amount of the payment. For example, 200 x .50 = \$100.00 (or $200 \times \frac{1}{2} = 100$). Both of these repayment plans model the first meaning of division above — i.e., take \$100 and form "groups" — first of \$2 and then of $\frac{1}{2}$ dollar. The problem in both cases is how many "groups" — or payments — of the given amount there are in \$100, a measurement representation of $100 \div \frac{1}{2}$. To see a partitive model of the same expression, try the following: Imagine that you are packing apples in crates. You have 100 apples and find that they fill exactly half the crate. How many apples will it take to fill the crate? In this case, number $\frac{1}{2}$ represents the number of groups and the question is how many will be in each group. Notice that this representation corresponds directly to the algorithm "invert and multiply": in fact, one would think about it in terms of doubling the number of apples it takes to fill half a crate.

[49] It is worth noting that the procedure "invert and multiply" is not unique to dividing with fractions. For instance, $6 \div 2$ yields the same result as $6 \times \frac{1}{2}$ (where one has inverted the 2 and mutiplied the result by the dividend). Yet rarely, if ever, is this made explicit to pupils.

earlier examples, the answer could always be multiplied by the divisor and equal the original total. Division by zero is undefined: It has no solution that fits sensibly within the meaning of division and its relationship to multiplication.

The last context in which I looked at teacher candidates' knowledge of division was in algebraic equations. The equation

$$\frac{x}{0.2} = 5$$

gives information that permits one to identify the correct value for an unknown number, denoted as x. In common language, the equation says that when one divides this unknown number by .2, one gets 5 as the answer. Knowledge of division makes clear what this means: that there are 5 groups of .2 in the number, or, more colloquially, .2 "goes into" the number 5 times. Reasoning conceptually about division in this way allows one to identify the number without performing any manipulations on the equation. The answer is 1. 1 can be divided into 5 groups of .2; .2 "goes into" 1 5 times.

Knowledge of Division in Division By Fractions

To learn about teacher candidates' knowledge of division, I asked them to develop a representation of the division statement $1\frac{3}{4} \div \frac{1}{2}$:

==

$$1\frac{3}{4} \div \frac{1}{2}$$

Something that many mathematics teachers try to do is to relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content, or examples or models that make clear what something means. Sometimes this is pretty challenging.

Could you think of a good situation or story or model for $1\frac{3}{4} \div \frac{1}{2}$? (i.e., something real for which $1\frac{3}{4} \div \frac{1}{2}$ is the appropriate mathematical formulation?)

How does that fit with $1\frac{3}{4} \div \frac{1}{2}$

Would this be a good way to help students learn about division by fractions?

==

The traditional algorithm for dividing fractions that most students learn in school is "invert and multiply" — that is invert the divisor and multiply it by the dividend. In addition, any mixed numbers must be converted to improper fractions. $1\frac{3}{4} \div \frac{1}{2}$ becomes $\frac{7}{4} \times \frac{2}{1}$. Multiplying the numerators and denominators produces $\frac{14}{4}$, which should be expressed as $3\frac{1}{2}$.

What does the answer $3\frac{1}{2}$ mean? Herein lies the essence of the algorithm's conceptual background. As in any division, $1\frac{3}{4} \div \frac{1}{2}$ has to do with groups — in this case, groups of a certain size: $\frac{1}{2}$. How many groups of that size can be formed out of $1\frac{3}{4}$? Asking prospective teachers to create a story problem, model, or other representation for which $1\frac{3}{4} \div \frac{1}{2}$ is the mathematical formulation gave me a glimpse into how they thought about its meaning.

An appropriate representation should show that the question is "how many $\frac{1}{2}$s are there in $1\frac{3}{4}$?" For example:

> A recipe calls for $\frac{1}{2}$ cup butter. How many batches can one make if one has $1\frac{3}{4}$ cups butter? Answer: $3\frac{1}{2}$ batches.

Why? Because there are $3\frac{1}{2}$ $\frac{1}{2}$-cup portions of butter in $1\frac{3}{4}$ cups of butter. This story makes clear the referent for the answer $3\frac{1}{2}$ — it refers to $3\frac{1}{2}$ <u>halves</u>.

<u>Correctness and explicitness</u>. The prospective teachers, the elementary candidates as well as the secondary students who were majoring in mathematics, had significant difficulty unpacking the meaning of division with fractions. Few elementary or secondary teacher candidates were able to generate a mathematically appropriate representation of the division. These results fit with evidence from other parts of the interview and that suggests that their substantive understanding of mathematics is both rule-bound and compartmentalized. I categorized the teacher candidates' responses as follows: appropriate, inappropriate, unable to generate a representation (Table 3.1 shows the distribution of responses by elementary and secondary teacher candidates). Below I discuss each of these categories of responses in turn.

<u>Appropriate representations</u>. Five secondary teacher candidates were able to generate a completely appropriate representation of $1\frac{3}{4} \div \frac{1}{2}$; however, this did not seem to come easily to any of them.

Carol said she didn't have any problems "doing it" (i.e., inverting and multiplying to get the answer) but that she wasn't good at creating story problems. She thought for a moment and then said she might use "representations on paper, on the blackboard, things like that":

> I guess I would, I could use a number line. You know, let's mark off where it is; here's one-and-three-quarters and there's a half, now how many times does a half count there. Then you get this far and then you have to talk about that. How much of a half is that?

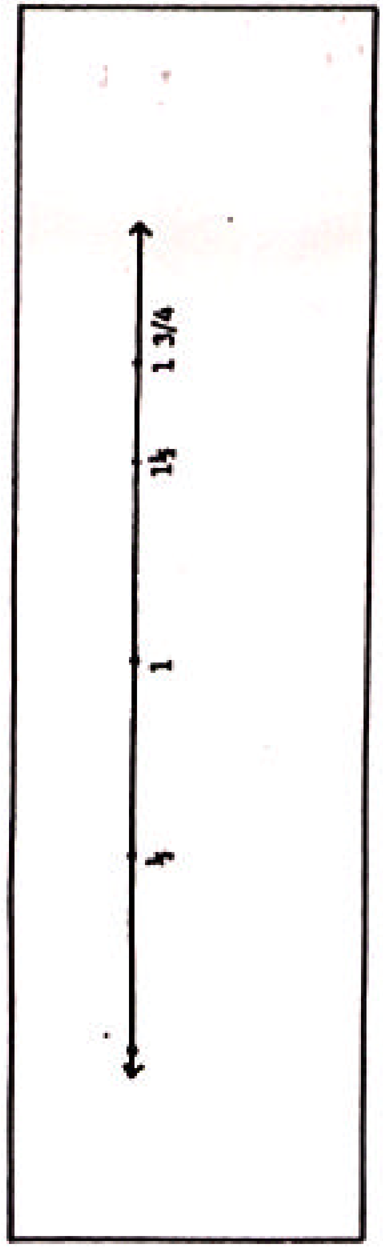


Figure 3.1
Carol's number line

Carol explained that this would be good because it showed that the question was: "How many halves are in one and three-quarters?"

The other four prospective secondary teachers, although they understood the meaning of division with fractions, showed more signs of strain in generating representations for it. Their problems had a forced quality. For example, Terrell first said he couldn't "think of anything specific." Then he said he would use pizza:

> If you took the pizza and took $\frac{1}{2}$ of a pizza and you took a whole pizza and $\frac{3}{4}$ of a pizza (that would be $1\frac{3}{4}$). You put the $\frac{1}{2}$ of the pizza on top of each piece. So first you'd take the whole pizza and you'd put it on top of it. Then you'd take that off, whatever it fits on and you'd do it again. Only take it off if it fits the whole thing. If . . . . both pieces are equal. Then you go through the $\frac{1}{2}$ a piece and do the same for that. Take that off. Then you get that last piece and you. . . well, that's the way I'd explain it.

Terrell then explained what the answer ($3\frac{1}{2}$) meant in this context:

> You'd take the $\frac{1}{2}$ and the answer would be how many times you got a whole half (if you want to say that). Of the. . . whatever's left over, what part of it is <u>of</u> the half, I guess you could say. You'd have a $\frac{1}{4}$ left, which is $\frac{1}{2}$ of a half.

While it is rather confusing to follow, this does make mathematical sense, although laying pieces of pizza on top of whole pizzas forces the imagination a bit and suggests that coming up with a realistic story did not come easily for Terrell.

All four teacher candidates' examples appropriately represented division with fractions. While a bit awkward, their answers indicated that they understood the problem in terms of division, that they did think about it in terms of how many halves there are in one and three-fourths. They were in the minority, however.

Table 3.1
Division by fractions: $1\frac{3}{4} \div \frac{1}{2}$

Category of response	Teacher Candidates: Elementary	Secondary	TOTALS
Appropriate representation	0	5	5
Inappropriate representation	3	2	5
Unable to generate a representation	6	2	8

<u>Inappropriate representations</u>. Five teacher candidates generated representations that did not correspond to the problem. The most frequent error was to represent division by 2 instead of division by $\frac{1}{2}$. For example, Barb, a mathematics major, gave the following story:

> If we had one and three-quarters pizzas left and there were two of us dying to split it, then how would we be able to split that?

She proceeded to give a careful conceptual explanation of how to split the pizza equally:

> If there's three-quarters over here and first you want to tackle that part before we break into the nice pizza that's still there. . . and then if each of us eat half, ate half of a three-quarters for each of us, then each of us ate half of the whole, then we'll eat all the pizza.

She said each person would get $\frac{7}{8}$ of a pizza altogether.

Anne, an elementary candidate with a much stronger mathematics background than many of the other elementary majors, told a story that was similar to Barb's but with a different result:

> Let's do the basic. You're having pizza, you have a pizza and, one and three-quarters pizza, okay. And you have two people that are really hungry and they both want to split the pizza that you have. You have to divide it between the two people. Um, you could go as basic as to draw, you know, draw the pizza (she draws two circles).
>
> And you want to divide that much pizza between two people which, you know, so you'd each have half of the pizza. I think everybody, kind of, assumes what half is. You could, um, you would convert one-and-three-quarters to a fractional form. . . .It should probably be in fourths because the denominator is in four. And then you'd have the seven-fourths.

Anne marked off three-quarters of the second circle without drawing in fourths. "The Pac-man pizza," she chuckled.

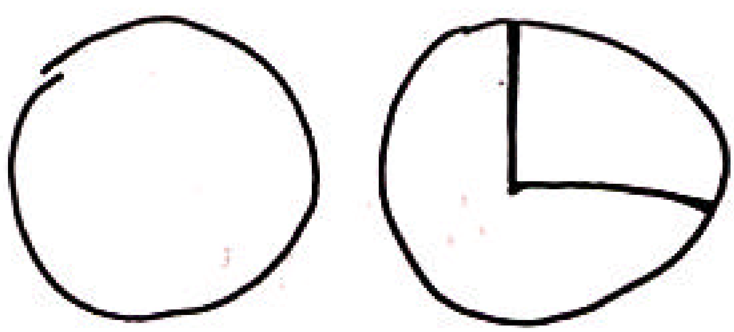


Figure 3.2
Anne's solution to $1\frac{3}{4} \div \frac{1}{2}$, first step

Then Anne divided the whole pizza into fourths. Turning to the three-quarters pizza, she divided the region into four parts.

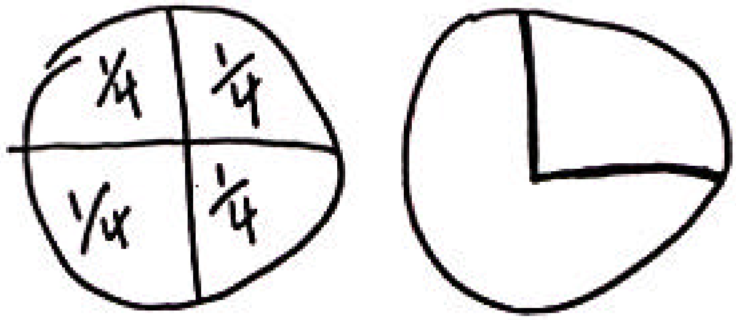


Figure 3.3
Anne's solution to $1\frac{3}{4} \div \frac{1}{2}$, second step

"So, and then this is one-fourth of this," she explained. Then she divided the remaining fourth of the second pizza into two parts and counted the total number of pieces: six.

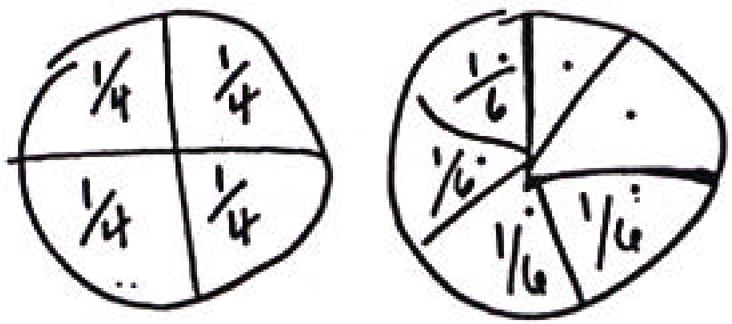


Figure 3.4
Anne's solution to $1\frac{3}{4} \div \frac{1}{2}$, third step

"These are sixths — is that right? And then you add." Anne commented that this was "a long drawn-out way to this" but that it shows "what we're really getting at" when performing the procedure of division.

Anne's efforts to provide a "basic" illustration of division with fractions led her into difficulty. Not only was she involved with division by 2, instead of division by $\frac{1}{2}$, but she "believed" her drawing in a way that misled her. When she "saw" six pieces in the second pizza, she decided that she was dealing with sixths instead of fourths in that pizza. Anne did not check the result of her drawing (and its representation of the solution as $\frac{2}{4} + \frac{2}{6}$) to see if it made sense — even in terms of division by 2.

Sandi, another elementary major, who also represented $1\frac{3}{4} \div \frac{1}{2}$ as $1\frac{3}{4} \div 2$ was the <u>only</u> person who noticed a discrepancy between the answer to her story ($\frac{7}{8}$) and the answer produced by the standard calculation ($\frac{14}{4}$). Her comments on this helped me to interpret how she understood division with fractions. She looked at the $\frac{14}{4}$ again and said, "obviously half of this [$1\frac{3}{4}$] is not <u>this</u> [$\frac{14}{4}$]." She began to doubt her calculation because she was troubled by the fact that $\frac{14}{4}$ was <u>greater than</u> $1\frac{3}{4}$. Her doubts about the calculated answer were evidence that Sandi thought that $1\frac{3}{4} \div \frac{1}{2}$ meant $1\frac{3}{4}$ divided <u>in</u> half, instead of <u>by</u> one-half. $\frac{14}{4}$ is "obviously not" "half of" $1\frac{3}{4}$. But, pondering the discrepancy, Sandi also revealed another factor in her thinking about this problem: "Obviously I don't like fractions. It's not something I'm comfortable with," she explained. For Sandi, the fact that this did not make sense to her was evidence of her own inadequacy. Apparently she did not feel that she would be able to resolve the problem by thinking hard about it (see Chapter 4).

Allen, an elementary major with 27 credits in college mathematics (through calculus) had a different problem than the other teacher candidates:

> Somebody has one and three quarters <u>apples</u> or something like that and they wanted to double up their, double the amount of apples they have, um, just give 'em an equation . . . using only fractions. Other than that I couldn't think of any, any situation where you could, where you would <u>logically</u> divide a number like that by one half instead of just multiplying by two. You would have to have, be, be working on, on dividing like fractions and setting up equations using that. Right, you just have to say.... "Well, you know, use this and figure the story problem out but, only use fractions."

Allen's story modeled $1\frac{3}{4}$ x 2 — the <u>procedure</u> used to divide fractions. Using the frame of reference of the procedure "invert and multiply," Allen did not seem to focus on the <u>concept</u> of division by $\frac{1}{2}$.

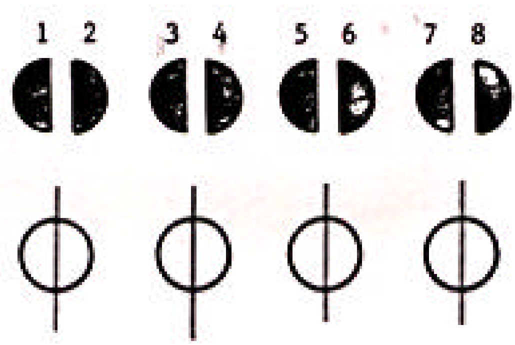
Overall, Barb's was the most common error — to represent division by 2 rather than division by $\frac{1}{2}$. Most of the prospective teachers used round food — pizzas or pies — and described sharing the food between two people. The reasons for this are understandable but nonetheless troubling. Two are particularly striking: I refer to them as <u>preoccupations and confusions with fractions</u> and <u>confounding everyday and mathematical division language</u>.

1. *Preoccupations and confusions with fractions.* The teacher candidates' comments showed that they saw the question as one about <u>fractions</u> instead of about <u>division</u>. When asked, for example, what made this difficult, most commented that it was hard (or impossible) to relate $1\frac{3}{4} \div \frac{1}{2}$ to real life because, as one said, "you don't think in fractions, you think more in whole numbers." Not only did their explanations reveal that they framed the problem in terms of fractions, but also that many were uncomfortable with fractions as real quantities. Several commented that they didn't "like" fractions.

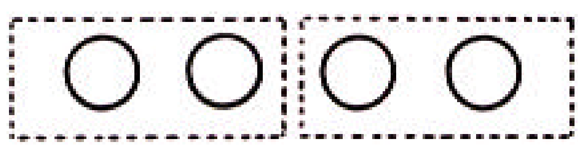
Round models account for most pictorial representations of fractions in school textbooks. Focusing on fractions therefore inclined the prospective teachers toward circular representations. Food was common because they assumed that it would be intrinsically more interesting to students than other divisible circular objects (see Chapters 5 and 6). In responding to my questions, the prospective teachers tripped over their interpretation of the problem as essentially "about fractions" and their limited conception of and repertoire for representing fractions.

2. *Confounding everyday and mathematical language.* The prospective teachers tended to confuse dividing in half with dividing by one-half, and they did not seem to be aware of the difference. This confounding went unnoticed even though the answer they got to their story problem ($\frac{7}{8}$) differed from their calculated answer ($3\frac{1}{2}$).

The teacher candidates did not notice this discrepancy because it was masked by an slippery change in the referent unit from wholes to fourths. Here is a typical example: Suppose you have $1\frac{3}{4}$ pizzas which you want to split equally between two hungry teenagers ($1\frac{3}{4} \div 2$). Each pizza is divided into 4 pieces, so you have 7 pieces. Therefore each person gets $\frac{7}{8}$ of a pizza, which is $3\frac{1}{2}$ pieces of pizza. However, to divide something in half means to divide it into two equal parts ($\div$ 2); to divide something by one-half means to form groups of $\frac{1}{2}$.



$4 \div \frac{1}{2} = 8$



$4 \div 2 = 8$

Figure 3.5
An illustration of the difference between division by one-half and division by two

The teacher candidates' error may have resulted from a common but problematic confounding of everyday with mathematical language. Orr (1987) writes about the mismatches between linguistic and mathematical use of prepositions. Awareness of such confusions is at the heart of what teachers must know if they are to help their pupils understand mathematics.

Stumped. Eight teacher candidates could not generate a representation at all for $1\frac{3}{4} \div \frac{1}{2}$. Teri, an elementary major, couldn't remember how to find an answer to this kind of problem:

> First, I'd, first I'd multiply the denominator of this one number by one so I'd get four times one. And then add three, so it would be seven-fourths divided by one-half, and then I'd probably make the, cause halves are easy to do, but I probably still make this into fourths. So it would be easier. So they'd be like, common denominators.

Then Teri pondered what to do next. "Cross-multiply?" she wondered aloud. "I can't — I don't remember." She said she couldn't think of any way to make a story problem or other representation for the problem.

Others tried to use the numbers ($1\frac{3}{4}$ and $\frac{1}{2}$) but didn't represent division. For example, Marsha tried the following:

> Um, about a classroom on a field trip and one-and-three-quarters of the class went one way and one-half of the class went the other way, and we're solved, oh, wait a minute, that wouldn't work, would it? One-and-three-quarters divided by (pause), um I think I would still use it as a group of people (pause), divided by half of it (pause), half of a wall or something that they walked by and then how much of the class was left on the one side of the wall?

Marsha's story seemed to be a confused attempt to represent only the <u>numbers</u> — $1\frac{3}{4}$ and $\frac{1}{2}$ — independent of the division involved. I asked, "Half of a wall? Is that what you said, or half of a <u>class</u>?" She tried again:

> Half of the wall, like, that wouldn't work either. One-and-three-quarters (Laughs), I guess I can't do it that way. I don't know that would be really hard to do because you couldn't, you can't give like a half of something will divide the class, or you know, something like that.

She said she felt stuck and couldn't remember how to get the answer either (i.e., through computation). She explained that she hadn't done this since high school:

> God, I don't even remember how to do this. (pause) Shoot, no, I have to, I don't know how to do it. Because like I don't know what I'm remembering here that I did, I found the common denominator and I did this, but I think what I have to do is go 4 and 1, 4 and then plus 3 is 7, fourths, no I think that's what I did, one-half, but then, see, I don't know what need to divide. I don't even remember that. . .I remember doing these for a long time though and trying to get these down, and so I remember bits and pieces and then I try to apply it generally, and I can't do it.

Marsha's story revealed that she had fundamental confusions and discomfort with fractions. Simply trying to represent the <u>numbers</u> (i.e., the fractions) themselves in some real context was difficult for many of the prospective teachers like Marsha. One teacher candidate commented that it was hard to

> [find] a model or a story to teach division by fractions. To me fractions are hard to begin with and then the division part of it is confusing to the mind I think, to <u>me</u> it is, it can be confusing.

These prospective teachers who did not generate a representation at all seemed to fall into two groups. Some did recognize the conceptual problem. They initially proposed stories or models which represented division by 2 and then realized this themselves. For example, Janet, a secondary candidate, began with a story about $1\frac{3}{4} \div 2$, and then tried again. She said $1\frac{3}{4} \div \frac{1}{2}$ would be "definitely different" and she could not think of anything because

> I guess when I think of division problems, I think of dividing it between people and it's hard to have $\frac{1}{2}$ a person! (laughs) or between bins or between discrete things, as opposed to between a fraction of a thing.

Janet's response suggests that her understanding of division was limited. Because she had only one way to think about the meaning of division (the partitive interpretation), she was constrained in trying to generate an appropriate representation for $1\frac{3}{4} \div \frac{1}{2}$. Had she been able to think of division as about dividing a quantity into groups of a certain size (measurement interpretation), she might have been able to do the task.

Others seemed to think that it was not a feasible task — that $1\frac{3}{4} \div \frac{1}{2}$ could not <u>be</u> represented in real world terms. Pam, for example, could not generate a representation for division of fractions because of the way she thought about fractions. She said it would be hard to think of anything "real," because her "favorite thing" was animals and she could not think in terms of $\frac{3}{4}$ of an animal. Pam did not seem to understand fractions as representations of quantity. Unable to think of anything "real" that could be represented by a fraction, she never focused on how would <u>divide</u> such a quantity.

These teacher candidates who were "stumped" included people with qualitatively different understandings of the mathematics. On one hand, those who saw the conceptual issue (that this was about division by $\frac{1}{2}$ which is not the same as division by 2) revealed a better grasp of the idea than those who constructed a story that represented division by 2. Still, despite this recognition, they were unable to figure out what division by $\frac{1}{2}$ <u>meant</u>. On the other hand, those who thought it was an impossible task revealed a view of mathematics as a senseless activity, out of which meaning cannot necessarily be made.

<u>**The teacher candidates' understanding of division in division by fractions**</u>. Although few of the prospective teachers even mentioned division explicitly while talking about the fractions exercise, the difficulties all of them experienced (including those who succeeded in generating an appropriate representation) suggest a narrow understanding of division. While they worried about the fractions in the problem, they also only considered division in partitive terms: forming a certain number of equal parts. This model of division corresponds less easily

to division with fractions than does the measurement interpretation of division.

In a study of preservice elementary teachers' understanding of division, Graeber, Tirosh and Glover (1986) found that teacher candidates tended to think only in terms of this partitive interpretation. Few of the preservice teachers in their study were able to write story problems that modeled a measurement interpretation of division. This finding offers another insight into why the task of making meaning out of $1\frac{3}{4} \div \frac{1}{2}$ was so difficult for the prospective teachers in this study.

Knowledge of Division in Division by Zero

I also explored the teacher candidates' understanding of division with a question about division by 0:

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Suppose that a student asks you what 7 divided by 0 is. How would you respond?[50]

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While some might argue that this question deals with an esoteric little bit of mathematics, I contend that it deals with four important ideas in mathematics: division, the concept of infinity, what it means for something to be "undefined," and the number 0, all significant content. Prospective teachers' responses depended on their understanding of the specific content at hand as well as of mathematical knowledge and mathematical ways of knowing. When they are explaining, their responses reveal the explicitness of their knowledge. What they focus on also provides information about what they think counts as an "explanation" in mathematics.

Correctness. Of the 19 teacher candidates, 5 explained the meaning of division by zero. Most of the prospective teachers responded by stating a rule, 5 of which were incorrect. Two did not know. Table 3.2 shows the distribution of types of responses among elementary and secondary teacher candidates. I discuss each category below.

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[50] This question also contributed to my understanding of prospective teachers' ideas about mathematical knowledge. Responding to a pupil who asks what 7 divided by 0 is depends of course on a host of considerations — the age of the pupil, whether the idea has been taught before, and one's view of the teacher's role in answering student questions. I did not specify the context (e.g., the age of the pupil) because I was, for other purposes, also interested in what teacher candidates thought they might need to know in order to answer this question.

Table 3.2
Division by zero: 7 ÷ 0

Category of response		Teacher Candidates: Elementary	Secondary	TOTALS
Meaning		1	4	5
Rule	correct	2	5	12 (7 correct 5 Incorrect)
Rule	Incorrect	5	0	
Don't Know		2	0	2

Explanation: Focused on meaning. Four teacher candidates gave answers that focused on what division by 0 means. Two approaches were used: (1) showing that division by 0 was undefined and (2) showing that the quotient "explodes" as the divisor decreases.

Tim, a mathematics major, chose the first approach. He said that he would write 7 ÷ 0 "in mathematical form" on the board — i.e., with division bracket: $0\overline{)7}$. Then he would explain that

> you cannot divide 7 by 0 because there is nothing multiplied by 0 to get 7. In other words, everything multiplied by a 0 is 0, so if we had 7 over 0. Okay, if we had 0 divided by 7 there is nothing multiplied, there is no number up here you could put to get 0. There's no number you can put up here to get 7. And I would show them that. Whereas 6 divided by 2 there's a number you can put up there. And whenever you come across that case, you can't find a number to put up there, it doesn't exist, you can't do it.

Allen, an elementary major, explained division by 0 using the second approach:

> Dividing 7 by 3 and then divide 7 by 2 and then divide 7 by 1 and, uh, and when they get up to 7 divided by 1 is <u>7</u> and you were to go one step farther you'd, you'd have numbers that were keep getting larger and larger. . . .Um, one step farther you would have to, you know, say divide it by 0, because dividing by decimals or fractions of that type. . . . Then you'd start, I guess it would be better to start getting closer to zero using the <u>decimals</u> and see that dividing 7 by fractions makes numbers, you know, they keep getting larger and larger and, uh, that if you keep making that, the divisor closer and closer to 0, the number's just gonna keep getting larger and larger and <u>larger</u> and, uh, then I'd start asking them what the largest number they can think of is so then, that there is no largest number that, uh, that it, there is really no such, such statement as 7 divided by 0.

Mark and Allen both focused specifically on the case of dividing by zero. What their answers had in common was the aim of <u>showing why</u> the particular case of division by zero is impossible. Their explanations were mathematical ones, not the "explanations" used by many of the teacher candidates, which consisted largely of restating rules.

Jon, a secondary candidate, was the only person who focused on division more generally. His answer revealed an unusual depth and breadth of understanding. He explained that there were two alternative answers to this question: (1) that there is "no answer" to 7 ÷ 0, and (2) infinity. He decided to use the infinity answer because "infinity is a little bit easier, maybe, to explain. . .than no answer at all." He used a number line:

> Here's [a] number line [from] 0 up to 7. Now, when I divide something I try to find out how many groups of that, the thing I'm dividing with, are, are within the number. Say, if I'm dividing 4 by 2, I say, "How many 2's are in 4?" So I want to find out how many zeros there are in 7. Well, how much space does a zero take up on this number line and what. . . , and then, maybe, well, it doesn't take any space. So, how many of them can I fit in there? Well, as many as you want.

Jon's understanding of division by zero seemed to be connected to his more general understanding of division. His understanding of division was, in turn, connected to his understanding of fractions. He was the only person whom I interviewed who talked about dividing by zero within such a concept of division in general; his explanation of division with fractions was consistent with this as well.

<u>**"You can't divide by zero."**</u> Seven teacher candidates explained division by zero in terms of a rule such as "you can't divide by zero." Unlike those who focused on meaning, these prospective teachers did not try to show <u>why</u> this was so. Instead they emphasized the importance of <u>remembering</u> the rule. Terrell, a mathematics major, said emphatically,

> I'd just say. . . "It's undefined," and I'd tell them that this is a rule that you should <u>never</u> forget that anytime you divide by 0 you can't. You just can't. It's undefined, so. . . you just can't.

He added, "Anytime you get a number divided by 0, then you did something wrong before." Andy, another mathematics major, said, "You can't divide by 0. . .It's just something to remember." Cindy, also a math major, said she would tell students that "this is something that you won't ever be able to do in mathematics" — even in calculus.

<u>**"Anything divided by 0 is 0."**</u> Five other teacher candidates responded in term of a rule. Like the prospective teachers quoted above, their notions of "explanation" in mathematics seemed to mean restating rules. What made their responses different, however, was that the rule they invoked was not true. Linda, an elementary major, was perhaps the most emphatic:

> I'd just say, "Anything divided by 0 is 0. That's just a rule, you just <u>know</u> it." Or I'd say, "Well, if you don't have anything, you can't get anything out. You know, it's empty, it's nothing, so you can't get anything out of it." Anything multiplied by 0 is 0. I'd just say, "That's something that you have to learn, you have to know." I think that's how I was told. You just know it. . . I'd just say, you know if they were older and they asked me "Why?" I'd just have to start mumbling about something, I don't know. . . I don't know what. I'd just tell them "Because!" (laughs) That's just the way it is. . . that's just one of those rules. . . something like that. . . you know, in English. . . sometimes the C sounds like K or. . . you know,

> you just learn. I before E except after C, one of those things, in my view.

Interestingly, although Linda mentions multiplication by 0, she doesn't connect that understanding (n x 0 = 0) with the problem of dividing by 0.

Pam, another elementary teacher candidate, emphasized that "0 is always going to end up 0":

> That's almost just a rule you learn. And I suppose you'd have to go back to your basic rules that when you use a zero, it ends up zero because it can't go any further. And you could divide or multiply zero by any kind of number, but you will never get anything bigger than that zero. I'm not sure about how I would approach it, but there's nothing to multiply, there is not another number to multiply it with, so you can't get anymore. . . I guess you'd have to go back to your basic rules, it's just one of those basic rules that just gets pounded in your head when you're young and you just kind of remember it, or hope you do.

Cathy, also an elementary major, echoed Linda and Pam with "anything divided by 0 is 0 no matter what number it is." She said she would give different examples: "What's 4,000 divided by 0?" What's 0 divided by 0?, so that it gets like embedded into their head."

Like those who stated "you can't divide by 0," these prospective teachers all emphasized the absoluteness of the rule and the value of getting pupils to remember it. Explaining and knowing in mathematics were all rule-focused. However, these teacher candidates did not realize that what they were saying was not true.

"I don't remember." Two prospective elementary teachers said they could not remember the answer to 7 ÷ 0. Mei Ling said simply, "7 divided by 0? Isn't that — isn't there a term for the answer to that? I can't remember."

Rachel, who had taken a little more math, more recently and more successfully, than most of the other elementary majors, was simply stumped by this question. "Seven divided by 0," she mused. "I'm having trouble. . . is that 0 or is that 7? I'm trying to think myself." Rachel considered that the answer might be "you can't divide 7 by 0," but then she said,

> If you have 7 and you divide it by 0, and you don't divide at all, why isn't it just 7?" If you don't divide it at all, that's what I'm thinking and maybe . . . if that's right then it is right. I don't think it is. . . . something tells me it is not but I am not sure. . . otherwise if you have 7 and you divide it by 0 they are saying that you cannot divide it so you just end up with 0. . . so I don't know.

Although neither Mei Ling nor Rachel knew the answer to 7 ÷ 0, Rachel's struggle revealed a different approach to knowing than Mei Ling's. Mei Ling could not remember — for her, this meant she did not and could not know the answer. Rachel could not remember either, but she did not think this meant that she couldn't know the answer. She used two approaches to try to answer the question. She tried to retrieve bits of information that might help her remember (e.g., something with division and zero produces infinity, something in graphing that wasn't a line). She also tried to reason about the meaning of the problem (e.g., "If you have 7 and you divide it by 0, and you don't divide at all, why isn't it just 7?")

Teacher candidates' knowledge about division by zero. Division by zero comes up frequently in college mathematics; math majors have had more and more recent experience with dividing by zero than have non-math majors. As such, it was not surprising that the secondary candidates were better prepared than were the elementary to deal with this question, both in terms of providing mathematical explanations and in terms of knowing the correct rule.

Still, most of the teacher candidates, whether right or wrong, whether focused on meaning or on rules, did not seem to refer to the more general concept of division to provide their explanations. Instead they recognized division by zero as a particular case for which there was a rule. Their "explanations" were simply statements of what they though to be the rule for this specific case. Furthermore, half the elementary candidates had the rule <u>wrong</u>. Because they did not think about the <u>meaning</u> of division by zero, they did not monitor the reasonableness of their answers.

<u>Knowledge of Division in Algebraic Equations</u>

A third interview question provided yet one more angle on the teacher candidates' understanding of division. This time I asked:

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Suppose that one of your students asks you for help with the following exercise:

If $\frac{x}{0.2}=5$***, then x=***

How would you respond?

Why is that what you'd do?[51]

==

In algebra classes, students are taught procedures for "isolating x" — i.e., for manipulating equations so that the unknown number is on one side of the statement and a number is on the other. This enables one to "solve" the equation, or figure out what number(s) x could be. For example, the ubiquitous procedural script for solving the equation discussed above is:

You want to isolate x, so you want to get rid of the point 2 in the denominator.

Multiply both sides by point 2.

$$(.2)\frac{x}{0.2}=5(.2)$$

[51] This question was presented to elementary as well as secondary teacher candidates. Its function in the interview was to extend the analysis of their understanding of division in different contexts. In other words, was division with fractions one case, division by zero another, and division in algebra something yet entirely different again? Lest critics argue that this content is too advanced for the elementary teacher candidates, I contend that it is not unreasonable to expect that teachers whose Michigan teaching certificate will extend through eighth grade in all subjects should understand division in simple algebraic equations.

The point 2's cancel on the left side; 5 times point 2 is 1.

So x equals 1.

Learning procedures such as these often seems to eclipse any focus on the meaning of the equations or the numbers. Furthermore, referring to .2 as "point two" does not emphasize the meaning of the number as <u>two-tenths</u>.

Scripts similar to this one were what the teacher candidates produced in response to this question. Overwhelmingly the teacher candidates "explained" it by restating the steps of procedures to solve such equations. Only one prospective teacher talked about it in terms of what it meant, and a few teacher candidates didn't know how to do it at all. The results of the teacher candidates' responses are summarized in Table 3.3.

<u>Focus on meaning</u>. Only one teacher candidate — an elementary major — tried to talk about the meaning of the equation. Sandi said that she would want the pupil "to understand what he's doing first." She said she would help the pupil understand "the idea that the .2 has to go into x." While her explanation was vague, she was trying to make sense of the problem by reasoning about division.

<u>Focus on procedures</u>. Fourteen of the prospective teachers, including all of the mathematics majors, focused on the mechanics of manipulating algebraic equations. Terrell, a secondary candidate, said

> I'd explain that somehow you have to get this x by itself without that .02, I mean 0.2. . . and then I'd ask her, I'd ask her. . . I'd tell her somehow she's going to have to get rid of that .02.

Then he laughed self-consciously — "the complex math terms that teachers use, like `get rid of.'" Mei Ling, an elementary candidate, gave more detail:

> I would show the relationship between the x and .2 and I would say "x is being divided. . . x is being divided by .2 so this equation here is equal to 5. They are saying you want to isolate x, you want to leave x all by itself on this side, okay? So what can we do to cancel this out? What can we do to both sides that will leave the x by itself, that will get rid of that .2? Okay?" Then I would say, "You want to do the <u>opposite</u> of what's happening here. So x is being divided by .2 so the opposite of that is — ?" and they might say, "Multiply" and then I would say, "That's <u>right</u>!" So you want to take this . . . x divided by .2. . . multiply it by .2 and those cancel out. But you have to do one side just like the other. They have to. . . you have to treat them the same because it's all the same problem. Then that will leave x by itself. Solve this one and you've got your answer.

The other teacher candidates gave similar answers. They all talked about getting "rid of" the .2, isolating x, and multiplying both sides by .2. They seemed to see the question as quite straightforward and unproblematic, unlike some of the other questions I had asked, probably because solving simple equations was something they had done themselves many times and they could, for the most part, remember how to do it.

"I have no idea!" Four elementary teacher candidates did not know how to solve the equation themselves. One was overwhelmed at the prospect of having to help a student solve an equation such as this one. "Oh, my God!" she exclaimed when I presented her with the question. She said she had no idea, although she knew "there's steps that you go through to do it." Another said she hadn't "done these" in so long that she just couldn't remember.

All four of the teacher candidates who could not solve the equation attributed it to not having done algebra problems in a long time and not being able to remember the procedures for solving equations such as this one. The only difference between these teacher candidates and the fourteen who focused on procedures was that these four could not remember the procedures. However, like the thirteen, they did not focus on the meaning of the mathematical statement.

Table 3.3
Division in algebraic equations: $\frac{x}{0.2} = 5$

Teacher Candidates

Category of response	Elementary	Secondary	TOTALS
Meaning	1	0	1
Procedure	5	9	14
Don't Know	4	0	4

Summary: Knowledge of Division.

Although the three interview questions all dealt with division, the teacher candidates did not seem to focus from case to case on the concept of division. Instead, most of them responded to each question in terms of the specific bit of mathematical knowledge entailed — division of fractions, division by zero, solving algebraic equations involving division. For all three questions, the responses were overwhelmingly focused on procedures. The prospective teachers, both the mathematics majors and the elementary candidates, tended to search for the particular rules — "you can't divide by 0" or "get rid of the denominator" — rather than focusing on the meaning of the problems presented. Their understanding seemed specific and unconnected. Below I consider two possible influences on this finding.

Confounding of remembering and understanding. Why were the teacher candidates' responses so overwhelmingly procedural and rule-oriented? Was the preponderance of procedural answers influenced by the nature of the questions themselves? Two of the questions — division by zero and division in algebraic equations — were formulated in such a way that teacher candidates could simply retrieve the correct piece of information (e.g., "division by zero is undefined"), as it was taught, from mathematics memory storage. These two questions examined "conventionally packaged" pieces of knowledge — knowledge that the teacher candidates had been taught in school. If they could remember the necessary piece, they could answer each question by stating the rule. In fact, many of them equated remembering with knowing (see Chapter 4).

One might argue that nothing in either question compelled them to reorganize or to reexamine their understanding; however, both questions did ask the teacher candidates how they would respond to a pupil who raised that question. The dominance of procedural answers would suggest that the prospective teachers favored giving pupils rules to accept and remember, rather than conceptual explanations. However, there is substantial evidence in their responses that the teacher candidates wanted to give the pupils more conceptual answers but could not do so, that their subject matter knowledge, lacking explicitness, was insufficient to act on that commitment. One of the math majors realized this and commented (on division by zero), "I just know that . . . I don't really know why . . . it's almost become a fact . . . something that it's just there."

In answering the questions, many of them agonized over not having a "concrete example" or not knowing why something was true. One of the math majors, for example, in answering the division by zero question, said she "would hate to say it is one of those things that you have to accept in math" but that she might have to in this case if she couldn't think of a concrete example. Another laughed wryly at himself for using the phrase "get rid of the denominator," but did not have accessible any alternative ways of understanding. The answers the teacher candidates gave — rules — were what they understood, what they remembered from what their teachers said.

Moreover, some of the teacher candidates could not remember the rules at all. Once forgotten, rules are not easily retrievable without the concepts to support them (Hiebert & Lefevre, 1986). Mere remembering only serves one well in displaying mathematical knowledge — until one forgets, that is. The prospective teachers' knowledge seemed founded more on memorization than on conceptual understanding. The secondary teacher candidates, having had more(and more recent) opportunities to maintain their inventory of remembered knowledge, were therefore more likely to have something to say, less likely to draw a complete blank.

Fragmented understanding. The prospective teachers' focus on the surface differences among the three cases of division suggests that their understanding comprised remembering the rules for specific cases, not of a web of interconnected ideas. Evidence for this is especially clear in the teacher candidates' efforts to generate representations for $1\frac{3}{4} \div \frac{1}{2}$. This task, unlike the other two division questions, did require them to do more than reproduce what they had been taught. Division with fractions is rarely taught conceptually in school; most of the prospective teachers probably learned to divide with fractions without necessarily thinking about what the problems meant. Indeed, most of them could carry out the procedure to produce the correct answer ($3\frac{1}{2}$) — a task that required them to remember and use the rule "invert and multiply."

Yet, when they tried to generate a representation for the statement, most of them either represented $1\frac{3}{4} \div 2$ or couldn't do it at all. Only 4 out of the 19 teacher candidates talked about "how many halves are in $1\frac{3}{4}$" — the others interchanged dividing by $\frac{1}{2}$ with dividing in half. This confusion went unchallenged, even though the answer to the latter ($\frac{7}{8}$) differs from the calculated answer ($3\frac{1}{2}$)..[52]

The results for this question suggest that, in almost all cases, the prospective teachers' understanding of division with fractions consisted of remembering a particular rule, and was unattached to other ideas about division. The results for the other two questions (division by 0 and division in algebraic equations) are consistent with this interpretation.

Critical to note here is that the standard school mathematics curriculum to which most prospective teachers have been subjected treats these ideas as discrete bits of procedural knowledge, a point worth noting for it underscores what prospective teachers bring and what they, in many cases, must overcome in learning to teach even "simple" concepts like division.

Knowledge of Slope and Graphing

Slope is a central mathematical idea, first introduced formally in first-year high school algebra. The slope, or steepness, of a line is represented by a ratio of the vertical change (rise) compared to the horizontal change (run). This measure of steepness is useful for describing the change in a line.

I had three reasons for choosing slope as a topic for exploration with the prospective secondary teachers. First, this is a basic topic in first-year high school algebra, a course prospective teachers are likely to have to teach, or even student teach. Second, slope and graphing are foundational to much mathematics that is studied after algebra — calculus, for example. Third, slope is a central mathematical tool for important ideas like change and covariation; graphing is central for working with solution sets and functions. Finally, I suspected that slope and graphing are often approached mechanically, that prospective teachers may have to reconstruct (or develop) a more conceptual understanding of this topic.

Finding slope. One finds the slope of a line by picking two points on a line and determining the ratio between how much the line is ***rising or falling*** and how much the line is ***increasing to the right*** between those two points. In other words:

$$\frac{\text{rise or fall}}{\text{run}}$$

The sketches in Figure 3.6 illustrate different slopes. Notice that lines that slant down to the right have negative slopes: This is because the line drops rather than rises, so the ratio comprises a negative value over a positive value (the horizontal run of the line).

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[52] This discrepancy was most often masked by an unnoticed change in the referent unit from wholes to fourths. Here is a typical example: Suppose you have $1\frac{3}{4}$ pizzas which you want to split equally between two hungry teenagers ($1\frac{3}{4} \div 2$). Each pizza is divided into 4 pieces, so you have 7 pieces. Therefore each person gets $\frac{7}{8}$ of a pizza, which is $3\frac{1}{2}$ pieces of pizza.

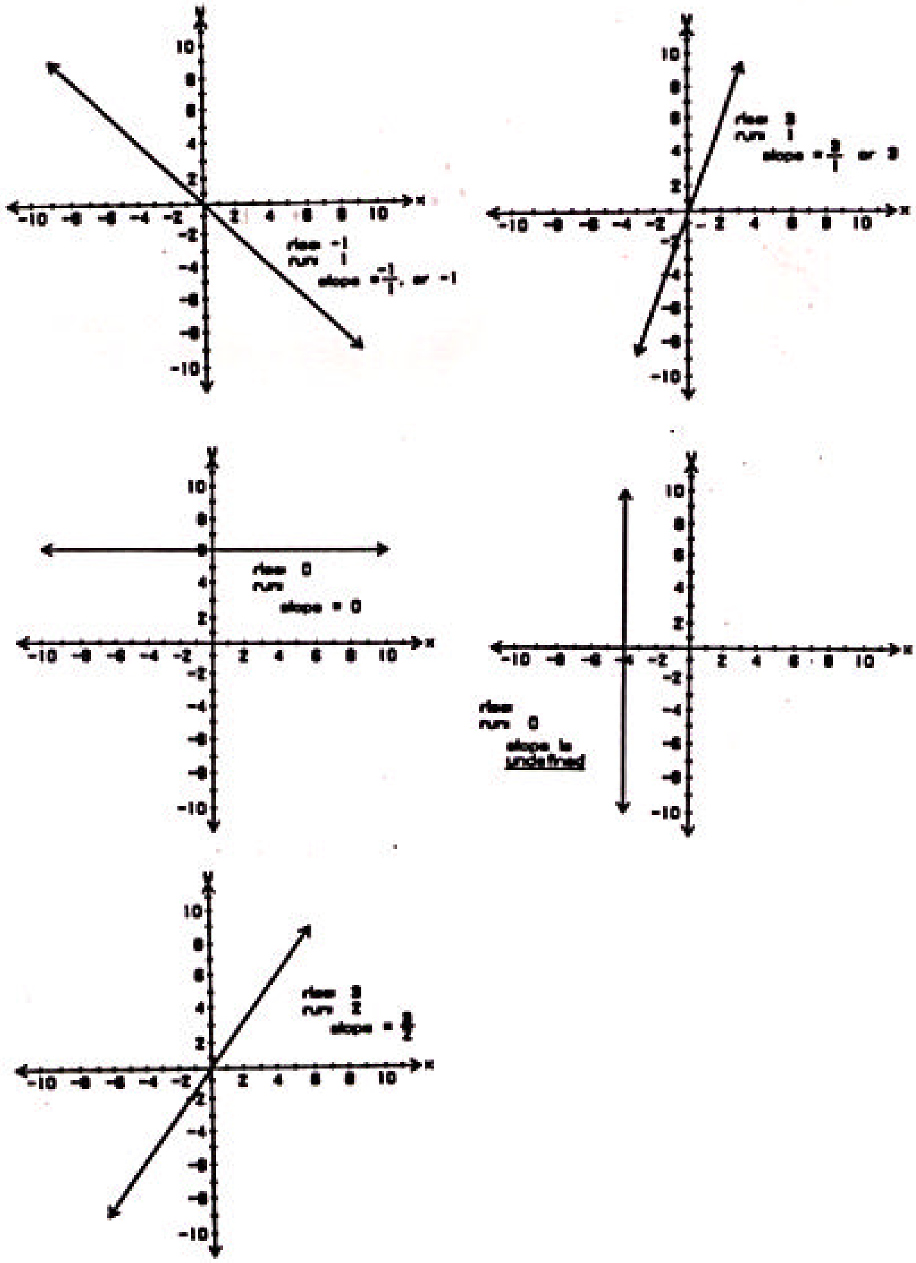

Figure 3.6
Lines with different slopes

<u>Slope, equations, and lines</u>. Assigning a numerical value to the steepness of a line allows one to compare the relative slopes of two lines, as well as to predict accurately other points on a given line. A line can be represented as an equation — this equation includes the slope. For example, the equation for the line in Figure 3.7 is $y = 2x$. In other words, the value of y at each x is double the value for x. If x is 2, y is 4. If x is 3, y is 6. The slope of this line is $\frac{2}{1}$

(or simply 2) because the line is rising 2 for each horizontal increase of 1. Now look at the line in Figure 3.8. This line is parallel to the line in Figure 3.7, but you can see that it is shifted up on the coordinate plane. Instead of intersecting the origin (0,0), this line crosses the y-axis at 3. So the equation becomes $y = 2x + 3$. Consider: When x is 0, y is 3. When x is 1, y is 5. y is always 2 times greater than x, plus 3. This form of writing equations for lines is called the slope-intercept form because it contains both the slope and the point where the line intercepts the y-axis. This equation form is also referred to in its general form as **$y = mx + b$**, where m is the value of the slope and b is the point of intercept. This form is the favorite of many students and teachers because it is most directly "graphable" — that is, because one knows the intercept and the slope, one can quickly graph the line.

This last example also reveals another feature of slope: lines that are parallel have the same slope, i.e., they are slanted in exactly the same way. The graph of $y = 2x$ has the same slope (2) as $y = 2x + 3$, but the second line is translated up 3 units on the coordinate plane. (See Figure 3.9.)

Besides the popular slope-intercept form of a line, there are also other forms of equations. Another, in its general form, is referred to as

$$\mathbf{ax + by = c}$$

So, for example, $-2x + y = 3$ is another way of representing $y = 2x + 3$. Consider some of the solutions to that latter: (0,3), (1, 5), (3, 9). Each of these of course also satisfies $-2x + y = 3$.

This form ($ax + by = c$) is often not favored by algebra teachers because it cannot be graphed as directly as the $y = mx + b$ form. Students are often taught to transform all equations into the $y = mx + b$ form (through algebraic manipulation, to "isolate" the y on the left side of the equation) so that they can graph the lines more easily.

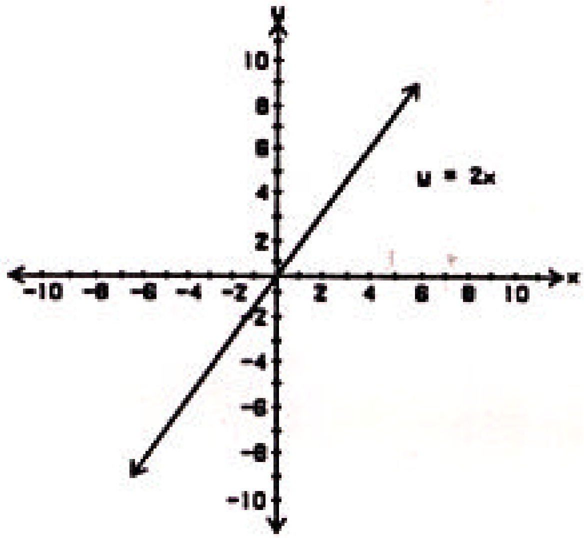

Figure 3.7
Graph of y = 2x

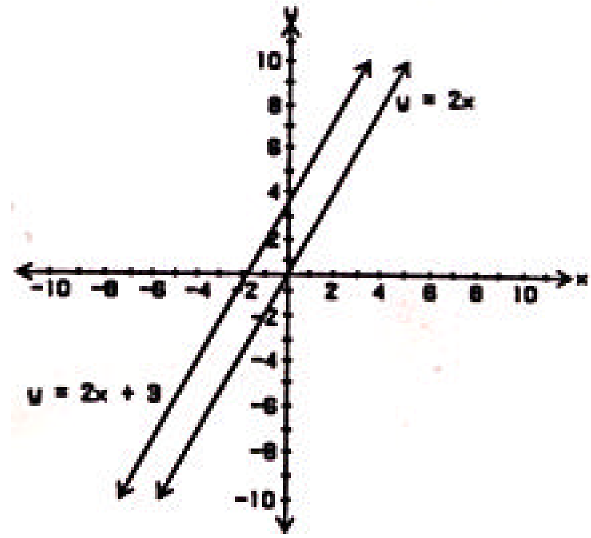

Figure 3.9
Comparison of graphs: y = 2x and y = 2x + 3

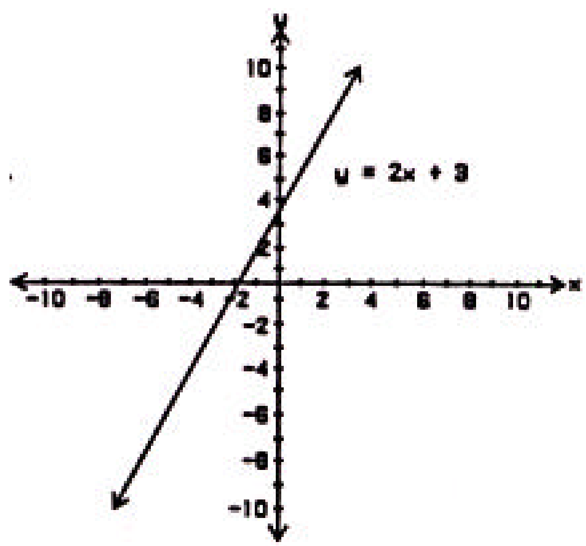

Figure 3.8
Graph of y = 2x + 3

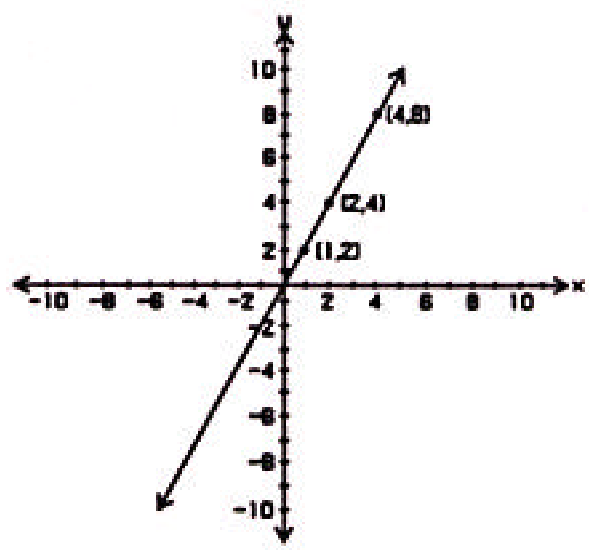

Figure 3.10
Graph of the solution set for y = 2x

What are lines and equations and what do they represent? One can think of lines and equations and the relationship between them in three ways. On one hand, an equation may be understood as a representation of an actual line. The equation allows one to describe symbolically the slant and orientation of that line. For example, y = 2x is a representation of the line in Figure 3.7.

On the other hand, a line can be thought of as a representation of the solution set for a mathematical equation. The line includes all the points that are solutions to the equation. For example, if the equation is $y = 2x$, as we discussed above, then (1,2) and (2,4) and (4,8) are some of the solutions to that equation. These points are all on the line that one draws based on $y = 2x$. (See Figure 3.10.)

Finally, a line may be a graphical representation of the relationship between two variables — amount eaten and weight gained, or children's ages and their heights, for example. Points may be plotted to represent the pertinent data, and an equation may be derived to describe the relationship between the two variables. In this case, both the line and the equation are representations of some real world relationship.

This polyfocal approach to thinking about the relationship between lines and equations is significant in thinking about slope and graphing, and especially in teaching it, for each perspective has implications for how this topic might be taught.

This portion of the interview, lengthier than other questions, was an occasion for the teacher candidates to engage in a larger piece of thinking about teaching mathematics. It helped me to explore how their ideas about teaching and learning interacted with their understanding of mathematics in performing a series of <u>connected tasks of teaching</u>: evaluating curricular material (as a representation of the subject matter), planning to teach, evaluating student work, responding to students' questions.

I presented the prospective teachers with a section on slope and graphing lines from a conventional Algebra I textbook (Dolciani, Wooton, & Beckenbach, 1980) and asked them to appraise it. This particular textbook section does not mention the conceptual mnemonic "rise over run," a typical approach to teaching slope. The book proceeds relatively rapidly, including examples of negative slopes expressed in ratio terms. It also deals with equations for lines. I wanted to know how the teacher candidates, all prospective high school teachers, judged this text. What did they think were its strengths? Its drawbacks? Next I asked them how they would go about teaching this topic if they were working on it with a class of pupils. After that, I showed them a student's work on the practice exercises from the textbook section. I asked them to evaluate the student's work and to discuss what they would do next if they were that student's teacher. Finally, I posed several unstraightforward questions students might ask and asked the prospective teachers how they would respond.

Although this exercise was helped me to learn about the prospective teachers ideas about teaching and learning, as well as about students, in this section I focus on my analysis of their understanding of the content: slope and graphing.

<u>Knowledge of Slope</u>

There was little variation in response among the ten prospective secondary teachers whom I interviewed. They talked about the idea of slope in similar ways: as the steepness of a line or as the relationship between x and y. "Rise over run" was what they remembered, and they criticized the textbook I showed them because it didn't mention this conceptual mnemonic. Still, most of them thought of slope primarily as a <u>tool</u> used in the technique of graphing lines. The teacher candidates thought that the reason for students to learn about slope was that it can help in graphing equations quickly. For example, Andy explained:

> Well, first if you get it into the form $y = mx + b$ you're gonna find the slope. . .

> and once you get it in that form, then the slope is right there for you anyway so you don't have to do all the points or plot all the points. When you get this y = mx + b, m is the slope, get it in that form, whatever the coefficient is in front of the x is the slope, so I think that's the main thing.

While Andy, and many of the others, saw slope as a tool to use in graphing lines for equations, Tim went even further, dismissing slope:

> You don't even need slope — if you can find the x and the y, you're going to graph it right. I mean slope is, if they're going to be a physics major, I mean it's just a thing that's nice and extra.

Tim's comment suggested that, for him, the main thing was to be able to graph lines and slope was not really necessary for that.

The prospective teachers emphasized the value of the slope-intercept (y = mx + b) equation form because it was easier to use for graphing. Cindy observed that the slope-intercept form was really the only form needed:

> I would want to show them how to go about graphing a line, how to take when you are given the slope and the y-intercept how to . . . they show you how to put that into the formula for line, and then move it around to this form — ax + by = c, but I think I would try to show them more explicitly how you use the slope in the y-intercept to find the graph of your line and find two points on the line so that you can draw a line. . .I don't know if I would ever show them, or how important it really is to show the ax + by = c form of a line. You know, because to me, you don't really ever use that when you are using anything that has to do with slope or graphing that I can remember. There may be some ways that when you use that form of it, that's just kind of the formal form, I think.

Because she thought of graphing as a technique, Cindy saw little reason to teach students an equation form that was not easily graphed without transformation.

For these teacher candidates, the reason to learn about slope was to be able to graph lines. Yet they also wanted, as teachers, to be able to give their students "concrete" or "real-world" examples of the content. Almost all of them expressed the desire to connect the content to "real life." They struggled to find "story problems" or other illustrations of the concept's usefulness. These applications were not readily accessible, for the prospective teachers' understanding seemed largely confined to the xy-coordinate plane and was focused on the techniques of "plugging in values" and graphing lines. One teacher candidate, after unsuccessfully searching for applications or illustrations, commented, "I was trying to think up some examples of. . .where you would use slope and I couldn't think of any right away." And another said, "God, I'd want some concrete examples of how this applies to real-life situations. . . [to have] some basis in reality. . .so it would be a little more valuable. . . I can't think of any at the moment."

Several teacher candidates mentioned ski slopes as a possible illustration of the concept's value. Anne's was typical:

> Maybe like skiing, you know, you're a beginner and you don't want to go down this hill because this is the slope of this hill, you know. . . You might want to try the bunny hill where the slope is, uh, whatever.

Although this example illustrated the correspondence between slope and steepness — i.e., the bunny hill would have a smaller slope since it was less steep — it was not an illustration of the value of the concept, nor a real world application of the mathematical content at hand. People do not calculate the slope of different ski slopes in order to decide whether or not to ski them. Another teacher candidate, Barb, also gave a real world situation using ski slopes:

> If you're trudging up this huge mountain and you have so many feet to go and you know it's this much farther and that much higher and how much will you have to trudge. . .to get up to the top.

Barb felt that her example would make students see that slope "does have some importance." She did not seem to recognize, however, that the story did not illustrate the concept of slope. Asking "how much will you have to trudge to get to the top" is not a question about the path's <u>slope</u> or steepness, but rather about its <u>length</u>, or distance.

One teacher candidate, Jon, went beyond the others in his search to identify illustrations of the concept's usefulness. Like the other teacher candidates, he used hills and sleds to discuss the idea of slope as a measure of steepness of a line. However, he said, he would want his students to realize that "slopes don't have to just be, you know, sleds and hills." He gave the following example:

> If I have an experiment that I'm doing in class and that I want to find out how much money every kid makes when he's 25, here's, like in dollars, and I want to see how that relates to how much money their father's making. Say, here's, here's everybody that has fathers that make less than $10,000 and here's everybody that has fathers that makes between $10,000 and $20,000, $20,000 and $30,000. And then I'd say, "Well now, let's see, now if I did this, this, um, survey and I found out that most of the kids, whose parents made less than $10,000, ended up making between $10,000 and $15,000, that the average of the kids whose parents made less than $20,000 was about $20,000, and the average of kids whose parents made $30,000 was about $40,000," and I'd say, "Now, now if I had to, kind of, draw a line somewhere around through here, I'd see that, that the amount of money that kids made is pretty strongly tied to the amount of money the parents made."

Jon's example was the only one that showed a genuine application of graphing. He illustrated that graphs can be used to represent and examine the relationship between two variables — in this case, the relationship between people's income and their father's income. He said he thought it would make slope "more interesting to the kids" and help them "see that it actually applies to different things besides people on sleds or Cartesian coordinates." Although Jon reached beyond graphing equations in algebra and beyond ski slopes, his explanation showed that he was confusing the concepts of <u>correlation</u> and <u>slope</u>. In describing how he would analyze the plotted data (income of parent and child), he said:

> Now if I had to, kind of, draw a line somewhere around through here, I'd see that, that the amount of money that kids made is pretty strongly tied to the amount of money the parents made. The slope is, is pretty strong, it's pretty steep, it's, you know, it's a, it's a good sized number.

Jon was right in thinking that the stronger the relationship between two variables, the greater the correlation. However, the numeric range for expressing correlations varies from -1 to 1, with the strongest correlations being -1 (to express a strong negative relationship) and 1 (to express a strong positive relationship). Thus, a "good-sized number" does not necessarily represent a strong relationship — for instance, a correlation of .2 represents a weaker relationship than -0.9, even though the number .2 is greater than the number -0.9. Furthermore, the value of the slope of a line does not express the strength of the relationship between two variables. Any two variables that, when plotted against each other, actually form a line would be in fact perfectly correlated — i.e., -1 or 1. But the slope of the resulting line could be any value — $\frac{5}{3}$ or $-1\frac{1}{2}$ or 44. Slope and correlation are not the same thing.

The teacher candidates' understanding of slope and graphing seemed centered on the role of these as tools in algebra. Although they wanted, in principle, to connect mathematics to the "real" world, their understanding of this particular content did not prepare them to do so. They couldn't locate examples of its application or usefulness; they were at a loss for a way to show its value beyond the further study of mathematics. As one prospective teacher commented, "It's hard in high school because a lot of math builds up to higher math which is used in things that are beyond the kids and that they don't — can't relate to." One of the questions I asked helped me learn how the teacher candidates' understanding of slope connected to "higher math" — i.e., the derivative in calculus. I posed the following question:

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Suppose a pupil says to you, "You said we can think of slope kind of like steepness. So I was thinking about climbing up a steep hill or something. But how do you tell the slope of a curvy line like a hill?"

What do you think you'd say (or do)? Why would you say (or do) that?

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I wanted to explore whether the teacher candidates would make a connection between the slope of straight lines and the "slope" of curves. One way to make this connection is to note that the slope of a curved line is different at different points, and that one might approximate the slope at different points by identifying the slope of the tangent at that point. (See Figure 3.11).

Figure 3.11
Drawing successive tangents to a curve to approximate the "slope" of the curve

Half of the 10 secondary teacher candidates said they would explain this in terms of tangent lines. Andy's explanation was typical:

> Well, you could just real quickly, maybe, draw a picture of a, say a parabola or something like that. And you can't necessarily take the slope but you can take the slope at each point along the parabola and just, you know, just real quick, goes say like that. `Cause say if I wanted the slope at this point, I'd take the line that's tangent to the circle and take the. . . tangent to the curve and take the slope of that. So the slope's gonna be different at every point around the curve. And, and that's the same way for everything. You know you take the slope of this and the slope is gonna be tangent to the curve. So the slope there is whatever the rise over run is at that, on that point there.

All the prospective teachers who gave this kind of response said they would show how the slopes of a succession of tangent lines were part of a process of approximating the slope of a curve, which is different at different points. Several of them said they would tell students that they would be learning about this "later on" in calculus and that it was called the derivative.

Two teacher candidates connected the student's question to "higher" mathematics, but said they wouldn't go into it at all with the student. Tim said he would just tell the student "you'll find out later" — that "that's calculus, it has to do with a derivative. . .but the formula we have right here and right now we can't apply to a sine wave or any curve." Anne said she would be "tempted" to show the student but that it would be "a waste of time" to go into it. Instead she would tell the student that "when you take geometry you will learn that." Anne seemed vague about what she was thinking about showing or what the student would learn in geometry that related to this question.

Finally, two prospective teachers said they would make clear that slope didn't apply to curves. Cindy explained:

> I'd stress the point that a curve and a line are not the same thing. When you are dealing with slope, you are dealing with things that are straight lines and that they don't follow a curve. Maybe I would give an example of oh, something that had a curve, like the Astrodome, the top of that was curved, you know, but the side of a tent. That would be a good example, two tents. Some tents that you see now have curved tops and those wouldn't have slope, but some have straight tops and are pointed, and those would have slope.

And Mark said he would want to make sure the students understood that only "equations or lines have a constant slope all the way through."

Of course, both Cindy and Mark were right — curves do not have slopes in the same sense as straight lines since the ratio between the rise and run of the curve varies. Still, neither of them went beyond this to make the connection to the derivative as a means of gauging the variable "slope" of a curve. It may be that this was a simple oversight. But it may also be that, as I found with other concepts, these teacher candidates' understanding of slope was disconnected from their understanding of the derivative: two more fragments of mathematical knowledge.

Learning Mathematics in Order to Learn Mathematics

Although the secondary teacher candidates whom I interviewed did seem comfortable with "rise over run" and with graphing equations, their understanding seemed limited to the use of these as tools in algebra. When asked how they would respond to a pupil who complained, "Why do we have to learn this boring stuff?" they said they would tell the pupil that it was required, necessary in order to go on in math. "It probably isn't the most interesting thing you'll do in your life, but this is what we've got to do right now." The teacher candidates did wish they had something else to say besides "`cause you have to learn it." Although dissatisfied with this response, however, they knew no other.

As a group, the prospective secondary teachers seemed to remember the relevant information about slope and graphing, but had few connections or applications of this content readily available. For them, learning one bit of mathematics had served primarily as preparation for learning the next bit.

Prospective Teachers' Substantive Knowledge of Mathematics: Rethinking Common Assumptions

My exploration and analysis of these prospective teachers' substantive knowledge of mathematics raises some serious questions about subject matter preparation for mathematics teaching, questions to which I return in Chapter 7. However, before leaving this chapter, I want to point out and challenge three all-too-common assumptions that underlie conventional arrangements for teacher education as well as, paradoxically, current proposals to reform teacher education and certification.

Assumption #1
Traditional School Mathematics Content is Simple

This first assumption underlies the following three and is based on the notion that topics such as place value, division, and slope, not to mention addition, decimals, graphing, and factoring, are not that complicated. Implicitly, the message is: If you can "do" them correctly, then you can teach these topics. Qualities of explicitness and connectedness are not usually considered as criteria for knowing. Given this assumption, it is logical for teacher educators to leave subject matter preparation to precollege and college math classes and to concentrate on teaching methods and knowledge of children instead (Buchmann, 1982).[53] The assumption is founded, however, on an unexamined conception of subject matter knowledge in mathematics, one that implicitly accepts ***remembering*** and ***doing*** as the critical correlates of mathematical understanding.

Assuming that the content of first grade mathematics is something any adult understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized. Throughout my interviews, many college students, including people who were majoring in mathematics, had difficulty working below the surface of so-called "simple" mathematics. Although they could perform the procedures, they seemed to lack explicit conceptual understanding of the content.

The discussions of place value, division of fractions, of zero, and in algebra — as well as of the concept of slope in the chapter show the insufficiency of this implicit conception of knowledge by illustrating what Duckworth (1987) refers to as the "depths and perplexities of elementary arithmetic." In her writing, as well as in Lampert's (1985, 1986, in press a), the "simple" content of the school curriculum is opened up and its mathematical complexity revealed. Teacher educators may convince prospective teachers that "teaching for conceptual understanding" should be the goal. However, without revisiting the "simple" mathematical content they will teach to develop the explicitness and connectedness, not to mention

[53] Many mathematics educators do include mathematical content in their methods classes. But, interested in preparing teachers who can break out of the conservative practice of math teaching, these instructors often concentrate on or nontraditional content and innovative teaching materials and tools. While they recognize that their students will teach multiplication as well as probability, they choose to emphasize the different instead of revisiting the familiar, more conventional content.

correctness, of their own understandings of the content, prospective teachers may be wholly unprepared to do more than teach "invert and multiply."

Assumption #2:
Elementary and Secondary School Math Classes Can Serve As Subject Matter Preparation for Teaching Mathematics

This assumption flows logically from the first. Whatever the contributions of upper level mathematics study to teachers' disciplinary knowledge, the fact remains that a large part of what they teach is material which they studied in elementary and secondary school. The fact that prospective teachers study little school mathematics content as part of their formal preparation for teaching implies that this second assumption is widely shared.

My study can shed some light on the validity of this assumption. In order to respond to the interview questions and tasks, the teacher candidates drew on what they had learned in school. Even the mathematics majors, most of whom had taken over 7 college math courses, had to think back to seventh grade for the subject matter embedded in the questions. When they did this, seeking particular mathematical concepts, procedures, or even terms, they typically found loose fragments — rules, tricks, and definitions — inexplicit and unconnected. Most did not find meaningful understanding, nor even the stuff with which to figure out such understandings on the spot.

While troubling, it is also unsurprising given the widespread criticism of the algorithmic understanding fostered in many math classrooms (e.g., Davis & Hersh, 1981; Erlwanger, 1975; Goodlad, 1984; Madsen-Nason & Lanier, 1987; Wheeler, 1980). My findings suggest that relying on what prospective teachers have learned in their precollege mathematics classes is unlikely to be adequate for teaching mathematical concepts and procedures meaningfully.

At the same time, the image of mathematics implicit in even good elementary and secondary mathematics classrooms (Schoenfeld, in press a), as well as in our culture, is one of a linear, rule and fact-filled body of knowledge in which perspective, interpretation, and argument are irrelevant. Prospective teachers are therefore unlikely to acquire an appropriate view of the discipline as a result of their precollege mathematics experience. The view they do hold is likely to shape not only the way in which they teach mathematics once they begin teaching, but also the way in which they approach learning to teach mathematics (Ball, 1988).

Assumption #3
Majoring in Mathematics Ensures Subject Matter Knowledge

Some people do not make the second assumption. In fact, some completely overlook prospective teachers' precollege mathematics experience and assume instead that subject matter preparation for teaching occurs at the university. Others would remedy the shortcomings of precollege mathematics education through university coursework. Many recent proposals for reforming teacher education and certification (e.g., Holmes Group, 1986; Carnegie Task Force, 1986) recommend that elementary teachers specialize in an academic discipline. Other reforms propose to certify college graduates who have completed an academic major but have had no teacher education. Underlying such proposals is the assumption that the study entailed in a college major can equip the prospective teacher with a deep and broad understanding of the subject matter.

In mathematics, however, I found narrower differences in substantive understanding between the elementary and the secondary teacher candidates than one might expect (or hope). Although the latter, because they are math majors, have taken more mathematics and do know more "stuff," this does not seem to afford them substantial advantage in articulating and connecting underlying concepts, principles, and meanings.

Some people propose that this is due largely to the poor academic caliber of teacher education students (see Lanier, 1986, for a discussion of this common assertion). However, interviews conducted by researchers at the National Center for Research on Teacher Education with mathematics majors who are not planning to teach do not support this suggestion. These math majors, too, struggle with making sense of division with fractions, connecting mathematics to the real world, and coming up with explanations that go beyond the restatement of rules. Furthermore, most of the secondary teacher candidates in this study were good students, with impressive college entrance exam scores and high grade point averages in their college math courses. A more plausible explanation for the problems experienced by the math majors is that even successful participation in traditional math classes does not necessarily develop the kinds of understanding needed to teach if, as is often the case, success in these classes derives from memorizing formulae and performing procedures. Moreover, studying calculus does not usually afford students the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will teach. Requiring teachers to major in mathematics, or even increasing the mathematics course requirements for prospective teachers, both currently advocated, will not necessarily ensure increases in their substantive understanding.

* * *

Analyzing prospective teachers' understanding of several specific topics (place value and numeration, the procedures of multiplication and subtraction with regrouping, division — by fractions, by zero, and in algebraic equations, and slope and graphing) sheds light on teacher candidates' knowledge of some of the <u>substance</u> of mathematics. The analytic categories of correctness, explicitness, and connectedness helped to illuminate some key qualitative dimensions of their understandings. However, I was also interested in examining what they understood <u>about</u> mathematics — what kind of a subject it is, what it's good for, what it means to "know" something in mathematics, how knowledge is justified, and what "doing mathematics" means. The next chapter continues the exploration of prospective teachers' knowledge of mathematics, focusing on these issues, analyzing the teacher candidates' understanding of theory and proof, and their ideas about the nature of the discipline.

CHAPTER 4

PROSPECTIVE TEACHERS' UNDERSTANDINGS AND FEELINGS ABOUT MATHEMATICS AS A DISCIPLINE

> Math is concrete. If I were teaching a elementary math class, there's never going to be a problem that doesn't have an answer. It has to be an <u>exact</u> answer. Which wouldn't be so with other subjects, I suppose. (Pam, elementary teacher candidate)

> Math is so abstract, you can't even. . .In other subject areas, you can use personal opinion and personal ideas into the subject and make it more interesting and more understandable. Math is something that you've got to <u>accept</u> it the way it is and it can be <u>real</u> scary sometimes. (Mark, secondary teacher candidate)

Pam and Mark, each separately, were talking to me about teaching. Pam was explaining her ideas about student assessment. She said that she would test students on a regular basis to see if they were "grasping" the material. She was comforted by the fact that, since answers in mathematics were definite, it would be easy to decide what students knew. Mark was explaining why learning mathematics was so difficult — that it was monotonous and that there were so many things to accept and not question. He said that "if the interest is there, you can do it — if it's not, then you are going to have struggles with it."

Pam's and Mark's remarks point up some ways in which understandings and feelings about the <u>nature</u> of mathematical knowledge influence pedagogical thinking. Pam's conception of mathematics as a domain of clearly right and wrong answers is the basis for her assumption that finding out what students know is easy. Mark's image of mathematics as scary, not always reasonable, and tedious shapes his notion that learning mathematics is especially difficult for some students. Throughout my interviews with teacher candidates like Mark and Pam, the prospective teachers' ideas, explanations, and justifications about teaching and learning were rooted in their assumptions about the subject matter.[54]

I argued earlier that understanding mathematics involves a mélange of knowledge, beliefs, and feelings about the subject. Chapter 3 focused on part of the prospective teachers' knowledge of mathematics — what they knew about particular topics. I have referred to this kind of knowledge as ***substantive knowledge***. This chapter focuses on two other critical dimensions of subject matter knowledge in mathematics: ***knowledge <u>about</u> mathematics***[55] and feelings — about the subject and about self in relation to the subject. Knowledge about mathematics includes understandings about the nature of knowledge in the discipline — where it comes from, how it changes, and how truth is established. It also includes what it means to "know" and "do" mathematics, the relative centrality of different ideas, as well as what is

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[54] I return to "pedagogical reasoning" and the warrants that govern it in Chapter 6.

[55] The same caveats about "knowledge" that I discussed in Chapter 3 apply equally to this dimension of understanding mathematics. People believe things about and have conceptions of mathematics that are not valid from a disciplinary perspective and might, therefore, not be considered "knowledge."

arbitrary or conventional versus what is necessary or logical, and a sense of the philosophical debates within the discipline.

While students are rarely taught explicitly about the evolution of mathematical ideas or ways of thinking, teachers do nevertheless convey many explicit and implicit messages about the nature of the discipline. If the teacher's guide is the dispensary of right answers, for example, then the basis for epistemic authority in mathematics does not rest within the knower. Teachers also communicate ideas about mathematics in the tasks they give students, from the kinds of uncertainties that emerge in their classes and the ways in which they respond to those uncertainties, as well as from messages about why pupils should learn particular bits of content or study mathematics in general. As such, teachers' ideas <u>about</u> mathematics are a significant dimension of their subject matter knowledge.

Finally, knowing mathematics is also colored by affect: How people feel about mathematics and about themselves as knowers of mathematics interacts with the ways in which they think and what they understand and shapes their participation in and experience of mathematics. The prospective teachers varied in how they felt about mathematics and about themselves in relation to the subject. These feelings were related to differences in the teacher candidates' understandings and ways of thinking about mathematics. I return to this later in this chapter, and turn now to an examination of the prospective teachers' ideas <u>about</u> mathematics.

While there are many dimensions to knowledge <u>about</u> mathematics (e.g., knowing about the history of mathematics, understanding different philosophical perspectives on the nature of mathematical knowledge), I selected three areas of focus:

1. How did the teacher candidates understand the sources of and justification of mathematical knowledge?

2. What did the teacher candidates think was entailed in "doing" mathematics?

3. What did the teacher candidates think was the purpose of or reason to learn mathematics?

I decided to focus on these three aspects because of their significance for the teaching of mathematics. Each section begins with an explanation of the issue from a disciplinary perspective and a justification for including it as an important component of teachers' knowledge and ways of thinking. Next I discuss what the prospective teachers know and assume about mathematics, and appraise their ideas from a perspective of mathematical pedagogy.

<u>Ideas About the Sources of and Justification for Mathematical Knowledge</u>

How does one know that a particular idea in mathematics is true? When one has a sudden inkling of a pattern or a relationship, how can one decide if it holds up in all cases? Once convinced oneself, how does one persuade others of one's idea? Answers to these questions matter implicitly to working mathematicians as they labor to construct and validate new theories.

But these questions are explicitly critical to teaching mathematics. Students bring up novel ideas and ask unusual questions. They make propositions and assertions. For example, a third grader may claim that, when throwing a pair of dice, all outcomes from 2 to 12 are equally probable. A sixth grader may invent a procedure for calculating the area of a triangle. A high school student might posit that the differences between each pair of successive squares are consecutive odd numbers.

When teachers are confronted with student claims like these, how might they respond? One possibility is for the teacher to maintain control of the direction of the class and to generally divert students from pursuing ideas outside the scheduled curriculum. A second possibility is for the teacher to be personally responsible for evaluating the truth of the student's assertion. This implies that the teacher must know or be able to determine whether what the student is claiming is valid. A third alternative is for the teacher to engage the pupil in <u>proving</u> the truth of his or her claim. How a teacher responds to these is all part of teaching mathematics — even if these student initiatives are ignored.

Concerns for "covering" the curriculum — real and pressing concerns for most teachers — suggest that one of the first two alternatives is most desirable. With either of these courses of action, little time is spent "off-track." Yet, this depends on what one considers the "track" to be, for both alternatives have significant costs in terms of what students learn <u>about</u> mathematics. If novel ideas are not taken up in mathematics classes, students do not learn that the process of conjecture, reason, and proof are central to knowing and learning mathematics. If the teacher is the source of validation and answers, this communicates to students that mathematical truth is something one <u>has</u>, not something one establishes with knowledge and reason. The answers are in books or stored in the heads of those with more education. Either of these alternatives, while perhaps efficient, misrepresents what it means to learn or to know mathematics, making it seem that the task is to acquire and store knowledge in one's head. Neither do these approaches help students acquire the skills and understanding needed to judge the validity of their own ideas and results — to be "independent learners" or to be "empowered," part of the rhetoric of education today.

The third alternative — that teachers engage pupils in <u>proving</u> the truth of their claims — communicates to pupils a very different picture of what "doing" and knowing mathematics entails. Moreover, helping students develop mathematical power is a central goal of mathematics instruction in this approach. Teachers can help students understand that the ideas they have, the flashes of insight, the things they notice from exploring specific cases, are not true statements, but conjectures, and that conjecturing is a critical part of learning mathematics. Teachers can help students acquire a repertoire of strategies for pursuing their conjectures — such as generating alternative representations of their ideas, searching for counterexamples, building an argument from things they already know, and constructing an argument from specific cases. Teachers can also employ the resources of all members of the class as a mathematical community by, for instance, expecting pupils to convince others of the validity of their claims.

Time is not infinite in classrooms, however. Every conjecture a pupil raises may not be equally worth pursuing. While this is a delicate matter, given the goal of helping students acquire power in mathematical reasoning, teachers must nevertheless make judgments and decisions. Some conjectures that students make are reinventions of ideas that others have already proposed and perhaps established — how to calculate the area of a triangle, for example. Other conjectures have already been proven false — that the probabilities of all outcomes on the dice are equal. Still others may be novel. In other words, as far as we know, these ideas have not previously been proposed. Conjectures of all three kinds may be worth spending time on, depending on the pupils and on the curriculum. Knowing as much as possible about the conjectures their pupils raise is likely to help teachers make judgments about how to help pupils pursue their ideas and about which ideas are most fruitful to "milk."

Substance Over Proof: The Prospective Teachers' Inattention to Justification

The teacher candidates seemed quite unattuned to issues of the justification of mathematical knowledge; it simply did not seem to be something they thought much about. In this section, I present a specific case of their thinking about the nature of mathematical knowledge that highlights this inattention to proof; I continue with some more general discussion of their notions about what it means to know something in mathematics.

In order to explore the prospective teachers' ideas about theory and about proof — or about what it means to <u>know</u> something in mathematics in a concrete situation — I presented them with the following scenario:

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Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove that what she is saying is true:

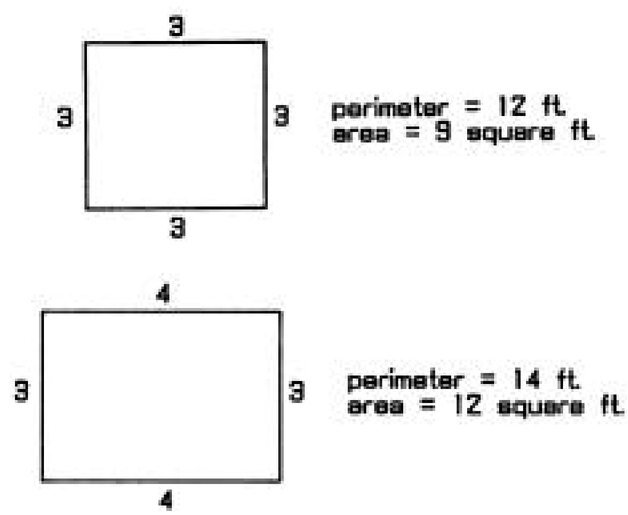

Figure 4.1
The student's "proof" of her theory about perimeter and area

How would you respond to this student?

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There are two dimensions to this question. One dimension has to do with the specific concepts of perimeter and area and their Figure 4.1 relationship, the substance of the student's claim.[56] The second dimension of this question has to do with mathematical knowledge and the justification of knowledge — "theory" and "proof." The student claims to have "discovered a theory" and offers a picture as a "proof." An example, however, does not establish the truth of a generalization. The student has illustrated, not proven, her claim. Her claim is a conjecture, not a theory.

I wanted to see whether the prospective teachers would pay attention to this dimension of the question. Would they be persuaded by the drawing or would they be skeptical? If they were skeptical, I wanted to know what they would consider sufficient evidence that the claim was true, and how they would proceed to find such evidence. Would they respond to the student's conception of theory and proof or would they focus on the substance of her claim?

I also wanted to find out whether the teacher candidates knew that there is no direct relationship between perimeter and area. I anticipated that many of them might not be sure, and I was interested in learning what made them unsure — whether it was the lack of good proof, whether it was not being able to remember, whether it was being unsure about how to decide if a claim is generally true in mathematics.

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[56] The ***perimeter*** of a figure or region is the length of its boundary, while the ***area*** is the number of unit squares it takes to cover the figure or region. For example, calculating the perimeter of a dog pen in your backyard would tell you the length of the fence. (Figure 4.2). Figuring the area would tell you how much space your dog had in which to play. (Figure 4.3) Intuitively, the two measures seem related. That is, it seems logical that longer amounts of fence would necessarily yield bigger spaces for the dog. This is what the student in the above scenario claims. While it seems intuitively right, however, it isn't. Imagine, for instance, a square, 3 feet on each side. Its perimeter is 12 feet and its area is 9 square feet. (Figure 4.4).

Now suppose you take 18 feet of fence. You might construct a narrow rectangle, 1 foot wide and 8 feet long. This rectangle with perimeter of 18 feet will have an area of only 8 square feet. In this case, the area is actually less despite a greater perimeter. (Figure 4.5) Two regions can have the same perimeter but different areas (see figure 4.6) or the same area but different perimeter (see Figure 4.7).

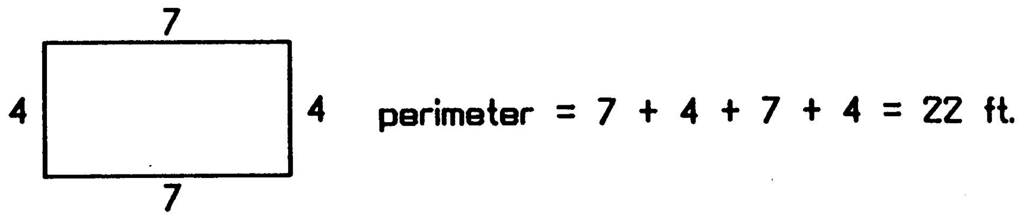

Figure 4.2
Perimeter

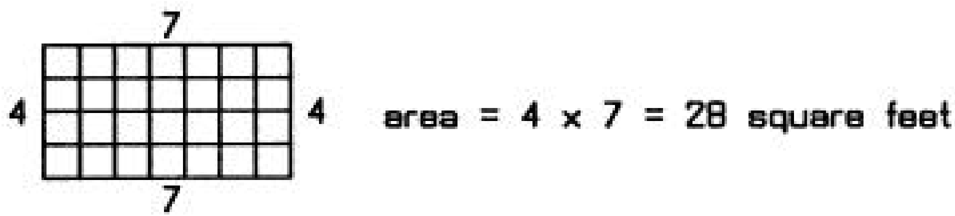

Figure 4.3
Area

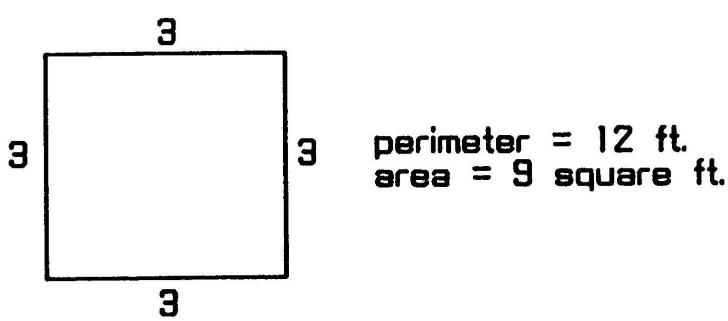

Figure 4.4
Perimeter and Area

Figure 4.5
Increased perimeter, decreased area (from Figure 4.4)

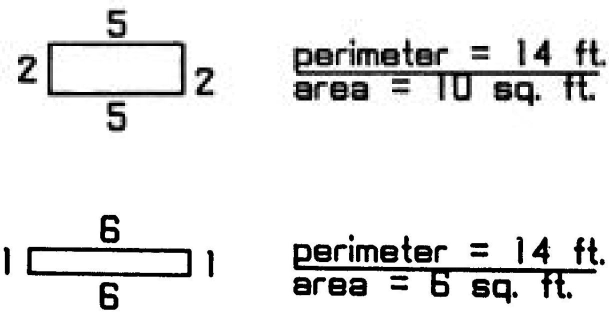

Figure 4.6
Same perimeter, different area

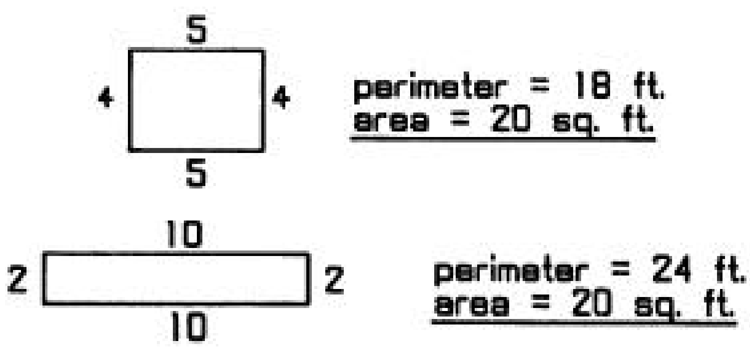

Figure 4.7
Same area, different perimeter

I analyzed the data to determine how many prospective teachers identified the student's "theory" as false or concentrated on the fact that examples aren't proofs, how many thought it was true, and how many were not sure about whether it was true or not. Only 3 of the 19 prospective teachers I interviewed knew that the student's claim about the relationship between perimeter and area was not true. The rest were either not sure or believed that she was correct. The results of this analysis are summarized in Table 4.1. Below I discuss each of these kinds of responses to the question.

Table 4.1
Evaluating a false student conjecture about perimeter and area

	Teacher Candidates		
	Elementary	Secondary	TOTALS
Thought the pupil's theory was NOT TRUE	2	1	3
Were NOT SURE whether the pupil's theory was true or not	5	3	8
Thought the pupil's theory was TRUE	3	5	8

The student is right. Cindy, one of the mathematics majors, said she would be impressed that the student discovered this. She leaned over and, after studying the example briefly, said:

> That's interesting. That would be true, first of all. I'd probably say, "That's a very good observation and that was very neat that you could think of that on your own at home."

Then she added that she could give the pupil another example to illustrate the theory:

> Our classroom, you know, the length of the walls is small, but in the cafeteria, look how long the walls are in the cafeteria. It takes you a longer time to walk around the end of the cafeteria than it does to walk around the walls of the classroom.

Cindy's example illustrates the case in which the perimeter and area are both larger, thus, despite the fact that it does not generalize, reinforcing the student's claim.

Mei Ling, one of the elementary candidates, was especially enthusiastic about the student's "discovery." Unlike Cindy, though, Mei Ling did not examine the illustration nor reflect on the student's claim before responding:

> I would say, "That's fantastic." I would say, "That is wonderful that you went home and you thought about this and you came back with this new idea." Then I would say, "Well, you know I didn't bring up in class, but you are absolutely right" and I would say "People have discovered this, but I didn't mention it. Today I'm going to mention it to the class because you are right, I didn't say it and it's an important fact that's true."

Six other prospective teachers agreed with Cindy and Mei Ling and were impressed that the student had discovered this relationship herself. Without hesitation, they said they would praise her; one of the other secondary candidates said he might be inclined to consider recommending the pupil for a higher class. These teacher candidates showed no sign of skepticism, but agreed immediately with the assertion. Apparently the thought that an example is insufficient proof never crossed their minds.

The student is wrong. Three of the prospective teachers said immediately that the student was wrong — that the asserted relationship between perimeter and area held some of the time but that there were many cases in which it was not true.

Jon, a mathematics major, said he would ask the student to search for a case in which the perimeter increases and the area does not. The other two teacher candidates said they would tell the student this wasn't true all of the time, and they would show her a counterexample. Still, the question of the insufficiency of a single example to establish the truth of a claim never seemed to come up.

I'm not sure if this is right or wrong. Many teacher candidates were unsure whether there was a direct relationship between perimeter and area. Some were not persuaded by the student's drawing but could not remember if the assertion was true either. These teacher candidates said they would have to figure it out for themselves. Terrell, one of the mathematics majors, was one of these. When he first looked at it, he assumed that the assertion was false, and began searching for a counterexample to show this. But as he worked on it, and couldn't come up with one, he grew less sure that he was right: "I am almost believing that this is true." Janet, another secondary candidate, said she would test it for herself, "to see if it's true. . . if I came up with an example where it wasn't true, then I would point that out to her."

Others who weren't sure said they would look it up or ask someone. They seemed less inclined than the others to puzzle over it themselves. Their uncertainties sometimes stemmed from not remembering the relevant concepts as well as not knowing whether the relationship held. For example, Marsha, an elementary candidate, wasn't sure at first what area was: "Is the area inside? What is the area?" Then she said that it seemed like "a reasonable assumption if you're going to increase the parameter (sic), of course the area inside is going to increase." But Marsha said she would have to "go and check on it" before she could say if it was "valid" or not.

Examples, proof, and the justification of knowledge. Perhaps the most significant result was that, in responding to this question, over half of the prospective teachers focused exclusively on the substance of the student's claim, and responded in terms of what they knew about perimeter and area and the relationship between them. They did not comment on the student's approach to establishing her claim. Instead, they were concerned with telling her that she was either right or wrong.

Not only didn't the teacher candidates worry about the insufficiency of the "proof," but a few prospective teachers even seemed to become convinced themselves by the student's drawing. They looked closely at the sketch, and a couple of them carefully checked the calculations of perimeter and area. They made comments like "it looks like it" or "it seems right."

Several teacher candidates were pleased that the student had figured something out for herself. One said it was good that the student had "put the two pieces together, especially on her own," that many students "never really tie it together" — they just "get this piece and that piece." Another said she would tell the student, "You are really using your thinking skills about mathematics."

These teacher candidates were preoccupied with the student doing something "on her own," a preoccupation common in mainstream American culture. An emphasis on ownership, individuality, and independence can jeopardize, as it does in this case, concerns for legitimacy. "Thinking on your own," if it is wrong, is in direct tension with acquiring disciplinary knowledge and skills. Yet most of the teacher candidates seemed to be oblivious to the problem of justifying knowledge claims on the basis of one example. To say that to prove something by showing an illustration is to use "thinking skills in mathematics" completely misses the point of what mathematical thinking entails.

Some teacher candidates did seem concerned with the issue of proof. These were all people who were genuinely not sure themselves about the validity of the student's claim. Five of them were skeptical because they were aware that the student had shown only one case. They knew that this one case did not establish that the relationship would hold in all cases and, since they didn't know whether her proposal was valid or not, they were concerned about this <u>one</u> example. Some of these teacher candidates talked about searching for a counterexample — a case in which this relationship did not hold. Others said they would look up the answer. However, these five prospective teachers would look for counterexamples or check in a book because <u>they</u> wanted to know the correct answer to the question <u>themselves</u>. In other words, it was part of their own personal search for knowledge in mathematics, not part of a desire to appropriately represent the issue of knowledge justification to students. Had they known the correct answer themselves, I suspect that they wouldn't have worried about the insufficiency of the proof.

Only four prospective teachers talked explicitly about proof, noting that the student had not actually <u>proved</u> her claim by showing a drawing. These four, all secondary majors, focused on the issue of proof in mathematics and said they would discuss it specifically with the student. Mark explained,

> First of all, I'd let her know that when you are talking about theories and stuff, you want to be very, very careful about you know, stating them right away. You almost have to have a good, solid direct proof of a theory before you just take a few examples and say "Yeah, that's true."

Terrell said he "would explain that examples are not proofs" and Janet said if the class had done some proofs, she would encourage the student "to go through a proof and see if she can prove it to herself and to me." Tim was the most elaborate. He said that this was "a misleading way to think about proof." He would tell her that "this shows that, but you have to get an idea of what a total proof is." He explained that a proof has to show that the claim holds for every case and that "to prove something is very, very, very difficult because when you prove something, you are proving it for everything." Tim emphasized that "that's something that all math students throughout the junior high and high school level need to be aware of — what to prove something means." Interestingly, Tim thought nevertheless that the claim was valid. He said he would tell the student that "you do understand that this is true" but that she had not proved it.

Summary: Teacher candidates' understanding of perimeter and area, theory and proof. What did the prospective teachers seem to know — about perimeter and area and about the justification of knowledge in mathematics? First, substance dominated the teacher candidates' responses to this question. Most concerned with the claim itself, almost half thought that perimeter and area were directly related — that as the perimeter increases, the area increases automatically. Interestingly, over half the secondary majors thought this was correct — only one of the secondary majors knew that this wasn't true. Altogether, only 3 out of the 19 teacher candidates knew that the claim was not true.

For most of the teacher candidates, the basis for justifying or refuting this claim lay in the reaches of their memory, in their accumulated knowledge of mathematics. In other words, they remembered that these two measures were (or were not) related.

Only four teacher candidates responded to the proof dimension of the question as a focus in its own right. These prospective teachers saw the justification issues — how one proves something in mathematics and what a theory is — to be equally significant to the question of whether perimeter and area are directly related or not. Their responses showed that they considered the student's conception of theory and proof an important part of her understanding of mathematics.

Those prospective teachers who were not sure about the student's claim also appeared to consider the question of justification of knowledge. They could not remember whether perimeter and area were directly related or not and they were skeptical from seeing just one example. Like the teacher candidates who focused on the proof dimension, these people seemed to be aware that the student's "proof" was insufficient to prove the claim. But unlike those who emphasized the question of proof, these prospective teachers were focused on the substance of the student's claim. I suspect that if they had been able to remember whether or not her "theory" was true, they would not have been concerned with her "proof," just like the 11 teacher candidates who "knew" that it was true (or not true).

Ideas About Sources of Mathematical Knowledge and How Knowledge is Justified in Mathematics

In addition to the exercise on perimeter and area, I also asked prospective teachers some more direct questions about the sources and justification of mathematical knowledge. In this section, I discuss the prospective teachers' ideas about the epistemology of mathematics. The teacher candidates' views can be expressed in the form of two main views, each of which is elaborated in the section that follows:

View #1: Mathematics is a mostly arbitrary collection of facts.

View #2: Mathematics? I've never really thought about it.

View #1: Mathematics is a mostly arbitrary collection of facts. Teacher candidates who held this view believed that the way to confirm an answer was to look it up. For example, when they were unsure about their own knowledge, they said they would check in a book or ask someone who knew more mathematics. For some of them, this stance grew out of their own sense of incompetence in mathematics (which I discuss in later section in this chapter). However, it also seemed to be based on a view of mathematics as a domain in which reason mattered little. For instance, Marsha, an elementary candidate, explained why she was unsure about whether the student's assertion about the relationship between perimeter and area was true:

> You never know if there's going to be some weirdo exception that if you double it, it'll never increase or something. I mean, you know, math is so contradictory.

Marsha didn't know whether the relationship was true or not, but she also did not think it was something one could figure out because one would be unable to know if there was some arbitrary "exception." The teacher candidates' responses to other questions rang with this theme as well. For example, when I asked them how they would explain division by zero, 12 out of 19 of them stated a rule and said this was something to be remembered.

Explaining something in mathematics means clarifying an idea by unpacking underlying concepts as well as giving reasons that reveal its meaning or logic. Yet when the teacher candidates talked about "explaining" (which they did frequently) they seemed to mean something much weaker, something that much more closely resembled simple telling — as in giving directions for the steps of a procedure, restating a rule, or saying a definition. The prospective teachers' weak use of the term **explain** suggests that their idea of knowing something in mathematics was narrower than a disciplinary view.

Almost all the prospective teachers in both groups agreed that remembering rules and facts was essential to "knowing" mathematics. The secondary candidates remembered these rules better than the elementary majors, and used them to provide explanations. For them, in many cases, to state the rules, procedures, or definitions was to explain. For example, the most common explanation for division by zero was to say that "it's undefined." Although true, this does not explain what that means or why dividing by zero is undefined; it is not a mathematical explanation.

Cindy's "explanation" was typical:

> I'd just say. . . "It's undefined," and I'd tell them that this is a rule that you should never forget that anytime you divide by 0 you can't. You just can't do it. It's undefined, so. . . you just can't. They should know that anytime you get a number divided by 0, then you did something wrong before. It's just something to remember.

In general, the prospective teachers' ideas about what it means to know something in mathematics centered on remembering rules and being able to use standard procedures. Obviously the prospective teachers' ideas about mathematics do not exist separately from their substantive understandings of particular concepts or procedures. Many of them lacked explicit and connected conceptual understanding of mathematical ideas and procedures. As such, many of them could do little else but respond in terms of rules and procedures.

Andy, like many of the secondary majors, said that division by zero was "just something to remember, you just can't divide by zero." Without any rationale for the rule, Andy and the others were not able to offer true explanations. However, they generally thought they were explaining.

For many of the teacher candidates, mathematics was something to be remembered or retrieved, and clearly not something to figure out or reason about. This was true of most of the teacher candidates, whether or not they "knew" the mathematics in the questions I asked. When they didn't know something during the interviews, they said they couldn't remember. For example, 4 out of the 19 prospective teachers did not know how to explain how to solve a simple algebraic equation involving division:

$$\frac{x}{0.2} = 5$$

Teri explained, "I haven't done those in so long, I can't remember." For Teri, and for the other three, not being able to remember the procedure meant that she could not do the problem. The other teacher candidates who did have an explanation used what they remembered — reminders to "isolate x" or "multiply both sides by the same number." Only one of the 19 teacher candidates, an elementary candidate, tried to reason her way to a sensible explanation of the problem's meaning. This view of mathematics as a collection of facts to be remembered was equally evident among those prospective teachers who thought they did know the mathematics embedded in the interview questions. Over half the secondary teacher candidates accepted without question the (false) "theory" that perimeter and area are directly related. Carol remarked, "Of course that's what happens" and several others commented simply, "That's true." They did not seem oriented toward questioning or validating mathematical claims themselves. If they remembered that it was correct, that was sufficient for confidence in the claim.

View #2: Mathematics? I've never thought about it. In general, the prospective teachers seemed to have given little thought to the origin or growth of knowledge in mathematics.[57] I would not infer from this that they thought that mathematics is "given," but rather that mathematics as a field of human endeavor was an issue about which they had simply never thought. Instead, their image of mathematics was a body of knowledge to be mastered.

Because I wanted to pursue these issues explicitly even when teacher candidates did not raise them themselves, I included some statements about these epistemological issues in the card sort. For instance, I tried a item that said, "Some problems in mathematics have no answers." In the first round of interviews, I discovered that this item meant something quite different to many of the prospective teachers than what I had been thinking. Mei Ling, an elementary candidate, said she agreed with the statement and she explained:

> Mei Ling: I'm thinking is this the obvious, like just some math problems you are going to find in the textbooks have no answers?
>
> Ball: Oh, no, it's like are there problems in the field that no one can solve?
>
> Mei Ling: As far as <u>teaching</u> it?
>
> Ball: No, as math as a subject. Like a math problem that no mathematician can solve. Problems like that.
>
> Mei Ling: I am sure there <u>are</u>. I don't know if we've found any, but the possibility <u>exists</u>. It's like saying, "Is there life on other planets?" (laughs), We don't know, but. . .it <u>could</u> be!

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[57] Only two teacher candidates, both mathematics majors, made any comments during the interview about the construction of knowledge in the field. Jon talked at length about how he would like to have students learn to appreciate and understand mathematics "in a historical context." He described how he got the idea from watching one of his English professors:

> He showed how different types of literature were, were effects of the culture, or affected by the times and affected the times and the culture and, so, um, it seems that if I could, if I could apply that to math, if I could say "Now, now this is the math the Greeks had and this way in which Greek culture influenced its math and how math influenced its culture." And then I could like, I could show how, how math changed during the middle ages or <u>didn't</u> change, actually, and the Renaissance and into today. The way that some of Einstein's theories or some of the different kinds of geometries that we're developing today are changing the way that we look at universe and not just physically look at the universe but the way we <u>think</u> about the universe.

Tim, another mathematics major, became very agitated describing calculus:

> All of calculus, now, that's based on an assumption, now, isn't it? That you can calculate the derivative of a smooth function, which has never been proven , and people don't - all these professors just don't realize it. Issac Newton didn't prove it, you know. It's <u>never</u> been proven. . . I love calculus, but, you know, I'm just waiting some day for it to wash right away.

Jon and Tim both exhibited an awareness of mathematics as a field in which people construct and validate claims about new knowledge. None of the other teacher candidates talked about mathematics in this way: In fact, Jon's and Tim's perspectives brought the other teacher candidates' — especially the other mathematics majors' — views into sharp relief.

The statement was intended to elicit the prospective teachers' notions about the relative certainty and stability of mathematics. Yet such questions were quite remote from Mei Ling's ways of thinking about mathematics. Laughing, she compared the notion that mathematics includes some unsolved problems with the possibility of life in outer space, evidence that questions about the origins of mathematics were far from her mind. Cathy, another elementary candidate, disagreed with the statement. She asked me, "How would you do it if there was no answer?" Both Mei Ling and Cathy focused on mathematics in school and in texts. The nature of mathematical knowledge did not seem to be something they thought about.

But I also think that many teacher candidates were actually unsure about whether there was ambiguity or uncertainty in mathematics. When I tried another item, "There are unsolved problems in mathematics" during the second group of interviews, one teacher candidate looked up at me and said, "I don't know what you mean." Some said that they just didn't know if there were any problems outstanding in the field. Several were unsure whether mathematicians create new knowledge or whether they simply work with "what's already there." A few prospective teachers acknowledged that "somebody must" create knowledge in mathematics, but they were tentative and did not elaborate their answers. Only a few prospective teachers were sure that mathematics was a field in which there was uncertainty and unresolved dilemmas, and a field in which knowledge grows and changes.

Many of the prospective teachers were uncertain about the role of proof. Some remembered proofs as they had encountered them in high school — as a series of statements to be memorized, statements that show why something is true. On the card sort, I used the statement, "Proofs are a means of making arguments in mathematics" and discovered that the word "argument" confused some of the teacher candidates. Pam, an elementary candidate, said proofs were "absolute" — they show something absolutely — and why would one be "arguing against something concrete"? Several teacher candidates said they didn't really know what proofs were for, but that they remembered hating them in their math classes. Most seemed to regard writing proofs as a skill that one must perform in math classes. For example, Cathy, an elementary major, said she never understood the point of proofs. She described them in this way:

> Proofs. . .I never got the grasp of it. There were so many different — what were they even called? Formulas, they were numbered, there were these rules that you had to memorize to apply to the proofs to get to each step. They were horrible. I never understood them, you had to memorize them all and they were all so similar! I couldn't understand. It would be like one little catch that would separate it from another and I just never got the grasp of it all through high school. I'm sure that way of thinking got me started off wrong from the beginning because I took the defeatist attitude: "Why should I do this, it doesn't make sense?"

Cathy understood proofs as rules and steps to be memorized. Because it didn't make sense, she found proof discouraging, and she saw little reason for doing it.

Tim, a math major, saw the point of proof as providing "certain established principles." Like several of the other math majors, Tim seemed to see proofs as a given part of mathematics, but perhaps not as central as did Jon and Terrell. For example, in responding to the pupil who presented the drawing as a proof, Tim said that she had come up with a true theory about perimeter and area and that she just had a "misleading idea about proof." "All you really need to do," he said he would tell her, is to write a conventional proof for her claim instead of just giving an example of it. For Tim, proof was a necessary part of presenting a new idea but not so clearly tied to validating its truth.

Overall, the teacher candidates seemed to focus on substance when they considered mathematical questions. Issues of proof and the justification of knowledge were not ideas they thought about. Even when they were directly confronted with an instance where a claim was made without sufficient proof, they either did not recognize it as such or were not concerned. This preoccupation with substance parallels Romberg's (1983) distinction between the "record of knowledge" on one hand and the activities of mathematics on the other. Romberg argues that schools should concentrate on the latter, but frequently exclusively emphasize the former. This emphasis was recapitulated in the teacher candidates' focus on substance even in cases where, one might argue, questions of justification were of greater significance. Although the prospective teachers' own understandings of mathematics — topics like division by zero, for instance — were influenced by the their assumptions about mathematics, these were implicit and not something that most of the teacher candidates, even the secondary candidates, mentioned.

Notions About What Is Entailed in "Doing" Mathematics

Doing mathematics, from a disciplinary perspective, involves a wide range of activities, including looking for patterns, positing conjectures, pursuing hunches, drawing on accumulated knowledge and generating new ideas, modeling real-world phenomena and applying theoretical ideas as tools in concrete situations. In mathematical pedagogy, a significant aim is to provide students with genuine experiences with doing mathematics.

I used three different approaches to explore the teacher candidates' ideas about what "doing mathematics" entails. One source of insight lay in their responses to particular content-focused interview questions, a second from a series of questions on what it means to be "good at mathematics," and a third from a card sort task which included statements about mathematics and the teaching and learning of mathematics. Two views, neither of which resembled the disciplinary one, were dominant:

View #3: Doing mathematics means following set procedures.

View #4: Doing mathematics means using remembered knowledge and working step-by-step.

While these may seem almost the same, they differ in conceptions of the goal. I examine each one below.

View #3: Doing mathematics means following set procedures. One dominant theme was that doing mathematics involves using standard procedures to get the right answer to assigned problems. One place this came through vividly was when I presented the elementary candidates with a pupil's invented procedure for subtraction:

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Suppose one of your pupils told you that he or she had come up with a new way to do this that didn't require "all that crossing out." The pupil came up and showed you the following:

$$\begin{array}{r} \mathbf{36} \\ \underline{\mathbf{-19}} \\ \mathbf{-3} \\ \underline{\mathbf{+\ 20}} \\ \mathbf{17} \end{array}$$

What would you make of this and what do you think you'd say?

==

Only a third of the elementary teacher candidates understood the procedure completely and were sure it would work all the time.[58] Most thought it seemed to "work" (i.e., produce the right answer) in this case and wondered whether it would always work. Cathy, for example, was quite overwhelmed. "Oh, my God! Does this work out with other problems?" She said it would "freak her out" if a student presented her with something like this because she didn't know if it would work out "with other problems" and she didn't know how to figure that out. She said she would go through several examples, and if they all worked out, "I don't know what I'd do!" Several others were equally uncertain about whether the pupil's algorithm would always work and some worried that it "would cause problems later on" in working with larger numbers. They explained that the student wouldn't know "the concept"[59] and would get confused.

Only one teacher candidate mused about the possibility that this pupil might be inventing "the thing of the future." Although he was skeptical about the usefulness of the algorithm, he was the only person whose response reflected the idea doing mathematics could be an inventive or creative process.

All but one of the prospective teachers said that, even if the algorithm would always "work," they would discourage the pupil from using it. They said they would like the student to do subtraction "the standard way." Pam said she would tell the student that exploring is okay, but that "there's got to be some kind of form, some kind of procedure." Rachel objected to the algorithm because although it produced the right answer, it was "a backwards way of doing it." Other simply said that the usual procedure was "more acceptable."

The prospective teachers' uneven understanding of the underlying concepts was interwoven with their shaky reaction to the alternative subtraction algorithm. Since most didn't know whether it was a reasonable alternative — i.e., whether it would hold up — they felt insecure about allowing students to use it. Had they been able to determine its validity, would

[58] What the pupil has done is to subtract each place of the numbers separately and then to recombine the results. In other words, 6 - 9 = -3 in the ones place, while 30 - 1 = 20 in the tens place; 20 + -3 = 17. The procedure is similar to the way many people add numbers mentally without regrouping — e.g., 58 + 26 = 50 + 20 = 70, 8 + 6 = 14; 70 + 14 = 84.

[59] In Chapter 5, I discuss the teacher candidates' broad use of the term "concept."

they have been more enthusiastic of the novelty? I cannot know this, although the three teacher candidates who did seem to understand its basis said they would nevertheless discourage the pupil from using it.

The elementary teacher candidates' responses to this question suggested that they tended to think that doing mathematics involved using established procedures to solve problems one is given. The secondary teacher candidates also seemed to see mathematics this way. For example, in responding to the question on solving algebraic equations, not one secondary teacher candidate focused on the <u>meaning</u> of the equation. Instead they emphasized the steps that should be followed in order to solve for x. In general, they tended to emphasize the importance of knowing and using standard procedures — lining up numbers correctly in multiplication, and transforming equations into **y = mx + b** form, for example. The teacher candidates' view of "doing mathematics" clearly reflected what "doing math" <u>in school</u> usually means. There was little sign that they thought much about what "doing mathematics" meant in any broader sense — to mathematicians, for instance, or in the mathematical community. This finding is not surprising; it also fits with what Thompson (1984) has reported from her study of practicing mathematics teachers.

<u>View #4: Doing mathematics means using remembered knowledge and working step by step</u>. The teacher candidates' views of what it means to do mathematics also emerged from their answers to a pair of questions focused on what it means to be "good at math." I asked the teacher candidates to think of a person they knew whom they considered to be good at mathematics and one whom they would say was not good at mathematics. Then I asked them to describe each of these people — what they do and why they think they are good or not good at math. I designed these questions to learn about the teacher candidates' ideas about mathematics and about what doing mathematics involves, as well as to explore their ideas about the sources of success in mathematics.[60] I thought that if I could get them to talk about people whom they thought were good at math that I would learn something about what it takes to "do math" — and what mathematics entails.

The prospective teachers' ideas about what it means to be good at math — what is involved in <u>doing</u> math well — seemed to fall into two categories: a <u>knowledge</u> view and a <u>problem solving</u> view. The knowledge view emphasized remembering procedures and formulas and "grasping" concepts. The problem solving view held that being good at math means being able to solve problems either by thinking slowly and in a step-by-step manner or by thinking flexibly, considering alternatives.

The <u>knowledge</u> view was associated with getting right answers relatively easily and with the importance of study. According to this perspective, people who are good at math take more courses, and they study, even, according to Marsha, to "an extreme." Taking courses implied acquiring facts and procedures; studying seemed to mean practicing them. Marsha described a friend, who

> literally took notes into the shower and would just study in the shower,

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[60] I discuss my analysis of the teacher candidates' ideas about sources of success in mathematics in Chapter 5.

> equations, and memorize equations. He would come in, like an empty classroom, and just completely write on the board over and over and over.

People who subscribed to the knowledge view emphasized remembering concepts, formulas, and procedures. Allen said that people who are good at math have "studied it and they understand it and they still remember it" and Anne said that "people who are really good at math can remember all the formulas, remember why, and how, and get the right answer."

The knowledge view seemed to imply that math is a static body of knowledge — of particular facts, formulas, and procedures. Doing mathematics, according to this view, is a matter of remembering and plugging in correct procedures.

From the problem solving perspective, people who are good at math are good at "solving problems." They can consider alternative strategies and perspectives. Cindy explained that they understand the ideas; they have not just memorized formulas. Several others discussed the importance of taking things step-by-step.

The problem solving view implied a view of mathematics in which the goal is to solve problems. Although this sounds sophisticated, the prospective teachers' notions of what a "problem" is or what it means to "solve" a problem varied substantially. For many, a "problem" was simply a textbook exercise in which to goal is to follow a set of predetermined steps to find an answer.[61] In other words, the person who can easily complete the "problems" at the end of the textbook section is good at solving problems by remembering the material that has been taught. For a few other teacher candidates, a problem implied a truly thorny question whose solution depends on using ideas one understands in novel ways.

Whereas responses to the content-focused questions and the questions about being "good at math" suggested that the teacher candidates believed that mathematics involved following established procedures step-by-step, my analysis of the card sort task revealed several striking contrasts. Teacher candidates agreed with statements that seemed to me to contradict the perspectives they had revealed in other questions. Below I discuss two examples.

In describing people whom they considered to be good at mathematics, most of the teacher candidates seemed to place a high premium on computational skill, accuracy, and speed. Yet over half of them disagreed with the statement: "Being good at mathematics means being able to perform computations quickly and accurately." They pointed out that being able to calculate did not necessarily mean that one understood the "concepts." Their use of the term "concept," however, was unclear. Although they stressed the importance of knowing concepts, the term sometimes seemed to mean the steps of a procedure.[62]

In the interviews, most of the teacher candidates emphasized the importance of learning and using standard procedures in mathematics, and said they would discourage students from using their own novel strategies. On the card sort, however, over half disagreed with statement #4: "It is better to use an established procedure than to invent your own way of doing a problem." Only 3 agreed with the statement, a result that seemed to contradict my

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[61] According to Butts (1980), an exercise is a task for which a procedure for its solution is known (to the solver). A problem is a task whose solution is truly problematic, where no approach to solving it is yet known (by the solver). So, for example, finding the product of 786 and 35 is an exercise for an adult, but may be a problem for an eight-year-old.

interpretation of their interviews. In analyzing their comments, however, I discovered that several prospective teachers disagreed with the statement because it seemed to undermine the value of the idea: "in your <u>own</u> way." This led them to interpret the statement as a general affront to individuality or a denial of the value of creativity. One of the elementary candidates said it sounded like, "Don't be different." A secondary candidate commented that using established procedures required "only memorization" while inventing required true creativity; another else pointed out that it could "build a student's ego" to find their own ways to work on problems.

The cultural value placed on individuality, discussed earlier in this chapter, was once again a significant factor here. Important to note is the way in which this view is held completely separate from views of mathematics. In other words, the teacher candidates did not confront or grapple with what might be <u>legitimate</u> ways of doing things on one's own or being "creative" in mathematics. In fact, they did not seem to examine the dilemmas inherent in holding goals of fostering both individuality and disciplinary knowledge.

<u>Problems of interpreting apparent discrepancies in the data</u>. What could explain the apparent discrepancy between what the teacher candidates said during the interviews and their responses to the card sort statements? I find three possible explanations quite compelling.

Differences of interpretation undoubtedly accounted for some of the discord. Despite piloting and revision of the card sort statements as I was constructing them, the teacher candidates clearly interpreted terms differently than I did, and differently from one another. One funny example of this occurred with the statement, "I have always been anxious about mathematics in school." I was surprised when one of the math majors, who had talked enthusiastically about his love of mathematics, agreed with this statement. When he didn't comment on it, I probed. He looked at me oddly and said that he had <u>always</u> been anxious to go to math class because he liked it so much! The colloquial equation of "eager" with "anxious" led to this discord in the data. Terms like "argument," "concrete experience," "liberally educated," and "problem-solving" also elicited varied interpretations. Undoubtedly there were others that I didn't stumble upon as well. When teacher candidates commented about their choices or when I probed, I learned how they understood certain statements. However, since their interpretations were often different than what I had been thinking when I developed the items, the results often seemed, at first glance, startling.

I do not think, however, that variance in interpretation of items completely explains why the results of the card sort task seemed at odds with the interview responses. Another possibility is that the interview questions and card sort task did not tap the same thing. The interview responses revealed various dimensions of knowledge and belief in interaction with one another — understandings of particular mathematical topics, ideas about the teacher's role, ideas about pupils and about how they learn, as well as ideas <u>about</u> mathematics. The card sort statements were more "bald" — that is, they invited people to respond to them more unidimensionally. One could argue that their responses may reveal the components of their beliefs and understandings that are composed when they respond to specific teaching

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[62] In Chapter 5, I discuss the teacher candidates' broad use of the term "concept."

situations.[63] In other words, if one teacher candidate's responses indicated that she thought inventing procedures was a legitimate and valuable part of doing mathematics and that young children can engage in problem solving, one might predict that, consistent with those beliefs, she would encourage the student who invented an alternative algorithm for borrowing. Still, that is not the pattern that emerged in the data. The teacher candidates' responses to the interviews did not seem to be the sums of their responses to the card sort task. In context, their ideas <u>interact</u>, one understanding or assumption shaping another. In the interview, teacher candidates were inclined to discourage the pupil from using the invented procedure for subtraction because they lacked sufficient substantive mathematical understanding to be able to make sense of it. Prospective teachers who believed that the teacher's primary responsibility was to provide students with basic knowledge and skill also seemed to construe the student's invented algorithm as a diversion from the required curriculum, despite their professed belief in mathematics as a creative process. This interaction of knowledge and beliefs *in situ* suggests that, compared to the interview, the card sort may not be as valid a measure of prospective teachers' ways of <u>thinking</u>. Still, it can contribute to understanding the pieces of belief held by prospective teachers.

Another possible explanation for the apparent discrepancy between the results of the card sort task and the interview responses is that the teacher candidates knew the "right" things to say. Perhaps they knew the appropriate positions about mathematics — that problem solving is central, that inventing new procedures is a legitimate part of "doing" mathematics, that getting right answers is not all there is to mathematics. The card sort analysis shows that many of the prospective teachers responded in ways that were quite consistent with currently favored views, in contrast with their responses on the interviews. But they may have just known that they were <u>supposed</u> to believe certain things. If so, it would make sense that they would respond to the isolated statements of the card sort in one way and yet reveal other points of view when talking on their own about particular situations in teaching. This interpretation suggests that the card sort analysis is helpful in uncovering the extent of prospective teachers' understanding of "intellectually appropriate" positions about mathematics (since the task required them to respond to already-formulated perspectives).

<u>Ideas About Purposes for Learning Mathematics</u>

Cathy, above, saw no reason to do proofs. In analyzing the prospective teachers' understandings <u>about</u> mathematics, one final issue I examined was the prospective teachers' ideas about the purposes for learning mathematics, about the value of mathematics as a subject. I pursued this at two levels. Most generally, what is the value of mathematics? Why study or know it? Specifically, what is the reason to learn a particular branch or piece of mathematics — constructing proofs, multiplication, or geometry, for instance? I used some card sort items for the more general level. To examine the teacher candidates' ideas at the more specific topical level, I posed a question from an obstreperous pupil: "Why are we learning this?" I used this question within the longer exercises on slope and graphing (secondary) and subtraction with

[63] Examining this assumption was part of my reason for using a card sort task.

regrouping (elementary).

Many of the teacher candidates had apparently never thought explicitly about purposes for studying mathematics or about the subject's value. Those who liked mathematics took it because they enjoyed it. Those who didn't like math stopped taking it as soon as they could. Because of the wide variation in mathematics background, the teacher candidates even meant different things by "mathematics." Thus, questions of value and purpose had different referents for different people. Mathematics majors wondered about the value of groups and rings or graphing functions while elementary majors were thinking about fractions. Still, their ideas about the value of learning mathematics can be summarized in three main propositions:

View #5: Mathematics is essential for everyday life.

View #6: Mathematics helps one learn to think.

View #7: Mathematics is just "there," but it is necessary for progress in school.

I briefly examine each of these in turn in this section.

View #5: Mathematics is essential for everyday life. This variation in the referent for "mathematics" was obvious in teacher candidates' remarks about the application of math in everyday life. Andy, a secondary candidate, for example, was thinking about the usefulness of calculus in physics, while others emphasized the need for computational skills to balance checkbooks and buy wallpaper. I asked the elementary candidates what they would say to a pupil who asked about learning to subtract with regrouping (or to "borrow"), "Why are we learning this? I already have a calculator and I can do these problems on there." The most common response was to tell the pupil that a calculator won't always be available, that it might run out of batteries or break. Their examples — being at the store and not being sure whether or not one has enough money to pay for groceries, balancing the checkbook, building a house on a desert island — did not necessarily involve subtraction.

Although all the prospective teachers thought that mathematics was useful in everyday life, they often had difficulty thinking of specific applications of particular concepts or topics. For example, several teacher candidates commented that fractions were difficult because it was hard to relate $\frac{3}{4}$ to something real. The secondary candidates wanted to make slope more exciting, interesting, and show ways that it could be used, but found it difficult to think of "real world examples." Some grasped at ski slopes (see section on understanding slope and graphing). Jon, however, said he would want to avoid saying something like, "This helps us understand how sleds go fast." These difficulties were evidence of the unconnectedness of their own substantive knowledge, discussed in Chapter 3.

View #6: Mathematics helps one learn to think. In the most general sense, many of the prospective teachers expressed the view that learning mathematics could help one learn to think. The statement, "Mathematics can help you learn to think better," drew unusual consensus. Barb, the only one who said she was unsure, nevertheless acknowledged that math can probably "help you think more logically, take a look at things in a more step-by-step fashion." Janet, a secondary major, said she thought studying mathematics could help a person who thinks in a more integrative way ("a social studies or English person") learn to "think mathematically" — i.e., "very analytically — to pick things apart." And Cathy said she would tell the pupil who asked why it was important to learn subtraction:

> You need math to make you think. . . Doing these problems makes you think. It develops your logic. . .It makes your mind work.

In a vague way, the teacher candidates reflected the commonsense "mental muscle" view of learning, according to which one's brain gets stronger from studying mathematics. This was, however, not a dominant theme across the interviews.

View #7: Mathematics is just "there" but it's necessary for progress in school. Since the teacher candidates were quite focused on school mathematics, this purpose came easily to them when they were thinking about particular topics. Rachel, an elementary candidate, said that subtraction with regrouping was needed to get into "higher advanced math classes."

I asked the secondary candidates how they would respond to a student who asked why they needed to learn slope. Andy said,

> You just have to try, you have to convince the students that this is gonna be something that they're gonna need later on. . . .That's why before you present most material or something, you should probably say what it's going to relate to or why you're gonna use it.

As he found it difficult to do this for slope, Andy said he would just tell the student:

> Well, this is a requirement of the class and we're presenting it for people who are going on. You have to do it. And if you don't want to do it, then I guess, you won't get a good grade.

Others found themselves in the same predicament as Andy. Carol said "he has to learn it 'cause he's in the class" and she would probably just admit that "this isn't the most interesting thing you'll do in your life, but this is what we've got to do right now."

The teacher candidates' default to the taken-for-granted view that mathematics was necessary for progress in school as a justification of its value was yet another sign of the generally rule-bound and unconnected nature of their substantive knowledge. Although they saw the general importance of mathematics, they were vague about specific values or utility of particular mathematical ideas or topics as a consequent of their own narrow understanding of those areas.

Is learning mathematics part of becoming an educated person? No one independently mentioned any intrinsic value of learning mathematics, nor its role in being well-educated. I had, however, included one statement on the card sort task aimed at this idea. The item, "To be a liberally educated person, studying major strands of mathematics is just as important as reading classic literary works," elicited considerable agreement. Only 5 teacher candidates chose to comment on this item (they were asked to comment on any statements they wished and especially on those about which they felt strongly). When I probed, I found, unsurprisingly, that the teacher candidates varied in how they understood the phrase "liberally educated person."

Some teacher candidates responded to this item in terms of what it took to be considered well-educated by others. Terrell, a secondary major, said that although he cared about studying math, he wasn't sure if knowing mathematics was essential to being considered well-educated. Linda, an elementary major, took a similar stance. She said she thought that "people should read a lot and know a lot about different countries and art and stuff like that. I just don't think that in everyday life someone can tell that you are good at math, but they can tell if you have read a lot."

Two of the mathematics majors were the only ones to focus on the liberal. Cindy thought people should learn "not how you integrate or differentiate, but maybe how mathematics as a science developed, personally. . .study people like Descartes." She thought that this might give people an appreciation that "math isn't just multiplying and dividing and things like that." This way of thinking about what it meant to be well-educated in mathematics was quite different from what Mei Ling meant when she said people should be strong in the all the "basic" areas of reading, writing, math, and science:

> The things you would use everyday, you know, everyday in just living — like adding, subtraction, division, multiplication, percentages, you know, things like that you would need when you got out. Outside of school. I think those would be the basic things that I would make sure that they knew.

Although they both agreed with the statement, Cindy and Mei Ling actually differed considerably in their ideas about what a good mathematical education entailed.

Summary: Prospective Teachers' Ideas About the Nature of Mathematics

Although the prospective teachers had many ideas about the nature of mathematics, these ideas were implicit, built up largely out of years of experience in math classrooms and from living in a culture in which mathematics is both revered and reviled. While these ideas influenced the ways in which they experienced mathematics, the prospective teachers did not seem to have given the matter much thought. Unlike their understandings of the substance of mathematics, which some of them wished to increase or deepen, the teacher candidates had never considered their understandings about mathematics. They did not seem dissatisfied, nor did they even seem to think explicitly about them. Many had probably never talked about these ideas before.

In the next section, I discuss a third major component of the teacher candidates' understandings of mathematics: their feelings about the subject and about themselves in relation to it. Like their understandings about mathematics, their feelings and inclinations about math shaped the way they thought about and experienced mathematics, and were tightly bound up with their substantive understandings of the discipline.

Prospective Teachers' Feelings About Mathematics and About Themselves

In addition to specific substantive knowledge and ideas about the subject discussed thus far, understanding mathematics is affected by one's emotional responses to the subject and one's inclinations and sense of self in relation to it. While knowledge of and about the subject is a framework for examining subject matter knowledge that applies equally in other disciplines (e.g., history, see Wilson, 1988; or English, Grossman, 1988)this affective dimension of understanding may be uniquely significant in mathematics.

Webster's Third New Unabridged Dictionary defines attitude as "a disposition that is primarily grounded in affect and emotion." A tradition of research interest in this area exists. Researchers have investigated teachers' and prospective teachers' "attitudes" toward mathematics (e.g., Bassarear, 1986; Brandau, 1985; Dutton, 1954; Ferrini-Mundy, 1986; McLeod, 1986; Smith, 1964; Thompson, 1984). They have developed scales with which to measure teachers' interest in, enjoyment of, and confidence with mathematics, generally treating attitude as separate from cognitive issues of understanding.

In mathematics, however, I argue that feeling cannot be separated from thinking and knowing. People's understandings of mathematics are interrelated with how they feel about themselves and about mathematics. This was clear over and over again in my interviews with prospective teachers. For example, when Sandi tried to generate a story to represent $1\frac{3}{4} \div \frac{1}{2}$, she found that her answer didn't make sense to her. Her calculations produced $3\frac{1}{2}$, yet her story yielded an answer of $\frac{7}{8}$ (see Chapter 3). She grew anxious about this discrepancy and began to doubt her calculation.

> Sandi: Is that right, seven, let's see that's what I was doing and it doesn't work out 'cause — that's unreal — $3\frac{1}{2}$ is not. . . I did something wrong.
>
> Ball: Why do you think it is unreal? What's bothering you about it?
>
> Sandi: I don't know, um, well obviously half of this [$1\frac{3}{4}$] is not this [$3\frac{1}{2}$], so I don't know what I've done, obviously I don't like fractions.

When I probed, Sandi said that fractions were "not something that I'm comfortable with."

This example, purportedly evidence of Sandi's understanding of division, is equally revealing of her feelings about herself and about the subject. In this case, when the answer didn't make sense to her, she doubted herself ("I must have done something wrong") and commented on how she didn't like the topic.

The relationship between knowing and feeling also shows up when the focus appears to be on feelings as well. Take the questionnaire statement, "I am attracted to mathematics because there is always a right answer" (Ferrini-Mundy, 1986). While this appears to be a measure of "attitude toward mathematics," agreeing or disagreeing with this statement depends on what one knows about mathematics as a discipline as well as how one feels about the subject.

Three approaches were used to explore prospective teachers' feelings about mathematics, and the relationship between those feelings and their ideas about mathematics. One simple way of learning how prospective teachers felt about themselves and about mathematics was to present them with statements to which they could respond in isolation. This method was not significantly different from what other researchers have done in using survey instruments or questionnaires. However, since I administered these in the context of an interview, I was able to learn something about how the teacher candidates interpreted the statements. This added to the value of the approach. Still, this strategy of presenting the prospective teachers with statements did tend to separate questions of feeling or attitude from issues of understanding.

A second method, better suited to learning about the relationship between affect and cognition, was to pay attention to affective dimensions of the teacher candidates' responses to all the interview questions, especially the ones in which they had to deal with pieces of mathematics. Giggles, sighs, exclamations, shaky voices, and explicit comments all contributed information about the interaction between their feelings and understandings.

Third, I suspected that a biographical angle would be useful. I asked the prospective teachers what they remembered about their experience with mathematics throughout school — in elementary, middle, and high school, as well as in college. I probed what they took, what they remembered learning, any parental influences, what stood out about their teachers or any other aspects of their experience with mathematics. Their accounts helped me understand how they felt about themselves and about mathematics, and how those feelings both affected and were influenced by their school experiences with mathematics.

As one would expect, I found considerable variation among the teacher candidates regarding their feelings about mathematics. Some seemed to really enjoy math and were eager to take additional courses; others were obviously tense about the subject. Some were confident in their own ability to do and understand mathematics; some believed that they were not good at math. In order to give the reader a sense of how teacher candidates felt and how these feelings interacted both with their personal history and knowledge, I categorized the teacher candidates into four groups, relying on their own portrayals of themselves. These groups represent (roughly) the predominant patterns that I found***: highly anxious, matter-of-fact, solid, and passionate.*** Table 4.2 shows the distribution of prospective elementary and secondary teachers among these four groups. In this section, I describe and illustrate each of these groups.

Table 4.2
Teacher candidates' feelings about mathematics

	Highly Anxious	Matter-of Fact	Solid	Passionate
Elementary	4	4	2	0
Secondary	0	1	5	3

The Highly Anxious Teacher Candidate

Twelve prospective teachers agreed with the statement, "I have always been anxious about mathematics in school." For some this just meant that they worried before tests, or found new content difficult. Others, especially prospective elementary teachers, were anxious about mathematics. In their interviews, they repeatedly used words like "petrified," "terrified," and "frustrating" to talk about themselves and math.

These prospective teachers remembered the least school mathematics, not surprising since their level of mathematics study was the least of the group. One had taken no mathematics in high school; the others had taken two years or less. Their performance on the university placement exam landed them in the remedial arithmetic and algebra sequence, but only one had taken the course by the time I interviewed her. The others were delaying it because the prospect of taking the course made them so nervous. Because the remedial course was required in order to take any subsequent required courses for Table 4.2 teachers, putting it off delayed their progress in the program. One said she really had a "phobia" about math which she was trying to get over, another said she was "psyching" herself up to take the course because she had heard that it was hard, a third described herself as "petrified." Of course they had taken no other college math classes.

Several of the teacher candidates said they disliked, even hated, math. One said if she could get through the rest of her life "without ever doing another math problem," she "wouldn't cry." However, the general pattern was that they liked math in elementary school, and even did well at it. They described it as easy. As a result, most of them felt confident about their understanding of addition, subtraction, multiplication, and division of whole numbers. Still, when it came to thinking about <u>teaching</u>, their fears extended even to this elementary content. Linda, who had been observing in a first grade classroom, said she was "honestly nervous" when the teacher asked her to check the children's math papers. Linda said she wanted to teach first grade because she would be worried about teaching anything beyond adding and subtracting — like long division.

Beyond this, however, they all felt unsure. With either algebra or geometry in high school, they reported that their feelings changed. Marsha said she couldn't stand "solving for x and y" and "plotting the line on that thing that goes like that" (in the air she sketched a coordinate grid). Linda explained simply that, in algebra, "I got to a point where I didn't understand it any more" and after that, she avoided math. While Cathy had enjoyed algebra, "proofs were just a nightmare" in geometry. While these anxious teacher candidates were willing to deal with what Cathy repeatedly referred to as "the Basics," they disliked and felt stupid in what they thought of as "higher math." Their anxiety seemed to grow out a convergence of feeling dumb and not liking math.

The highly anxious teacher candidates felt overwhelmed by some of the interview questions that dealt with specific mathematics and they gave up easily. They exclaimed things like, "Oh my God! I have no idea!" when presented with puzzling math problems, assuming that they couldn't figure them out. None of them could generate any kind of representation for $1\frac{3}{4} \div \frac{1}{2}$, nor could they solve the equation

$\frac{x}{0.2} = 5$. When they examined the student's conjecture about perimeter 0.2 and area, they were unsure and looked to me: "Is it true?" One said that things "like this" terrified her.

Part of their reaction to these questions stemmed from their view of mathematics as a collection of facts to be remembered. This view brought feelings of security and confidence in some situations and feelings of inadequacy in others. When they did feel sure they knew the answer, it was because they could remember a rule or principle. All of them, for example, confidently explained division by zero as "anything divided by zero is zero." Marsha said that it was "always a big relief to see a 0 in a problem because then [the answer] was always 0." When they couldn't remember, they said they didn't know and rarely tried to reason about the question. They said they would have to look it up or ask someone.

The teacher candidates' reaction also grew out of their feelings about themselves as "not good at math." Although three of them said there was no such a thing as a "mathematical type" when presented with the statement, "I'm not the type to do well in mathematics" (one of the four agreed definitely with the item), they nevertheless described themselves in the interviews as good at English and "creative things," and not good at math. Cathy said she didn't have "that kind of mathematical mind," that she was "more into the creative like writing." This dichotomous way of thinking reflected, once again, a view of mathematics as a body of facts to be accepted and remembered. Some of the teacher candidates believed that being good at math was generally also a sign of greater intelligence and defended themselves, remarking on things they were good at, such as reading, writing, and common sense.

Other people, boyfriends and parents had apparently influenced the prospective teachers' self-perceptions. For example, when Linda told her father that she was in the remedial math sequence at the university, he was not surprised: "Of course you are," and when she offered to help her younger brother with his math homework, her father "burst out laughing [because] he just thinks that's hysterical." Her mother just told Linda, "Do the best you can." Other teacher candidates described boyfriends who tried to get them to think logically and step-by-step, instead of "wandering."

The highly anxious teacher candidates had the weakest formal background in mathematics, a subject they viewed as a body of facts to be remembered. With outside influences contributing, they had developed an image of themselves as not "the type" to be good at math.

The Matter-of-Fact Teacher Candidate

Other teacher candidates were unsure about their competence with mathematics but much less anxious about the subject. I describe them as "matter-of-fact" because although they did not think of themselves as particularly good at math, they did not have intense feelings nor strong orientations toward the subject. They were considerably calmer than the highly anxious teacher candidates and were less bowled over by the interview questions. Their backgrounds were not especially strong and when they didn't know something, they were inclined to try to figure it out. The statement, "I feel okay about mathematics. While I don't feel especially strong in mathematics, I am not fearful of it either," captured their stance. One said, "That's it exactly!" when she sorted that item.

Five teacher candidates, four elementary candidates and one mathematics major, formed the "matter-of-fact" group. This group was, however, perhaps the most varied of the four groups, varying in their math background and understanding, as well as in how much they liked or disliked the subject.

The elementary majors believed that they were not well-suited to mathematics, although they mostly enjoyed and felt they were able to do "basic" math — i.e., up through algebra. They attributed not being good at math both to the nature of "higher" mathematics as well as to what they were like as individuals. Sandi, for instance, said she wasn't good at mathematics because, in math, you have to accept things on faith and she was the "type of person who always wants to know why." She always did well in her math classes, but simply didn't find it that intellectually interesting once she got beyond algebra: "It just got to be a chore." Mei Ling explained that her thoughts were "more artistic" — that to be good at math, she would have to be more "analytical" and be able to focus her attention one point at a time. Both Sandi and Teri had an view of mathematics as a rigid, linear subject and, in reflecting on themselves, decided that they were not inclined toward that kind of thinking. Rachel said simply that many people "do bad in math" because "it's something that is easy to mess up."

Barb, the mathematics major, had a similar perspective, although it was translated up several levels. Instead of making a cutoff in high school math, Barb's "basic level" of mathematics encompassed algebra, geometry, and basic calculus, all of which she enjoyed and felt she could do. However, she had "real difficulties in the upper level mathematics courses — abstract algebra, group theory, rings" and that no matter how hard she worked, something held her back. Like the elementary candidates, Barb had doubts that she could do well at higher level mathematics, but her notion on what counted as "higher mathematics" was different. These teacher candidates were matter-of-fact about their conclusions about themselves in relation to mathematics. They thought of these as differences among people, not as a sign of being more or less intelligent.

The elementary candidates in this group had much stronger math backgrounds than those in the highly anxious group. They had taken, on the average, a little more math in high school (three or four years). But, like the others, they said they stopped liking math in high school, mostly in geometry. Teri said she had "bombed" because of all the proofs and Mei Ling had a "mean teacher" who used to embarrass her in front of the class. Although they generally didn't like math, they were not avoiders. They did not drop out of math in during or after high school. All had taken some college math: Two had taken college algebra and trigonometry, one had two terms of calculus, two had taken the math course for elementary teachers, and all had earned A's or B's. Barb, the secondary candidate had completed all the course requirements for the mathematics major, earning mostly B's.

Although they had taken relatively more mathematics, this group could remember little of the substance of their courses either in high school or in college. When I asked Rachel about the algebra class she had recently taken (and in which she had done well), she was vague:

> I know we got into polynomials and got into. . .oh, what's that?. . .I could write it out for you, but I can't remember the name of it. (She writes the quadratic formula.) I know we went over that.

Mei Ling couldn't remember anything about her analysis class in her senior year, and Teri just explained that "in high school everything is more important" than homework and classes. Most of the prospective teachers could remember more about elementary school than about high school or college.

Their confidence with mathematics fluctuated depending on the situation and on the math that was involved. At times they felt they could do math, at other times, they felt unsure. Mei Ling explained:

> It all depends — on the situation. It depends on the people I'm with. What degree of mathematic ability we are talking about (laughs) You know, if it is a common everyday situation, I am very confident. . . Like I mean well, just basic problems. Something I would face in everyday life, how to balance a checkbook, simple things. Then when you start getting into analysis, I start to feel shaky.

This fluctuating confidence showed up during the interviews. On one hand, they listened and responded to questions calmly. Although they had difficulty with many of the questions, only one gave "I don't know" as an answer. When they didn't understand something in a question, they were inclined to try to puzzle about it. On the other hand, they did not always feel confident in their solutions. For example, several of them tried to figure out "why the numbers move over" for each partial product in the multiplication algorithm. After she tried to explain it, Mei Ling turned to me and asked, "Why do you put the zero there??" They also tried to figure out what division by 0 meant. Rachel worked aloud through several alternatives for over ten minutes. She considered various interpretations of the meaning of division by 0, stopping to check the sense of what she was doing along the way. Eventually she decided that there was no solution to 7 ÷ 0, but she was not confident that she was right and said she was going to ask her math professor.

The prospective teachers in this group were considerably less intense about mathematics than any of the others. They had a matter-of-fact orientation to the subject and to themselves in relation to it. While their stronger background in math had not helped them to construct the kinds of substantive understandings tapped in the interview, it did seem to have contributed to their sense that they might be able to figure things out in mathematics.

The Solid Teacher Candidate

The group of prospective teachers whom I dubbed "solid" had a more substantial mathematics background and a sense that they knew math well. Two elementary and five secondary teacher candidates belonged to this group. They all enjoyed mathematics, and said they had always liked the subject.

These teacher candidates had studied a lot more mathematics than the ones I described above. All had had four years of high school mathematics; several had taken advanced placement calculus as seniors. Anne and Allen, the two elementary candidates, had taken college algebra and trigonometry, and calculus, as well as the math course for elementary teachers. Anne's grades — A's and B's — were better than Allen's, but Allen had taken more courses. The secondary candidates had all completed the calculus sequence and had taken various other upper-level courses, such as number theory, geometry, and matrices. They had earned mostly A's and B's. While their academic records do not tell us about their understandings, the sheer difference in amount of formal mathematical experience between these teacher candidates and the anxious and matter-of-fact candidates is worth noting.

Their more extensive background showed up clearly in their sense of confidence. Completing all those courses seemed to be a kind of entitlement to mathematical security. While they knew they could be stumped or confused, they also believed that they knew enough mathematics to be able to become unstuck. They had had ample experience doing so. As Mark said, he had "<u>so</u> much math" that he felt "relaxed as far as the material goes." Most agreed with the statement, "I usually feel confident about my ability in mathematics." Cindy said simply, "I've always been good at it." Several mentioned that they know they can get the right answer to a problem and they attributed this to working hard and putting in effort, rather than being "naturally" inclined to do well at math. They talked a lot about "motivating" themselves to try and to do well in math. The flip side of this was that they also <u>expected</u> themselves to know the mathematics in the interview questions. Andy said he was nervous that he wouldn't know something — and this was only eighth or ninth grade "stuff." Cindy laughingly worried about whether any of her math professors would see the tape of our interview and say "that girl doesn't know any math."

These prospective teachers said they enjoyed being challenged and solving puzzling problems. Janet said that solving difficult problems gave her a "really good feeling — yeah, I can <u>do</u> this" and Anne said that that is what she liked about mathematics. During the interviews, though, they seemed to find few questions problematic. They often knew an answer for the situation described in the question and many of them searched no further. When they had come up with an answer, they seemed quite satisfied with what they had done. For example, over half of them were content to tell a student $7 \div 0$ was undefined and that they needed to use 0 as a placeholder in the multiplication algorithm. All of them explained how to solve an algebraic equation by enumerating the steps of the procedure.

The difference between these prospective teachers and the others I have described is that they felt they <u>knew</u> the content — thus, they had confidence in what they retrieved from memory. Like the others, though, they did seem to regard mathematics more as a body of knowledge to be mastered — and in their own case, stuff they should already know.

The Passionate Teacher Candidate

Three secondary teacher candidates exhibited a degree of enthusiasm and excitement about mathematics that surpassed the way in which the "solid" prospective teachers simply liked math. Their passion extended to the doing of mathematics, beyond "knowing" it, like the previous group.

These teacher candidates had solid mathematics backgrounds, similar to those in the previous group. Like the others, they had taken four years of high school math and had taken many college math courses. Their grades, however, were slightly lower.

Like the previous group, while these prospective teachers said they really enjoyed puzzles and challenges; however, they tended to see more dimensions and problems in the mathematics of the interview questions. Confident in their own knowledge, they also seemed interested in understanding and having students understand more about the questions beyond what they could remember learning in their classes. For example, in responding to the student who proposed that area and perimeter were related, all of them considered the insufficiency of one example to establish a theory. Terrell became completely caught up in trying to figure out the conjecture. They were intrigued with trying to generate representations for $1\frac{3}{4} \div \frac{1}{2}$ and to explain $7 \div 0$. Jon was excited about a student asking about the slope of a curved line and when Tim looked at examples of student work, he spoke fervently about the importance of having students consider the reasonableness and magnitude of their answers.

These prospective teachers also appeared to be the most relaxed about the mathematics of anyone I interviewed. They didn't seem to have the same expectations of themselves as did the "solid" types. Although Terrell said that it was "scary" not to know "this simple stuff," he thought it was fun to work with. They were at the same time, the most successful at generating conceptual answers, explanations, and examples for the mathematics in the questions.

Discussion:

Teacher Candidates' Feelings About Themselves and About Mathematics

The teacher candidates' feelings about mathematics and about themselves were interwoven with how much mathematics they felt they knew, the view of the subject they had, what they believed about mathematical ability, as well as how they thought about themselves. Not surprisingly, the most anxious teacher candidates also had the weakest formal background and thought of mathematics as a collection of arbitrary facts. They generally also thought of mathematical ability as innate. This combination of ideas seemed to spawn intense feelings of dislike, fear, and anxiety. In contrast, the teacher candidates whom I called "solid," thought they knew mathematics, thought mathematics was a body of knowledge, and thought mathematical ability was largely a matter of effort and desire. This set of ideas produces a calmer feeling of confidence and control.

These complex combinations of ideas and feelings also influenced their responses on the interviews. Cathy's fear at having to respond to a student who proposed an alternative method for subtraction grew out of the way she thought about knowledge in mathematics, the way she understood what doing math entails, the way she understood subtraction with regrouping, as well as her sense of herself. She didn't know whether the proposed procedure would always work nor how to find out. It scared her to have to think about this and made her feel "dumb." At least, she thought, she could teach the standard procedure — and after all, doing well in mathematics requires remembering formulas and algorithms. Therefore, she concluded, she really should teach the child the standard algorithm. Prospective teachers' feelings are part of the way they participate in and understand mathematics, not a separate affective dimension called "attitude."

Summary: Prospective Teachers' Ideas and Feelings About Mathematics

What does all of this contribute to understanding the ways in which prospective teachers know and think about mathematics? Three issues stand out from the results discussed in this chapter: the teacher candidates' primary emphasis on mathematical answers and knowledge and concomitant disregard of the nature of mathematics, teacher candidates' sense of the "right" answers, and the interweaving of knowledge of and about mathematics together with their feelings about mathematics. Each of these suggests an area of focus for teacher educators trying to learn about teacher candidates.

Focus on Substance

What place is occupied by knowledge **about** mathematics in teacher candidates' understandings of mathematics? To what extent do prospective teachers see connections between mathematical knowledge and ways of knowing in mathematics? On a question that I designed to elicit their ideas about theory and proof, only three teacher candidates seemed to consider the issue of justification of mathematical knowledge as an student outcome worthy of explicit attention. The others focused on the substance of the proposed "theory," and seemed to assume that they should know whether the theory was correct or not.

This is not to say that the prospective teachers did not have ideas about mathematics that were evident in their responses. Assuming, for example, that mathematical knowledge is an arbitrary collection of facts, some teacher candidates were not inclined to reason about things they didn't know. Similarly, because they thought mathematics was a domain in which using standard procedures mattered, they were loath to let students use invented algorithms. As one person explained, "math is so contradictory" — there are so many "weirdo" exceptions — that there was little point in trying to decide something on logical grounds or use alternative strategies.

Although the teacher candidates had ideas about the nature of mathematical knowledge, about what it meant to "do" mathematics, and about the purpose of learning math, these ideas seemed to be the backdrop for their mathematical activity, and were taken for granted as givens about the subject. While many of these implicit assumptions misrepresented the essence of mathematics, there was no evidence that the prospective teachers were dissatisfied with their perspectives nor that they were concerned with revising their understandings about mathematics. Teacher educators might try to learn the views of mathematics held by their students, looking both at explicit statements and at implicit views revealed through the student's actions.

Knowing the "Right" Answers

In any study of people's ideas and feelings, the fact that people know the "socially appropriate" positions interferes with learning what they "really" think or feel. This is especially problematic when trying to uncover people's reactions to people of different ethnic groups, for example. One might not have expected this characteristic of human behavior to show up in a study about college students' understandings of mathematics. Still, it seems likely that some of the teacher candidates' responses were colored by their sense of the "right" things to say about mathematics. They talked generally about the importance of "concepts," "problem solving," and alternative strategies, and yet focused on the importance of rules and standard algorithms when they dealt with the specific teaching situations in the interview.

Not necessarily contradictory, this finding may suggest that although they know the "right" words, teacher candidates' understanding of the terms may differ from what teachers or teacher educators think when they use the same terms. "Problem-solving" seemed to mean solving "problems" such as 4 apples plus 2 apples, or 5 + 3 to some prospective teachers, while "concept" seemed to mean any piece of mathematical knowledge — such as crossing out in borrowing.

Discrepancies may also arise out of the gap between teacher candidates' beliefs and their knowledge. Prospective teachers who want to emphasize "conceptual understanding" and making connections to the real world may be nevertheless substantively unprepared to do so. The secondary candidates who wanted to give real world applications of slope and graphing simply didn't have any examples available.

A final interesting piece of what can be learned out of the teacher candidates' awareness of the "right" answers lies in the inverse domain — the things about which they did not seem to know the "right" answers. For example, several prospective teachers thought that the important uses of mathematics involved balancing one's checkbook and calculating totals at the grocery store. Others thought that some people cannot learn mathematics. There are important clues for teacher educators in what the prospective teachers didn't know to say as well as in what they did. About what things, for example, do prospective teachers talk as teacher educators would want them to? Where do they say things that are quite different?

The Interweaving of Feeling and Thinking in Knowing Mathematics

Throughout the interviews, I was struck with how much the teacher candidates' knowledge, ways of thinking, beliefs, and feeling interacted in their responses. Their approaches to figuring out a puzzling problem were colored by their self-confidence, their repertoire of strategies, what they could remember about the related concepts, as well as what they believed about the fruitfulness of trying to figure out the problem in the first place. Together these comprised their understanding — the way they participated in and experienced mathematics. I argue that consideration of prospective teachers' feelings and knowledge go hand in hand, that teacher educators might try to examine how teacher candidates approach and participate in mathematical activity.

* * *

In the next chapter, I let the teacher candidates' understandings of mathematics form the backdrop for a discussion of their ideas about the teaching and learning of mathematics, and about the people they will eventually teach — the learners. This chapter, however, is a critical backdrop for that discussion, for, as I discuss in Chapter 6, these ideas and understandings — about mathematics, about teaching and learning, about learners — interacted in the teacher candidates' thinking about concrete teaching situations.

CHAPTER 5

PROSPECTIVE TEACHERS' IDEAS ABOUT TEACHING AND LEARNING MATHEMATICS AND ABOUT STUDENTS AS LEARNERS OF MATHEMATICS

> I'm going to be teaching this next Tuesday morning at 8 o'clock. They'll be coming up to the board — they're not gonna like it, but. . . I'm going to do a lot of talking, too. I'll just present the xy-coordinate plane as a beginning thing: "Well, remember we had the number line and then we had 0, 1, 2, 3. Well, everything's not just on the one line. What if you, you know, what if you have a circle or something? You can't put <u>that</u> on the number line, or a square or whatever, so you have, um, two — " I'll say that, and then I'm gonna have the xy-coordinate plane already drawn on a flip-over chalkboard and then I'll flip it over and say, "<u>This</u> is . . . ", [said dramatically] Then, you know, I'll present like this is the y, this is the x, first quadrant, second quadrant, you know, just basic stuff like that. . . And then just practice plotting points. And then I guess you could go from there. I have some ideas about, you know, just <u>presenting</u> things.
> (Andy, secondary teacher candidate)

Listening to this prospective teacher describe a teaching plan reveals that, although he is just entering a teacher education program, he already has an image of himself as a teacher. He has concrete notions about things he will do to introduce students to the xy-coordinate plane, including what he'll say and how he'll set up his presentation. These notions seem like "basic stuff" to him. He will be "presenting" — talking and asking his students questions. To ensure that they learn, he imagines he'll give them practice, although he expects that they won't like coming up to the board. His plan trails off: He will just "go on from there." Andy's image reflects ideas and assumptions about teaching and learning and about pupils that he has picked up outside of any formal professional training.

In addition to knowledge, beliefs, and feelings about mathematics as a subject, prospective teachers like Andy bring ideas about the teaching and learning of mathematics. They hold images of what it will be like to be a teacher and of what they will actually do. Their ideas and images, both explicit and tacit, shape the ways in which prospective teachers think about teaching and learning to teach mathematics. This chapter focuses on a set of these ideas, drawn from my interviews with the entering teacher education students in this study.[64] I begin with a discussion of the sources of and influences on their images of and ideas about mathematics teaching and learning. Next I describe the ideas and images themselves, examining common and commonsense themes that run throughout. In the third section, I discuss their ideas about students as learners of mathematics. The chapter finishes with a look at what the teacher candidates did <u>not</u> seem to know.

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[64] I explain my rationale for focusing on these particular ideas in Chapter 1.

Where Do Prospective Teachers Learn About Teaching Mathematics?

American prospective teachers' ideas and images about mathematics teaching stem largely from two sets of overlapping influences: prior school experience and mainstream American culture. While others have noted the influence of prospective teachers' own schooling (e.g., Lortie, 1975, Feiman-Nemser & Buchmann, 1986), fewer have paid attention to the role played by cultural images[65], or to the specific influences of either of these sources on the teaching of particular subjects — in this case, mathematics. As students themselves, their time spent in mathematics classrooms gives prospective teachers a specialized "apprenticeship of observation" (Lortie, 1975). Watching their math teachers, they may have acquired scripts for teaching topics such as long division and factoring (Putnam, 1987), learned explanations and representations of concepts like fractions and place value, or stored away strategies for assessing student progress. Judgments formed from both good and bad experiences have led teacher candidates to ideas about what good math teachers are like and what they do. Prospective teachers have also developed ideas about who can learn mathematics and about the factors that influence success in learning mathematics. Their own childhood memories contribute to their assumptions about what children enjoy, what they are interested in, and "what works" with kids of particular ages or backgrounds.

Studies of mathematics teaching suggest that the mathematics classrooms which prospective teachers have experienced have many common denominators. To set a context for the "apprenticeship of observation" and what prospective teachers learn from it, I pause to take a look at where they have been doing most of this apprenticeship. Since American schools are set in a historical and cultural context whose values and traditions shape the experience of school mathematics, I consider both school and culture in this discussion.

The Apprenticeship of Observation in the "Ordinary Math Class"

The reader will undoubtedly be able to validate from personal experience the following description of what Davis and Hersh (1981, p. 3) call the "ordinary math class":

> . . .the program is fairly clearcut. We have problems to solve, or a method of calculation to explain, or a theorem to prove. The main work will be done in writing, usually on the blackboard. If the problems are solved, the theorems proved, or the calculations completed, then the teacher and the class know they have completed the daily task.

The teacher (or the textbook) is the authority, theorems are proved by coercion — not reason — and confusions are addressed by repeating the steps in "excruciatingly fine detail" (Davis & Hersh, 1981, p. 279). While it makes mathematics educators wring their hands (Kline, 1987), this mode, elaborated below, represents the dominant approach to mathematics teaching in the United States and comprises the "folkways" of school mathematics in which prospective teachers are steeped (Buchmann, 1988).

Stodolsky (1985), drawing from her own and others' findings (Fey, 1978; Goodlad, 1984;

[65] Two exceptions are Cohen (in press) and Waller (1932). Cohen especially, argues that cultural views of knowledge are a critical and often overlooked force shaping the nature of school teaching and learning.

Schwille, Porter, Belli, Floden, Freeman, Knappen, Kuhs, & Schmidt, 1983), provides a picture of modal practice. Classrooms are dominated by a recitation and seatwork pattern of textbook-centered instruction. In about 20 of the classes that Stodolsky observed, for instance, students worked individually at their own pace, although most time was spent on whole-group instruction. Rarely did students work in small groups or with partners. Generally, math teachers "introduce new concepts to children and teach and tell them how to do the arithmetic . . . Once material has been presented to the students, extensive periods of practice are provided" (Stodolsky, 1985, p. 128).

Textbooks dominate this approach to mathematics instruction. Although teachers sometimes omit topics they perceive as "extras," they rarely add mathematical content not covered in the textbook (Schwille et al., 1983). Stodolsky's (1988) analysis of elementary math textbooks suggests that concepts and procedures are often inadequately developed, with just one or two examples given, and an emphasis on "hints and reminders" to students about what to do. She argues that this suggests that it is the teacher's responsibility to develop the ideas in class. Yet, she reports, researchers observe little use of manipulatives or other concrete experiences. Instead, students spend most of their time doing written practice exercises from the textbook.

Madsen-Nason and Lanier's (1986) case study of Pamela Kaye, a high school math teacher, provides vivid detail of what happens in the ordinary math class. After giving a brief presentation which consisted of rules and "here's how to do these" explanations, Ms. Kaye typically assigned her students a large number of exercises from the book (e.g., 124 ratio and proportion exercises) to "cement it into their little minds" (p. 17). Although these assignments were long, they were repetitive and did not require much effort — just persistence. She spent most of each class period monitoring students' work on the assignments — keeping them on task and providing individual assistance. When Ms. Kaye taught something to the whole group, she focused on telling the students "how to do it" and gave them algorithms to follow that she thought would help them to be successful.

Spending as many as 12 years in "ordinary math classes" is a powerful apprenticeship for learning to teach mathematics. The years of redundant, often highly cathected, experience thus help to define and limit prospective teachers' images of mathematics teaching and learning.

Known through the common experience of having "been through it," the folkways of school mathematics assume qualities of both obviousness and necessity which "command a moral and cognitive loyalty" (Buchmann, 1987b, p.155). Buchmann (1987b, p. 155) argues that, "in learning the folkways, people do not simultaneously internalize the disposition to take a hard look at what they do and what the consequences are." On one hand, teacher candidates who have been successful in mathematics may think that the patterns they have seen are appropriate and therefore may be uninterested in alternative ways of teaching. Those who struggled in math may nevertheless assume that this is the way mathematics <u>must</u> be taught and that they are simply among the "have-nots" in mathematics. On the other hand, they <u>may</u> also aspire to teach differently. But even if prospective teachers are critical of their own past teachers for teaching badly and for making them feel stupid, many of them lack alternative images of mathematics teaching, having had no other models.

Mathematics Teaching in the American Cultural Context

Prospective teachers' ideas about the teaching and learning of mathematics come not only out of school experience, but also from our mainstream culture. Images of teaching and the teacher's role, of what goes on in mathematics classes, and of people who are good (or not good) at math permeate American movies, books, television shows, and jokes.

In these images, math teachers always seem to stand at the front of the room, write formulas all over the board, tell information, and give assignments. Students sit at desks, working exercises. Since mathematics is dull, they require firm management and control in order to keep them on task, for math is portrayed as a dry and boring subject. The only people who seem to like it are boys with pimples and black-framed glasses with thick lenses, who are seen as highly intelligent but freakish. A mixture of awe and derision is accorded such "math types." They solve the critical mysteries and come up with brilliant solutions to impossible situations (usually through some magical formula), but their pants always also seem to fall down in class.

In our culture, mathematics is both revered and reviled. Being good at math is considered an indicant of a certain kind of respected intelligence. Mathematics is supposed to be essential for many careers and for understanding the world around us. Yet, many people take pride in pointing out that they <u>never</u> use or need mathematics. Perhaps more than any other subject, even well-educated people are willing to admit freely that they are not good at math (Hilton, 1980). Being good at math is widely portrayed as a genetic endowment. Expressing enjoyment of mathematics or, worse, being seen doing or reading mathematics invites adverse comments, expressions of distaste, and disclaimers both by the "accuser" and the "accused." Mathematics is broadly perceived as a cold, hard domain which is paradoxically also mysterious, in which answers are always either right or wrong, and in which little reason for debate or discussion exists (Buerk, 1982; Guillen, 1983).

It is striking how much of this external influence is shared by those who enter mathematics teaching in the United States, for this degree of commonality in perception does not necessarily characterize other subject areas — English or biology, for instance. In these areas, prospective teachers are more likely to have had varied school experiences, and our cultural images of English or science teaching seem far less salient. Given this context, a critical question to be addressed in this chapter is: What do prospective math teachers make out of these experiences and messages and how much variation in their ideas and assumptions exists?

I have divided this discussion into two main parts: Prospective teachers' ideas about teaching and learning mathematics are discussed first, followed by their ideas about learners of mathematics. In each section I examine several themes in the ideas of the prospective teachers whom I interviewed. In addition, I discuss some areas that seemed to baffle or worry them. In this analysis, as in the analysis of their subject matter understandings, I have tried to push below the level of their terms, their slogans, to learn how they were thinking about questions of teaching and learning. If they talked about making mathematics relate to the real world, for instance, I tried to avoid assuming what they meant and, instead, sought evidence that would help to reveal their ideas. I return to this issue at the end of the chapter.

<u>Ideas about Teaching and Learning Mathematics</u>

Despite the personal and biographical differences among them, as well as the variance in their understandings of mathematics, the teacher candidates' ideas about the teaching and learning of mathematics were surprisingly similar. Four main propositions stood out prominently:

> ***1. Teachers stand in front of the class and talk and show students how to "do" math, by "going over" examples on the board.***
>
> ***2. Good teachers help students "figure things out on their own."***
>
> ***3. Learning mathematics is a process of acquiring and storing "concepts" (i.e., information).***
>
> ***4. Teachers are supposed to know all the answers and tell them to students whenever asked.***

These notions form a logically consistent view of mathematics teaching significantly different from the view of mathematical pedagogy advanced in Chapter 1. In this section, I unpack each one in turn and then discuss a distinctive theme that weaves throughout, a theme that makes plain the influence of the folkways of teaching on teacher candidates' ideas.

<u>Proposition #1:</u> Teachers stand in front of the class and talk and show students how to do the math by "going over" examples on the board.[66]

Corollary: Teachers need to be good at talking and at knowing how to "word" things.

When the prospective teachers imagined themselves teaching math, most of them pictured themselves standing up, talking, and writing on the board. They described themselves presenting, telling, reminding, "going over," explaining, pointing out, and showing things. This was the dominant image for 16 out of the 19 teacher candidates, but all of them fell into the telling mode at least some of the time.

Throughout their interviews, the elementary candidates exhibited a strong tendency to tell and show mathematics to students. For example, in describing how they would teach subtraction with regrouping to second graders, all of them said they would explain and show and do examples and then have students do "some more" following their examples. Sandi explained:

> I would just show them how you do it, really. I would just say if you have 2 tens, you can take 1 away and add it to the ones so you're adding ten here and not one. You would have to make sure that they understood that you were taking ten and not one so this would be 14, and not 5.

[66] I adapted this style of presenting the teacher candidates' ideas as propositions with accompanying corollaries from Schoenfeld (in press a).

$$\begin{array}{r} \cancel{2}\,{}^{1}4 \\ -\ \ 1\ 6 \\ \hline \end{array}$$

She said she pictured herself at the board and added that "you have to tell them — I mean obviously if you have 4 here you can't take 6 away, you have to explain that you only use it at certain times."

Other teacher candidates' descriptions were similar, although they varied in detail and the degree of focus on the underlying concepts. Cathy, on one hand, focused exclusively the steps of the procedure. She said she would explain how to do it (borrowing). She would give the students "little tricks like crossing it out so you can keep it all straight" and would advise them "not to try and keep it all in their head." On the other hand, Linda had a metaphorical explanation that personified the numbers into beings that needed to borrow "more" from "next door":

> Explain to them that if the top number is smaller, then you need more from somewhere, so you look next door and he looks like he's got more (if he's more than 0). So, you know, you can borrow 10 — but how would you explain that it would just be 1? They might say that 2 doesn't really have 10 to give away. So you'd say, "Yeah, but that's really 20 and that would be 10 , so you cross it out and put a 1 and 10 over there, that makes 14 - 6.

Linda admitted that this sounded confusing but said she felt that this was the only way it could be explained. She said borrowing couldn't be explained "visually." Rachel thought she could use blocks or marbles to "show them conceptually." Once the children got the idea that you can't subtract 6 from 4, she would "go to work on the board" and do a few more examples.

Across the other teaching questions as well, the prospective elementary teachers tended to tell, explain, and show the mathematics. They would tell students that squares have four equal sides, explain that "anything divided by zero is zero" (sic), and show students to put in zeros as placeholders in multiplying large numbers. Some of them worried about being able to find the best ways to explain and show mathematics so that children would understand it. Cathy, for example, explicitly connected telling and teaching, asking, "How do you word it — how do you teach someone math?"

The elementary teacher candidates' descriptions of what they would do varied in detail and specificity. The vaguer ones simply said that they would "point out" that squares were not the same as rectangles (sic), while, at the other end of the continuum, a couple of prospective teachers produced elaborate scripts, complete with intonation and parts for what pupils should say. Only three elementary teacher candidates gave any rationale for telling and showing as an approach to teaching mathematics. All three explained that teachers should make sure pupils understand; one recalled that her best teachers — the ones from whom she felt she had learned the most — were the ones that explained everything to her. The other elementary teacher candidates did not explicitly talk about why they would tell, demonstrate, answer, and point out. Rather than an examined and conscious choice, this approach to teaching mathematics was apparently taken for granted. Assuming this approach without question is a function of the ways in which prospective teachers acquire their ideas and the consequent commonsense and obvious nature of this teaching knowledge (Buchmann, 1987b).

In responding to the teaching scenarios presented during the interview, the secondary teacher candidates exhibited even stronger tendencies to <u>tell</u> and <u>show</u> than did the elementary majors. As these secondary teacher candidates described how they would help students understand particular mathematical ideas or procedures, 7 out of 9 consistently portrayed themselves as <u>telling</u> or <u>showing</u>. They repeatedly used a wide range of "telling" verbs — say, explain, talk about, present, and tell — as well as "showing" verbs — give, show, demonstrate — in describing what they would do. Carol, for example, said that when she was "lecturing" about slope, she would do lots of examples of graphs on the board and Andy said he would "begin with rise over run" and then "go over" several examples.

Cindy's sketch of what she would do highlights the predominant telling/showing mode:

> I think that I would probably start out with the idea of rise over run and . . . ***develop*** the idea of slope, I would ***give examples*** of things that are on a slope. . . I think I would ***address*** the difference in the y-coordinates over the y-coordinates and try to ***develop*** it from that point of view and then go in and I would ***introduce*** the slope-intercept form of the line and ***show how*** that was really kind of an easy one to graph from just because you are given, you know, ***point out*** that one is along the y axis, the x coordinate is 0, which would cancel that out which would mean that y would equal b at that point, try to ***explain*** that. I would go on and do the rest (laughs a lot). ***Introduce*** the . . . well, they went to the notions of having zero slope and no slope, which is something that is important to understand and then I would ***introduce*** different forms of lines.

The highlighted verbs also reveal a picture of Cindy as the dispenser of knowledge about slope. Giving out knowledge using traditional practices, she seemed to see herself telling and showing students about the topic. The other secondary teacher candidates talked in similar ways, describing in considerable detail what they would tell or show students. Most of them would answer students' questions directly ("7 ÷ 0 is undefined — you can't ever do it") and would correct incorrect responses or statements ("a square has four equal sides," and "use zeros as placeholders in multiplication").

Expecting that telling and showing might be a significant dimension of their image of mathematics teaching, I had constructed an item for the card sort that said, "When I teach math, I expect that my major task will be telling — or showing — students how to do the problems." Cathy was baffled: "What other task would there be?" That telling and showing were the central tasks of teaching mathematics seemed entirely obvious to her. The other elementary candidates largely agreed with this statement and made few comments about it. Like Cathy, they seemed to find the notion self-evident, and their reactions to the item largely fit with what they did and said during the interviews.

Overall, however, the secondary teacher candidates' responses to this item were surprisingly negative: Almost all of them disagreed with it or were unsure. They resisted this statement because they claimed that they wanted students to learn to "figure things out on their own." Despite what she seemed to describe in responding to the teaching scenario, Cindy said that she thought that giving examples was a good idea, but that the only way that students will develop an "understanding and appreciation of the way a procedure works" is if they "work through it on their own." Terrell explained that he would show students how they <u>could</u> solve certain problems, but that he would want them to figure out their own solutions. All the other secondary teacher candidates except one echoed Andy's and Cindy's protestations, using almost identical explanations and terms.

What at first appears to be a provocative difference between teacher candidates who have majored in mathematics and those who have not is made much more fuzzy by the interview responses. When Cindy, for example, did the card sort, she disagreed with the view of the teacher's role as one of telling or showing students how to do the problems. She said that she hoped she wouldn't just be "a voice up at the front of the class" — that she wanted to "actively involve" students. What she said she wanted to do seemed quite different from what she did on the interview (see above). Like the others, she portrayed herself showing and explaining.

One possibility is that, while Cindy <u>wants</u> to teach differently, she really only knows how to teach in a telling mode — after all, that is probably the dominant image to which she has been exposed. Telling students how to do the math is therefore currently her "default" mode.

That teacher candidates operate in a common "default mode" is entirely consistent with Buchmann's (1987b) analysis of the folkways of teaching in general. Lortie (1975), too, describes what students learn from their apprenticeship of observation as "intuitive and imitative rather than explicit and analytical" (p. 62). Indeed, although most of the secondary candidates assumed a telling and showing role throughout the interviews and gave relatively elaborated descriptions of what they would say and demonstrate, none gave a rationale for this approach. They appeared to assume or fall into this pattern of teaching, without explicit consideration of the role they were taking on. Cohen (in press) would find this unsurprising, indeed entirely predictable:

> Contemporary institutional practices embody an ancient instructional inheritance. In this inheritance, teachers are active; they are tellers of truth who inculcate knowledge in students. Learners are relatively passive; students are accumulators of material who perform prescribed exercises. And knowledge is objective and stable. It consists of facts, laws, and procedures that are true, independent of those who learn, and entirely authoritative. These ideas and practices have deep and old roots in academic habit. (p. 15)

Another possible explanation for the apparent discrepancy between what the secondary candidates <u>advocated</u> and what they <u>did</u> in their interview responses may lie in what the teacher candidates meant when they talked about "active involvement" and "getting the students to figure it out for themselves." In many cases, although the teacher candidates claimed that they were helping students to solve problems independently, the enactment of that idea looked much more like <u>telling</u>. I discuss this below.

<u>Proposition #2</u>: Good teaching is <u>leading</u> students to "figure out" the right answers "on their own."

Although the teacher candidates did a lot of "telling" and "explaining" and "pointing out" in the interviews, some of them talked about helping pupils "figure things out for themselves." Recall that few of them talked explicitly about the value of "telling"; in fact, most said that this was an undesirable way to teach. Cindy argued, for example, that the only way "students are really going to understand the way a problem works is to do it on their own and to work through it on their own" and Terrell said that the important thing is to get students to figure things out themselves.

Looking more closely at what the teacher candidates did as they responded to interview questions suggests three critical issues buried within the teacher candidates' image of "helping students to figure things out for themselves": one having to do with what it means to figure something out in mathematics, another having to do with what they thought it meant to do something "on one's own," and a third having to do with their notions of how teachers could help someone do that.

Their responses to the interviews suggested that the teacher candidates seemed to view "figuring something out in mathematics" as being able to carry out procedures. "On one's own" seemed to mean without reminders, and "helping" seemed to mean quite directive leading.

Janet's explanation of how she would help a student solve the equation

$$\frac{x}{0.2} = 5$$

revealed that her goal was to get the student to know the steps independently. She said that she would use a pattern of questions to "lead them to the answer":

> I would say to them . . . "Well, what do you need to do in this problem ?" Hopefully, — I'm guessing that their answer is going to be something along the lines "Well, I need to isolate x on one side of the problem." I would say, "What do you need to do that?" The student would probably say, "Well, I need to get rid of the .2 underneath the x and put it to the other side." I'd ask, "Well, what can you add or subtract or multiply or divide to get it to the other side — to eliminate the .2?" Or, actually I might say "Well, how do you isolate it?" The student would hopefully respond with something like "Well, I need to do mathematical functions to get rid of the extra stuff." If they couldn't come up with that answer, I would suggest that to them. Then I would ask "What function would make the most sense in this case? Would it make sense to add or subtract or . . . ?" They would probably say "No." Then I would ask "Well, what should you do?" At that point, I'm guessing that they might see that they have to multiply by .2.

Janet's rationale for doing this was that "they are getting that for themselves, as opposed to being told the answer, just taking it and not learning." Janet believed that cluing the student through the steps would help the student learn how to solve this type of equation on his or her own, in a way that was significantly different from being told the steps. Other teacher candidates shared this view. Marsha, for example, would teach subtraction with regrouping by showing them the steps and then putting other problems on the board and "getting input" from the class about how to solve them.

The strategy the teacher candidates used toward this end was to lead students with questions, hinting and giving clues about the correct answer or conventional solution path. Pam, an elementary candidate, explained that "instead of telling them what the answer is, [it is] kind of working with them to try and come to some conclusion through me asking questions and them doing some sort of reasoning, working out the answer." Terrell, one of the secondary candidates, used a metaphor for what he was trying to do: He said he was "beating around the bush" — trying to hint without telling, allowing the student maximum opportunity to solve the problem on his or her own:

> I try to beat around the bush as much as possible in some orderly way, where first you beat around the bush. You are furthest away from the bush, if you want to say it. Then if that doesn't work — if they are really having trouble, then further away from the bush is the point where they are not really having trouble. Then as you get closer and closer to the bush, you see that they really are having trouble. If you get right there and they still don't understand, then you've got to go back to learning the basics again. Learning how you get up to that point.

While many of the teacher candidates said they valued having students figure things out on their own, they had particular ends in mind: specific conclusions they wanted students to draw, answers they wanted them to get, and, usually, particular methods they wanted them to use. This was evident in the little scripts the teacher candidates constructed for their dialogues, scripts designed to get students to know and be able to do the mathematics at hand. These scripts served as teaching templates, with small holes to be filled with particular knowledge. The students would, they hoped, say the right things to each one of the sequence of questions, but if they didn't, the teacher candidates were prepared to simply give in and tell them the right answer.

Cindy's response to a first grader illustrates this. In this scenario, a first grader has identified a square as a rectangle. The pupil is right — a square is a rectangle — but Cindy, a math major, thought of a rectangle as having two long sides and two short sides, and so she believed the pupil was wrong. She had a script for leading the pupil to the "correct" understanding of rectangles and squares:

> I'd ask him to look at the sides one more time and remember <u>another</u> figure, ask him, "Do you know any other figures that are <u>like</u> a rectangle?" Hopefully, he will say a square! [laughs] Then I will ask him if he knows what makes a square different from a rectangle. If he says, "No, I don't," then I suppose I would say, "Well, a square has sides that are the same length." And I would say, "Do you suppose that this could be a square, maybe?" If he said yes, I would say, "Well, a square has sides that are the same length." And I would say, "Do you suppose that this could be a square, maybe?" If he said yes, I would say "Why?" and hope that he would say, "Because the sides look the same length."

Cindy wanted the student to change his mind — to believe that the square is not a rectangle. Her questions were designed to lead him toward this conclusion. She <u>hoped</u> he would say the right things at the right times, but if he didn't, she was prepared to tell him. Terrell, too, if he got right "up to the bush" and the students still did not understand, would resort to telling students. Both Cindy and Terrell were dispensing information; however, they tried to structure the situation so that students would say the right things before they were told.

The teacher candidates believed that this approach of leading students with questions would help them to internalize the procedures so that they could solve problems (i.e., "problems" as in exercises given in the book) "on their own." Out of 19 teacher candidates, 7 gave an explicit rationale for this approach: 3 said that students would understand and remember the content better, 2 said the that students would be more interested in learning, and 2 said that students would feel more sense of accomplishment and confidence. Others who spoke up for this way of working with pupils simply reiterated their view that the goal was to help students become able to do mathematics "on their own." Mei Ling's explanation summed up the comments of the other teacher candidates nicely. She said she liked to teach through asking questions:

> Because I want them to feel like they are . . . I want them to get the feeling of accomplishment that they understand it, that they realize and it is a decision that they made and they were right. Besides my just telling them that "this is a square, this isn't a rectangle, you're wrong."

Not surprisingly, a card sort item which read, "As a teacher, I would like to avoid telling. Instead I would try to lead my students to the answers by asking pointed questions," received agreement from all but one teacher candidate. Carol, a secondary candidate, was not sure. She explained that, "Sometimes you can't avoid telling someone, and if you're teaching high school, you're going to spend some time lecturing — and that's what you're doing, you're telling them."

Overall, many of the teacher candidates were enthusiastic about helping students "figure things out for themselves" and largely critical of "telling and showing students how to solve problems." Yet, what they did when they related how they would teach or how they might respond to a student question revealed that the differences between these two views of the teacher's role were more rhetorical than actual. In "figuring it out on one's own" the teacher candidates focused on the "on one's own" part; "figuring it out" simply meant getting the answer using already-taught content. The romantic value placed on independence in mainstream American culture helps to explain the appeal of figuring something out on one's own. Still, the teacher candidates' notions about what it means to "figure something out" in mathematics were generally superficial. This weak interpretation of "figuring something out on one's own" fits the dominant model of the mathematics teacher as knowledge-dispenser and demonstrator. It is also consistent with the prospective teachers' notions about mathematics as a body of facts, procedures, and definitions. But their tendency to tell, show, and point out was also influenced by their view of what is entailed in learning mathematics, discussed next.

Proposition #3: Learning mathematics is a process of acquiring and storing "concepts" (i.e., information).

Corollary: Math teachers should drill material into students' heads.

In three different ways — the value placed on repetition, common metaphors for learning mathematics, and an emphasis on learning facts, procedures, and definitions — the teacher candidates' language suggested a view of mathematics learning as a sometimes arduous process of acquiring "stuff" that enters through the eyes and the ears to be stored up in the head.

This view of learning fit naturally with their tendency to teach by telling and showing. In many cases, they hoped that they could get pupils to figure out these ideas in ways that made them feel that they had come up with them "on their own," but the teacher candidates' main focus was on getting the correct ideas and skills into students. Toward this end, telling and showing often seemed most efficient.

One piece of evidence that the teacher candidates tended to view mathematics learning as acquiring and storing knowledge was their overwhelming emphasis on **repetition**. Almost everyone talked about the importance of having students practice mathematics and they also agreed with a card sort item about the importance of practice. Andy, for example, one of the secondary candidates, explained that students should have homework every day because this repetition is essential: Once students practice and acquire a procedure or a fact, they will never lose it, he believed.

The teacher candidates also favored repetition when students have difficulty. They assumed that when students aren't understanding, teachers should "re-go over" the material and assign more problems for practice. For example, Anne said that if her students were having trouble with slope and graphing, she would

> try to be very clear when I re-approached it and take another day in class to go over it again and assign some more problems. I'm sure that after two or three days of going over it . . . a majority of them would get the concept.

And Rachel remarked that, if students weren't learning, she would "go through a few more": "It just takes repeatability," she remarked.

A second sign of the teacher candidates' conception of learning mathematics as acquiring and storing "concepts" was the salience of **mechanical metaphors** to refer to the learning of mathematics. Some talked about "drumming," "drilling," and "embedding" facts and procedures into pupils' heads; others described learning in terms of "grasping" or "grabbing" content. These metaphors all conveyed an image of mathematics knowledge as "stuff" that must get inside learners' heads. Cathy, for example, explained how she would answer a student who asked what 7 divided by 0 was:

> I'd explain it to them and then tell them to remember that rule — that anything divided by 0 is 0 — and give different examples: "What's 4,000 divided by 0? What's 0 divided by 0?," so that it gets like embedded into their head.

Cathy's little script illustrates both her mechanical image of learning as well as her confidence in repetition. If she repeats a number of questions, she believes, the "rule" (actually incorrect) will get inside her pupil's head.

I wrote an item intended to provoke disagreement or interesting comment: "In order to learn mathematics, many basic principles must be drilled into the learner's head." Only one teacher candidate rejected the statement, objecting to the term "drilling," and another was not sure. All the others agreed. I was surprised — such a level of agreement demonstrated the widespread cultural acceptance of this view of mathematics learning.

A third source of evidence for this conception of learning was the teacher candidates' focus on **making sure that their pupils knew standard procedures, definitions, and terminology**. This was implicit in the amount of "pointing out" the teacher candidates did. They would tell students the definition of a square, point out how to solve problems, and remind pupils of the steps in solving for x in algebraic equations. Teaching math, to many of the teacher candidates, entailed a hefty dose of giving pupils information.

Unlike the two views of the teacher's role and about teaching, this view of mathematics learning was quietly uncontroversial. No one spoke explicitly about it; thus no rationales were given. One interpretation of this is that this conception of learning may be simply a logical subsidiary of the prevalent image and enactment of mathematics teaching as telling (see Cohen, in press). Teachers who tell, after all, assume that children need to be given mathematical knowledge. This trio of assumptions leads to the last, focused on the teacher and the teacher's role.

***Proposition #4:* Teachers are supposed to know all the answers and tell them to students whenever asked.**

Corollary: When you don't know, tell the students you will look it up and will tell them tomorrow.

Almost all the teacher candidates thought that teachers should know mathematics and should be able to explain the content, answer students' questions, and evaluate their responses. They believed that knowing was fundamental to being a teacher. Still, their notion of what it means to "know" mathematics was equivalent to being able to come up with the right answers. While I do not know how they felt about their own preparedness prior to our interviews, many of them, especially secondary candidates, confronted during the sessions the possibility that their own knowledge of mathematics might not be adequate to meet the requirements of teaching.

Before examining this proposition, I discuss the teacher candidates' assessment of themselves in relation to what they considered to be the knowledge demands of teaching. Their fears reveal that they assumed that teachers should be able to give students answers.

Terrell, one of the mathematics majors, commented afterwards that the interview contained "some pretty scary stuff." When I asked what was scary about it, he said that he was scared by the fact that he didn't "know for sure this simple stuff" — i.e., the mathematics embedded in the interview questions. Cindy, a math major with a 3.7 average in her math classes, found herself stuck at several points during the interview. For example, when she tried to decide if a student's work on a ratio problem was correct or not, she giggled nervously, "Is this one right or wrong? I'm supposed to know this, aren't I?" Throughout the interviews, she kept commenting that she hoped none of her math professors would see the videotapes of our sessions because they would suddenly realize that "that girl doesn't know any math." Other math majors had similar experiences as they tried to use their mathematical knowledge in teaching (see Chapter 3). Barb, trying to explain division by zero, realized that she "just knew it" — that it was simply "a fact" to her — and that she had no idea of why it made sense.

Unlike the mathematics majors, many of the elementary teacher candidates lacked confidence in their knowledge of mathematics before they participated in the interviews. Most of them assumed that they needed "review" — to look over the material — and that they would need to "do homework" before they could teach most upper elementary topics, like long division or ratio and proportion. The highly anxious teacher candidates (see Chapter 4), however, were quite terrified with what they would need to understand in order to teach. Confronted with novel student answers, they exclaimed things like, "Oh, my God!" and were often completely overwhelmed by the content.

The interviews challenged the teacher candidates' assumptions about their preparedness for the knowledge demands of teaching mathematics. These assumptions, like the teacher candidates' conceptions of learning, were logically consistent with their images of and beliefs about mathematics teaching. They tended, as I discussed above, to conceive of teaching as giving students knowledge. This view places the teacher in the role of dispensing knowledge and monitoring students' knowledge acquisition, a role which demands that teachers know that which they giving to students. Since the teacher candidates tended to assume that subject matter of mathematics consisted of specific concepts[67] and procedures (see Chapters 3 and 4), labeled with "technical" terms, that is what they assumed they needed to know.

On one hand, many of them found that they couldn't even live up to their own expectations of what mathematics teachers need to know and be able to do: to provide (or lead pupils to) key information and to answer students' questions. Linda, one of the elementary candidates, said that she needed to review "all those rules" and several mentioned needing to review terms. Others couldn't remember how to perform procedures like dividing with fractions or solving algebraic equations. Some of them assumed they could just "brush up" on these "rusty" spots in their subject matter knowledge; others were concerned that it would take more than that and worried that the one math course included in the teacher education program would be insufficient.

On the other hand, the interview tasks also gave the teacher candidates glimpses of another sense of "knowing mathematics." Realizing that students were going to ask them "why" questions made the teacher candidates consider that they might also need to understand the content in ways that went beyond telling a rule. Once again, the teacher candidates were brought up short, for many of them felt that the underlying "whys" were things that they hadn't ever learned. As Mei Ling commented about multiplication, "I know how to do it, but I don't know the ideas behind it." They were also unsure at times whether particular ideas did have reasons.

Why the teacher candidates thought knowing mathematics mattered for teaching offers clues to their assumptions about the teacher's role. They explained that they didn't "want to teach anybody anything wrong." Andy explained simply that "you cannot present a topic" if you do not understand it yourself. Part of their concern stemmed from the assumption that teachers are the dispensers of knowledge in the classroom. But a related concern was that

[67] The teacher candidates used the term "concept" to cover a wide range of things — from procedures to ideas to simple information. For example, several of them referred to the "concept" of remembering to cross out the number in borrowing, and "lining up numbers" in multiplication was also considered a "concept" by some. One teacher candidate, in examining a student paper on subtraction with regrouping on which the child had made many critical errors, said, "She seems to have the concept, she's just forgetting to use it."

students' overall respect for teachers is derived from the authority of knowing the right answers.

So strong was this conviction that several teacher candidates said they would never tell students that they didn't know something. Intuitively aware of the "merger between social and epistemic authority in the teacher's role" (Buchmann, 1984), Marsha asked, "How effective would I be then?" Cathy, an elementary candidate, explained that it wouldn't be "professional" to tell a student that she didn't know, "That's what you are getting paid for, that's why you are there — to answer their questions." Cathy said that it would have let her down if any of her teachers had ever said that they didn't know. Janet, a secondary candidate, said that students want to believe that their teacher is a person to whom they can go with their questions. She believed that the teacher's authority in the classroom is fundamentally connected to the teacher's knowledge of the subject:

> I think as far as controlling the classroom, if I don't know the subject matter, the kids aren't going to respect me as much and they are probably going to act up more.

Some teacher candidates were willing to admit to students that they didn't know something, but most still saw it as their responsibility to "look it up" or ask someone and to tell the student the next day. For example, many prospective teachers were unsure about whether perimeter and area were directly related. Some said that they would have to find out and would promise to tell the student tomorrow. One way or another, these prospective teachers felt themselves responsible as the administrators of mathematical knowledge in the classroom.

Discussion: Comparing Strong and Weak Senses of Explaining, Figuring Something Out On One's Own, Showing, and Doing Mathematics[68]

The teacher candidates' use of the terms ***explaining, showing,* and *doing*** in mathematics highlight some significant differences between their image of mathematics teaching and the view advanced in Chapter 1. In each case, their meanings for the term were significantly weaker than the corresponding discipline-based meaning. These weaker uses present challenges and raise questions for teacher educators.

Explaining is the teacher's trade.[69] Buchmann (1987a) writes that, "as a pedagogy, explanation relies on the related strategies of exposition, amplification, and argument" (p. 181). Explanations open up ideas and make them more accessible by clarifying terms or delving into details. Explaining something in mathematics means clarifying an idea by unpacking underlying concepts as well as giving reasons that reveal its meaning and the logic. When the

[68] I am particularly indebted to Margret Buchmann for the development of the ideas in this section — in particular, the distinction between "strong" and "weak" meanings.

[69] Being able to explain mathematics is essential knowledge even for teachers who do not choose to teach by telling. Facilitating students' construction of mathematical understanding, for instance, involves selecting fruitful tasks, asking good questions, judging which student ideas should be especially pursued. All of this demands explicit and analytic knowledge, the kind of understanding entailed in constructing explanations.

teacher candidates talked about "explaining" (which they did frequently) they seemed to mean something much weaker, something that much more closely resembled simple telling — as in giving directions for the steps of a procedure or repeating a definition. Neither did that telling ever seem to include telling about the nature of the mathematics at hand: Its sources or what made it make sense, for example.

Similarly, ***figuring something out on one's own*** in mathematics involves puzzling over a problem, making a conjecture, pursuing it, and checking to see if it is valid. The children in the third grade class in Chapter 1 were figuring out the solution to the dog pen problem on their own: Using concepts and procedures they had learned, they invented others, they checked the mathematical reasonableness of their solutions, and they were able to revise those solutions independently in light of new mathematical evidence. When the teacher candidates talked about "figuring something out on one's own," however, it was a weaker version, one that had more to do with remembering the steps of the procedures independently, without reminders.

Showing, in a strong disciplinary sense, entails proving the validity of an idea using other mathematical ideas as the evidentiary tools. These ideas can be logically ordered into arguments or, in teaching mathematics, can be concretely represented. For example, regrouping can be "shown" by using bundles of popsicle sticks and loose sticks to represent two-digit numbers. The generalization that the sum of two odd numbers is always even can be "shown" with a drawing (Brown & Cooney, 1982, p.16). Yet, the prospective teachers used "showing" weakly to mean demonstrating a procedure or pointing something out in front of pupils who would watch.

A parallel contrast existed between stronger and weaker senses of ***doing*** mathematics. Doing mathematics, from a disciplinary perspective, involves a wide range of activities, including looking for patterns, positing conjectures, pursuing hunches, drawing on accumulated knowledge and generating new ideas, modeling real-world phenomena and applying theoretical ideas as tools in concrete situations. This was not what the teacher candidates meant when they talked about "doing," however. When they said they would show pupils how to do mathematics or they would look to see if their pupils could do the math they had been taught, the prospective teachers meant "perform procedures" and "get correct answers."

Distinguishing between these strong and weak senses of explaining, showing, and doing mathematics reveals significant features of the teacher candidates' ideas about learning mathematics and about mathematical knowledge. The belief that students learn what you tell them or what is put before their ideas suggests a "bucket" view of learning (Buchmann & Schwille, 1983) in which teachers get knowledge into students' minds. This Lockian view is part of the legacy of the folkways of teaching (Buchmann, 1987b). Watching teachers show and tell knowledge and strictly reinforce paying attention, prospective teachers understandably develop the assumption that children learn what they are shown and told. It seems, after all, to work — that is, their classmates and they themselves appear to be learning mathematics.

Unquestioned images of hearing, watching, and practicing as ways of coming to know also imply a view of mathematical knowledge as a set of given procedures which teachers dispense to pupils. All these views — of learning, of mathematics, and of the roles of teachers and students — are logically interrelated and consistent. Based on the potency of firsthand experience, these views are further warranted by both common sense and tradition (Cohen, in press). Given this, it is not surprising that the weak forms of explanation, showing, and doing run thematically through the prospective teachers' ideas about teaching and learning mathematics. Combined with their assumptions about learners, these ideas about learning and about mathematics form the framework for their images of teaching and learning math, images which contrast and conflict with mathematical pedagogy. I turn next to a discussion of their ideas about students. Following a discussion of the things they did not know, that they found baffling, I return to consider some bigger questions raised by this analysis.

Learners in Mathematics

The prospective teachers seemed to have more images of and ideas about the teaching and learning of mathematics than they had knowledge and ideas about students. In spite of the difference in breadth, the pervasiveness of two interrelated ideas about learners was striking:

***Proposition #5*: The ability to do well in mathematics is largely innate.**

***Proposition #6*: Many students do not like math.**

In this section, I discuss each idea and consider what they leave out in terms of teaching mathematics.

Proposition #5: The ability to do well in mathematics is largely innate.[70]

Dweck and Bempechat (1983) describe two common views of intelligence: an incremental view which holds that capacity to learn can be increased as a result of personal efforts, and an entity view which holds intelligence to be a stable attribute not subject to cultivation. The authors argue that teachers' implicit theories of intelligence influence their interpretations and actions. For example, a teacher with an entity orientation to intelligence may try to gloss over student errors and worry about pupils' feelings about failures. After all, pupils who don't "have it" cannot help it and, furthermore, their ability to learn is unlikely to change. Particular conceptions of intelligence vary depending on the performance context (Sternberg, Conway, Ketron, & Bernstein, 1981; Dweck & Bempechat, 1983). Consequently, teachers' ideas about what it takes to be a good reader may differ from their notions about what is entailed in mathematical competence. A teacher may, for example, believe that children's experiences can increase their ability to read and interpret text and yet assume that the capacity to learn mathematics is relatively fixed. Such differences, attributable to differences in teachers' ideas about reading and mathematics, are likely to affect what teachers do in each area.

The prospective teachers whom I interviewed tended to believe that success in learning mathematics depends on something within the learner. However, this issue was one about which they seemed to have mixed feelings and beliefs. What it takes to learn mathematics well

[70] Robert Floden was especially helpful in untangling the problems of analyzing the teacher candidates' ideas about sources of success in mathematics.

is a question that teachers ought to confront and consider; my results suggest that it may be a" soft spot" for prospective teachers — something with which they have not explicitly grappled and about which they hold different, potentially contradictory or conflicting, beliefs. My analysis of their ideas first separates some of the dimensions which were mixed within their views of mathematical ability and then examines the tensions among those aspects.

Two apparently distinct perspectives accounted for the prospective teachers' ideas about mathematical competence: one focusing on the learner's effort and the other emphasizing inborn traits. The ***effort*** view held that being good at math emerges directly from the learner's interest and effort and seemed close to an incremental perspective on ability. The ***trait*** view, an entity conception of intelligence, focused explicitly on personality and temperament, including qualities such as patience and a tolerance for frustration, confidence, a willingness to accept things without questioning, an interest in math, and a generally analytic and "logical" kind of mind. I will argue, however, that the effort view was actually a masked version of the trait view and that, at their root, both perspectives rested on the same assumption: that competence in mathematics derives from personal characteristics and qualities. Still, the ways in which teacher candidates reflected this assumption varied both across and within individuals. They seemed pulled among what they believed, what they wanted to believe, and what they thought they should believe.

An effort view of ability. The prospective teachers who expressed a view of mathematical ability as a function of the learner's effort, emphasized that people who are good at math take math classes and try very hard. One of the key things these people do is they pay attention. In contrast, "not listening" was a common explanation for learner difficulties in mathematics.[71] The surface implication was that mathematics was something that could be learned and that, with proper effort and dedication, one could become successful with math. For example, Teri, an elementary candidate, explained why her boyfriend was good at math: "That's all he's done is quantitative things, like take courses in calculus and math." His mind, she said, had been "molded" toward math as a result of hard work, attention, and desire. Teri's sister, in contrast, was not profiting from math classes because she did not try hard enough, she didn't think through "what they want."

The idea that being good at math depends on study and knowledge sounds as though it focuses on learning and thereby implies that anyone can learn mathematics if they only try. A distinction emerges here between mathematics achievement — i.e., being successful at mathematics in school — and mathematical ability — i.e., something broader about being able to easily learn and enjoy mathematics. Being a good math student means remembering the steps, the rules, the formulas and being able to do the work right that one has been taught to do. The prospective teachers seemed to regard being a good math student to be a function of effort.

Still, their expressions of the effort view were nevertheless tinged with strong innate trait notions. Although the prospective teachers who believed that people who are good at math have taken more mathematics and have studied a lot and thereby have become "molded" in a certain way, they also tended to believe that it takes a special kind of personality to do that,

[71] Both of these attributions — of success and of failure — reflect the commonsense epistemology discussed in the last section: Learning is a direct consequent of what students see and hear.

as mathematical knowledge is inherently difficult to acquire and retain and mathematics is a subject that many people dislike. Consequently, studying mathematics requires an extra measure of determination and tenacity in the face of frustration. The prospective teachers explained that, in order to study math, a person has to be very patient, not get frustrated easily, and have the will to persist. Pam explained this in terms of the inherent nature of math:

> You read it and read it and still don't understand it, and then you ask someone and still don't understand it. That's frustrating, and if you don't have the <u>drive</u>, you're going to give up.

To meet the challenge of math, she thought that people had to have the right "make up" — be intelligent, have "the drive," and be <u>able</u> to "grasp" the ideas.

Another common attribution for mathematical ability was "interest." Some teacher candidates talked about some people being more <u>interested</u> in math than others and, consequently, being more able to <u>do</u> it. Interest, though, like intelligence, was mostly conceived as something which people come by "naturally" — that is, that one is predisposed to be interested in certain things.

<u>The trait view</u>. The alternative to explanations based on effort or interest was the pure trait view, reflected throughout the prospective teachers' explanations of how people acquire mathematical knowledge or solve math problems. The idea that mathematical ability is innate, or fixed and only minimally improvable, permeated their comments. For instance, Cathy, describing a classmate who was good at math, explained, "She was born with it. I might study as much as her, but <u>I</u> just don't have the knack for it." The pure trait view seemed to be about something bigger than just being successful in math class: something about being able to learn math without effort, to be able to <u>think</u> in the requisite "mathematical" ways (which tended to be either mysterious or stereotypical).

One prevalent idea was that being "born with it" was associated with a certain kind of person or type of mind. Mark made a specific distinction that was also drawn explicitly by 11 other prospective teachers:

> I believe there are people who can think abstractly and do mathematics well and then there's other people who can work well with English and other types of subjects and . . . very rarely do you find one individual who can do both really that well. . . . Math's just a subject that you are really good at it or you have struggles with it. . . . I think some people think realistically and some people will think idealistically or more abstractly than others. . . I think it all has to do with the <u>minds</u>, really.

He explained, when I probed, that "idealistic" people were better at abstract things, like math, and "realistic" people were those who were good at English and could communicate well with other people. Mark seemed to use "idealistic" to refer to a preoccupation with <u>ideas</u>, "realistic" to an emphasis on the <u>real</u>, the concrete, which was a way of dividing the world in that gave credit to both "kinds" of people.

Others used other labels for this distinction between "two types of minds." Linda said that people are either good at "words or numbers" and Cathy distinguished between "logical" and "creative" minds. Some, like Janet, put it simply in terms of particular subjects:

> I just think that some people are just <u>naturally</u> better at English and social studies than they are at math and science.

Overall, this theory was dominant: that being good at math is related to individual traits or bents of mind with which people are born. The prospective teachers used it to explain why people were not good at math as well, saying that some people "cannot grasp concepts," or do not think analytically or logically. Cathy described a classmate whom she considered to be good at math:

> There was this unreal problem that no one could figure out. This girl was really patient and she went through it and she said, "Well that doesn't work," and then she went back and tried like three different formulas. She had never <u>seen</u> this problem before. That's what amazes me. . . .just being given any kind of problem, you know, and knowing how to do it, that amazes me. . . .going through it in her mind each step and I don't understand how you could be <u>that</u> logical.

Cathy explained the difference between herself and this classmate, saying, "I just don't have that kind of a mathematical, logical mind." In high school, for example, she understood the material when it was explained in class, but still could not solve "just any problem." For example, proofs made sense when her high school teacher did them on the board, but when Cathy tried to do a proof on her own, she had no idea where to go. The missing link was her own lack of capacity for logical reasoning, according to Cathy.

Linda's somewhat awed description of her boyfriend seemed to fit with and elaborate Cathy's ideas about the role of logical thinking in solving problems:

> He is <u>so</u> logical. . . he takes it step by step and doesn't get frustrated when he can't figure it out. He looks back to see <u>why</u> he can't figure it out. He just basically takes it slow — I always just want to get it over with. He just logically — it all clicks in for him. But I can't, I never see the same steps. I never ask myself the same questions that he asks — he will ask himself a question and it will lead him to a step and I just — it doesn't click in for me. He would get to one part and he would go, "Now what do you do <u>next</u>?" I'd go, "Well, <u>I</u> don't know." He'd go, "Well, just look at it <u>logically</u>," and he'd say, "That goes over there," or whatever. He just thinks so logically about it.

Linda's and other teacher candidates' use of the word "logical" suggested linearity, not reason. The very fact that they often pitted "creativity" against logic is evidence for this interpretation. "Creativity" was "allowing your mind to wander" and think of different alternatives; whereas, to them, logic was following an algorithm. Mei Ling, who also described a friend as "logical," explained

> He is <u>very</u> logical . . . his mind, his thoughts are just straight arrow. When he gets on a point, he won't let go.

She said that he was able to focus and "direct his thoughts," while she, more "artistic," tended to "incorporate everything instead of just focusing on one specific." Although their boyfriends may well have been reasoning logically — pursuing hunches, trying alternative strategies — Linda and Mei Ling did not necessarily see that — to them, "logically" meant methodically, step-by-step.

This capacity, with its components of intellect and temperament, was something her boyfriend was born with, Linda believed. He was just able to see what to do, and no matter how much she understood the material in her books, she would not be able to solve problems.

Other teacher candidates agreed with Linda's view that people are either naturally inclined to be good at math or not.

Mark stated it most baldly: "I believe mathematics is a God-given talent. . . I don't think it's a skill you can develop, I think it just a skill you have."

If being able to do math is a gift of the gods, there is little point in pushing students in order to make them good at math — as several teacher candidates argued, this tactic would discourage those who lacked the natural bent and make them feel bad. Andy, a secondary candidate, explained,

> Usually they've probably got other things that motivate them, other things in life that they want to do and I feel as if you teach them some important concepts and don't overly emphasize the importance of math, that they can be content with what they know and what they learned and don't discourage them.

He drew a parallel, explaining that he had the same experience in "English-type" courses. Andy thought of himself as not good at writing and didn't assume he could learn to do it (or like it) and his reaction to being pushed was "Why bother?" When he would get back papers with red ink all over them, he just felt discouraged.

The trait view implies that being good at mathematics is something to which only special people can aspire, not something that anyone can learn. In line with this, Mark expressed his "dependency" theory, which called for mathematical types to perform the necessary math for those individuals whose work requires math but who don't have the "right kind of mind" to do it. In this context, he explained,

> a lot of your architects are people who can draw really well, who can make incredibly wonderful designs on buildings and stuff [but] do not understand math.

Most of the prospective teachers attributed success at learning mathematics to immutable qualities of intellectual disposition or temperament. Interest in mathematics, a "logical" mind, and the ability to grasp concepts, as well as tenacity, patience, and tolerance for frustration were all cited frequently. The prospective teachers tended to talk about these as invariant attributes, unlikely to change significantly through development or learning. I will return to this in a moment.

Discussion: Tensions and Contradictions

Although most of the teacher candidates voiced aspects of the trait view at times, they were hesitant to agree with explicit expressions of it when they encountered them in the card sort task. Wording was critical; certain formulations seemed perhaps too blatant when they looked at them in print, despite the fact that the teacher candidates had often articulated the same ideas themselves. For example, one item on the card sort stated directly, "Many people are simply not good at mathematics; one needs to be mathematically inclined in order to do well at it." Only about 4 of the prospective teachers unhesitantly agreed with this, while 5 were not sure and 8 disagreed. Among those who disagreed were several who themselves had talked specifically about types of minds and innate characteristics as the essential determinants of mathematical success. Pam, for example, explicitly said in her interview that being good at math was "a natural thing," that one had to have a "really strong drive," an inborn desire and greater intelligence than others in order to do well at math, and yet she did not agree with the formulation on the card sort. Mei Ling disagreed with the card sort statement because she thought it made mathematical ability sound "genetic." However, in her interview, she said that her boyfriend was good at math because he had a logical and analytic mind: "When he gets on a point, he just won't let go." She explained that she was <u>not</u> good at math because she was more "artistic." These qualities, Mei Ling believed, accounted for the differences in mathematical success between them, but she nevertheless rejected the bleak statement of these views in the card sort.

Still, over two-thirds of the teacher candidates did agree with another formulation of the trait view: "Many people simply cannot `get' mathematical ideas." Even more (over three-fourths) disagreed with or weren't sure whether "there is <u>not</u> such thing as a mathematical `type' of person."

Another sign of ambivalence was that the teacher candidates, even those who spoke most unequivocally about needing to be born with "it," also tended to explain most student difficulties in terms of problems with effort and motivation. As math teachers, they said they would help students by going over material patiently, more slowly. They would try to make math class more interesting in order to engage pupils. On one hand, this may not be in conflict with a theory of innate mathematical ability, for, as I discussed above, some may consider "effort" and "motivation" in mathematics to be "natural" dispositions. Thus, the prospective teachers were simply trying to compensate for natural lacks in students by using helpful pedagogy. On the other hand, the teacher candidates' wont to rely on "effort" and "motivation" as explanatory levers may suggest that they also have conflicting ideas about the influences on successful mathematics learning.

The prospective teachers seemed sensitive in their responses to questions about the factors that influence mathematical ability. The question of what makes someone good at math is a sensitive issue in our culture and, as this analysis shows, this was clearly reflected in the interviews. Many cultural messages convey the notion that mathematical ability is innate and even synonymous with intelligence. In addition, both school practices and personal experiences support the idea that some people do better than others in math, for reasons that often seem mysterious.

At the same time, the idea that some people are simply born to be good at mathematics flies in the face of cultural dogma that anyone can succeed, anyone can learn — all one needs do is try. As if this weren't confusing enough, mathematics, while widely disliked, is nevertheless a domain of considerable prestige in our culture. Thus, it is unpalatable to believe that some people are naturally more intelligent and have the birthright to special capacities that afford them prestige.

Each explanation of the bases of mathematical competence has some understandable attraction for people. Many think they are not good at math; thus to interpret mathematical ability in terms of an innate trait lays the blame for lack of success in mathematics outside themselves. After all, how can one help it if one is simply not "built" to be good at math? At the same time, the belief that effort can conquer all also runs deep in our culture. Some prospective teachers, like Pam, dealt with these competing assumptions by welding them together, such that interest and effort in mathematics became static personality traits.

I argue that the issue of how one gets to be good at learning math is a soft spot for prospective teachers. Their ways of talking about this are confounded so that "effort" can mean something that is within the control of the individual or something one is innately predisposed to put forth. Conflicted about what they really think and what they want to believe and caught in the crosswinds of competing cultural values, many of them may vacillate in their explanations and attributions, their ideas about the roles of students and teachers, and their assumptions about the effects of teaching.

What do the effort and trait views imply for the teacher and for mathematics teaching? If pupils are either naturally good at math or not, then the teacher's role is not central. If ability to learn mathematics is based on invariant attributes, it is unlikely to change significantly through development or learning. If effort is what makes the main difference, though, according to teacher candidates, the teacher's role is to make math interesting and fun so that pupils will pay attention and try as hard as possible. This alternative is founded on the ubiquitous commonsense assumptions about learning and about mathematics. Because beliefs about capacity to learn and the factors that influence student achievement are foundational to teaching, prospective teachers' ideas about ability should be of considerable concern to teacher educators.

***Proposition #6*: Teachers should try to make math more interesting and fun.**

The teacher candidates believed that mathematics is not a favorite subject for many students. A few, mostly those who liked math themselves, had only recently come to this conclusion. Cindy, a mathematics major, said that it wasn't until she got to college that she discovered, to her surprise, that other people were "disgusted" with math and "just hated it." She explained that, in high school, the other students in her math classes were there because, like her, they liked math.

Over two-thirds of the prospective teachers agreed with the card sort statement, "For many of my students, math will probably be their least favorite subject." This level of agreement may be yet another reflection of the trait view. Those who disagreed apparently did so because they interpreted it as a prediction about their own teaching ability. Janet, for example, said she hoped she would be "doing better than that." And Cathy exclaimed,

> I hope not!! I don't want them to think that, because I want to make it fun for them. . .I think the teacher has a lot to do with that.

Cathy's desire to "make it fun" was widely shared by the prospective teachers. Expecting that many of their pupils would not enjoy math, they hoped to be able, as one of the math majors said, to "put a little sugar with it." Just as parents sweeten the spoonful of bitter medicine they dispense to their children, so the prospective teachers thought they should dispense mathematics more palatably. Three kinds of "fun" or interest were mentioned: adding features to the context that were irrelevant to the mathematics at hand, using materials that would be inherently engaging to pupils, and showing pupils how the math they were learning relates to the real world.

Sweetening the bitter pill: Making math class fun. Many of the prospective teachers — even those who loved mathematics — assumed that they would need to find ways to "sweeten" mathematics for their pupils by making math class entertaining. An unsurprising finding among those teacher candidates who disliked mathematics themselves, the idea that math class must be made "fun" was, however, equally prevalent among the math lovers, revealing perhaps another scrap of the trait view. Mark, a secondary teacher candidate, explained why it was critical:

> In other subject areas you can use personal feelings, personal opinion, and personal ideas into the subject and make it more interesting . . . Math is something you just have to accept the way it is and it can be real scary sometimes.

Mark's ideas about the nature of mathematics contributed to his assumptions about what would be hard about working with students. Other teacher candidates shared his view that mathematics itself was inherently dry and difficult to learn and they hoped that they would find some ways to mitigate students' inabilities and consequent anxieties.

Inspiration for their ideas often came from favorite teachers they had. Terrell said he really enjoyed math in high school because many of his teachers were funny and he had "a ball." He aspired to be fun, too, and make his students enjoy his class. And Jon described how one of his teachers had made a game of throwing chalk around his back and hitting the corner wastebasket. "It was always a lot of fun to watch him do that." Jon worried that "math class is an easy one to nap in, and if you have colored chalk flying in your face and people talking about sleds flying down hills [referring to his ideas about teaching slope] and throwing chalk around the room, maybe you'll want to stay awake."

Like many of the other teacher candidates, these men would resort to some strategies outside of mathematics to engage their students. In general, although the teacher candidates believed that making math class fun was a priority, they did not have a lot of ideas about how to do this, and many were concerned about it.

Using "fun" materials or examples. The prospective teachers tried to use materials or examples that they thought would engage students' interests. Their sources for ideas included their own imaginations (based on conjectures about what pupils of certain ages would like) and things they remembered their teachers doing. Food and money were by far the most frequently used materials.

Often the teacher candidates did not seem to be as concerned with the pedagogical or mathematical appropriateness of their examples as they were with making learning math more enjoyable. For example, Allen, an elementary candidate, suggested using mangoes to represent subtraction with regrouping. He described how he would make piles of the fruits — 2 groups of 10 mangoes and 4 other mangoes to represent 24, for instance: "And then move 10 mangoes over to the righthand side and then you would have 1 and 14." He smiled. I asked what made him think of mangoes. He explained proudly that that was why he was going into elementary education:

> I have a, a creative mind when working with that age. I do relate very well to, to youngsters . . . . you know you have to realize that, you know, bring in mangoes, they don't, you know, probably never heard of a mango before but it <u>sounds fun!</u> And if you have a mango there and you say, "Hey, this is a <u>mango!</u>' And you start juggling mangoes all around they're gonna enjoy it and they're gonna learn it.

Allen's view of "creativity" in teaching emphasized fun as the main avenue to learning. He did not seem to consider that bringing in mangoes might just confuse or distract the students.[72]

Many prospective teachers shared the assumption that children love food and that, therefore, food would make a good learning vehicle. As one explained:

> Hum, I've heard that if you want to get a concept over to children, all you need to is to relate it to food and then they will grasp it right away.

Eight of the ten prospective teachers who were able to generate some sort of representation for $1\frac{3}{4} \div \frac{1}{2}$ used food — mostly pizzas and pies. Certainly, the fact that round models account for most pictorial representations probably inclined the prospective teachers toward circular representations. But food was common also because they assumed that it would be intrinsically more interesting to students than other divisible circular objects. The elementary candidates used food to explain subtraction with regrouping as well (e.g., going next door to the neighbor to "borrow" hot dogs).

Second in popularity was money. Several of the teacher candidates thought that using money in problems would engage students' interest and attention. Others favored "cute pictures" or colorful materials. Mei Ling emphasized the importance of making student materials "cute, fun to look at, with colors — something that will help keep their attention." Underlying this emphasis on fun was the assumption that if students were enjoying themselves,

[72] Teachers should of course try to engage students in the subject matter by making it interesting, even fun. However, unlike Wilson's (1988) historians, who assumed that their subject matter would hold intrinsic interest for pupils, the prospective teachers whom I interviewed seemed to be trying to "get around" the subject matter. Their representations, therefore, sometimes misrepresented or missed the content. I discuss Allen's mangoes example and the larger issue of pedagogical thinking and warrants in Chapter 6.

they would pay attention. Paying attention was key to "getting" it, given the commonsense sensory view of learning assumed by many of the teacher candidates.

Relating mathematics to the pupils' lives. A third, and related, approach to making mathematics more interesting to pupils was to show them how what they were learning was useful by highlighting its relationship to the real world. While this may seem to be a goal overarching the preponderance of food and money examples (which are, after all, "real world" examples), it was instead a variant that emphasized usefulness and value. Food and money examples, in contrast, emphasized interest and fun. All the prospective teachers talked about wanting to do this. They explained that doing so would help keep students' attention and interest. For example, Janet, one of the secondary candidates, explained that she wanted to get her students more enthused by making the math less boring:

> Before I introduced any topic, I would want to come up with real world examples of where it comes in handy . . . . I always got annoyed when I'd ask, "Why are we learning this?" — "Well, because." You know, the non-answers. [I'd want to use] things that make sense to kids' terms and think of things where they could use this.

The elementary candidates also wanted to make the content "relevant" to pupils. They favored materials or problems that illustrated the content with something common in young children's lives. For this reason, most of them liked the textbook section that I used for the longer exercise on subtraction with regrouping. The workbook page (see Appendix B), glossy and colorful, was loosely connected to a story about a school fair. Consisting of 20 two-digit subtraction exercises, the page was illustrated with small pictures above each item (e.g., a teddy bear, a glass of lemonade) and each row posed a question about the items pictured, in an attempt to connect the numbers to a real situation. For example, above

$$\begin{array}{r} 41 \\ \underline{-14} \end{array}$$

was a picture of a beach ball. On that row of the workbook page the question was, "How many prizes were left?" The prospective teachers saw this as an example of relating mathematics to "something real." Mei Ling, for example, said she liked the fact that the workbook page used a story about a school fair because having "the cute pictures on the top gives them something to relate to and the pictures are cute, fun to look at, colorful — it's something that will help to keep their attention." And Pam explained that it "makes it more real to the child if they can relate it to something, not just numbers on a page." This idea that children wouldn't enjoy "just numbers" was echoed by many of the prospective teachers. Linda, another elementary candidate, liked the pictures of the fair that were on the page because it would

> make sense to the kids. They can see some reason for it — to see how many teddy bears are there left over and stuff instead of "11." They don't care about numbers as much.

Consistent with both cultural images of mathematics and the trait view, the idea that most ordinary children would not enjoy mathematics underscored the remoteness of mathematics even as the teacher candidates ostensibly tried to bring it closer to their pupils' world.

Both the elementary and the secondary teacher candidates believed firmly in relating the content to the real world. Still, this commitment was often difficult for them to act on and they stumbled in trying to <u>do</u> it.

It was difficult primarily because of the nature of their subject matter knowledge. The way in which they had learned the content did not include real world applications, and their understanding was not, on the whole, sufficient to generate appropriate connections. Certainly the elementary candidates were able to come up with consumer applications of arithmetic: figuring out totals at the store or buying wallpaper, for example. But they were unable to connect $1\frac{3}{4} \div \frac{1}{2}$ to anything real because they didn't understand what division by fractions <u>meant</u>; several even said that fractions "don't relate to the real world."

The secondary candidates struggled with division of fractions, but also with high school content: division in algebraic equations and slope. Andy's difficulty was typical. He tried to think of "some examples of later on where you would use slope" but couldn't think of any. He remembered that "a slope of a graph is equal to the mean of the density of a material in physics" but said a ninth grader wouldn't understand that and so rejected it as an example. He decided maybe he could just say that slope was useful in physics.

Others had the same difficulties as Andy. One worried, "God, I'd want some concrete examples of how this applies to real-life situations . . . [to have] some basis in reality . . .so it would be a little more valuable . . . I can't think of any at the moment." Several used ski slopes to relate slope to the real world. No one, however, was able to successfully take the concept of slope and appropriately connect the concept's mathematical significance to something real. Instead, when they did come up with something, it was an illustration of the concept of "steepness." And when asked to help students understand

$$\frac{x}{0.2} = 5$$

not one of the mathematics majors tried to connect it to a real situation.

Although the secondary teacher candidates believed that they should make the content relevant to students, they had difficulty. Janet's explanation about this reflected what many of the others thought:

> It's harder in high school because a lot of math builds up to higher math which is used in things that are beyond the kids and they don't — they can't relate to.

The mathematics majors, if they knew applications (and often they seemed not to), mostly knew technical or abstract ones even though they wanted concrete everyday examples or connections that would work with ninth graders. The way in which they understood the content just did not seem to overlap with their assumptions about high school students, so that demonstrating the usefulness or value of mathematics was difficult. Despite the fact that slope, division, and algebra all do have valuable applications to the real world which high school students could understand, and although they wanted, in principle, to connect mathematics to the "real" world, their understanding of the particular content did not prepare them to do so.

What does this <u>emphasis</u> on making mathematics fun and interesting through familiar connections reflect about mathematics teaching? This focus, yet another outgrowth of commonsense assumptions about learning, implies that the teacher's role is to get students to pay attention and, if they do, they will learn. Subject matter may be, at times, less the focus than simply a contextual factor. However, engaging students in mathematics goes beyond making math fun, and connections to everyday experience are not always appropriate or desirable (see Floden, Buchmann, and Schwille, 1987). I return to this in Chapter 6.

What *Didn't* The Prospective Teachers Know?[73]

Stated simply, the prospective teachers whom I interviewed appeared to have more ideas about teaching and learning than they did about learners, although there were a number of opportunities to draw on knowledge of children. Knowledge about students — what they know, what they can do, what they are like — is essential to the view of mathematics teaching described in Chapter 1. Yet, when the teacher candidates tried to think about students, their own experience figured prominently. They often exhibited, therefore, an egocentric perspective on learners — founded on and justified by their assumptions and understandings about themselves.[74] Some of them tried to project that experience onto all students, assuming, for example, that if they had had no trouble learning a particular topic, that others wouldn't either and that the topic was inherently straightforward. They also thought they knew what students would be interested in, although their assumptions about student interests were quite limited. Most of them were unsure, however, about what students would know and what they could expect them to understand. They were also unprepared to interpret student work beyond determining the correctness of the responses, and they seemed to lack a framework for such analysis as well as a repertoire of alternative hypotheses.

Not Seeing Mathematics Through the Eyes of the Learner

One of the most striking places where the lack of breadth in their understanding of students was evident was when they tried to appraise student written work. In order to figure out what students are doing and what they understand, knowing the content from one's own

[73] In analyzing their understandings of mathematics, what the prospective teachers knew and what they didn't were both clearly part of the discussion in Chapters 3 and 4. Yet with respect to teaching and learning and learners, I have discussed only those things about which the prospective teachers had ideas. The interviews are, however, also a source of information for what prospective teachers did not know.

Inferring lacks in knowledge must be done with considerable caution, however. If the teacher candidates did not have sufficient opportunity to talk about something, or draw on a particular idea or belief, then concluding that they did not consider it or know it would be irresponsible. At the same time, considering what prospective teachers do not bring to teacher education is as important as what they do. With caution in mind, I chose just one critical area: knowledge about students.

[74] Jackson (1986), refers to this as the presumption of shared identity. He writes, "Acting on this presumption, teachers take for granted that their students are like themselves in the way their minds work, in the way they think and feel, in what makes them laugh and cry and so forth. . . . In short, the presumption of a shared identity implies the existence of a match between teachers and students along several dimensions at once: cultural, psychological, and physiological" (p.22). My tentative use of the term "egocentric" as a label for this frame of reference is not intended to carry the pejorative connotation that the term's psychological use implies.

angle only is rarely sufficient (Buchmann, 1984; Dewey 1916b / 1964). In looking at students' math papers or considering students' questions, teachers draw on what they know about students as well as their understandings of the subject matter. Teachers must be able to see the content from the students' perspective (Buchmann, 1984; Dewey, 1916b; Hawkins, 1967 / 1974), and that means contemplating the learner as seriously as the subject matter (Buchmann, 1988). It also means both knowing and being disposed to look at how students think, what they understand, and what they typically find puzzling.

Within a longer structured exercise in which teacher candidates evaluated a textbook section on subtraction with regrouping (elementary) or slope (secondary), I presented them with a student's paper and asked them to look at it for as long as they needed and then to talk to me about what they thought the student was understanding or not.

The prospective elementary teachers examined a second grader's workbook page on subtraction with regrouping. "Susan" had not regrouped the subtrahends where necessary, except for three or four problems in the middle of the page which <u>were</u> done correctly. Instead, she had taken the <u>difference</u> between the numbers — e.g.,

$$\begin{array}{r} 52 \\ -\underline{\;29} \\ 37 \end{array}$$

This represents a significant, and common, conceptual error. From one point of view, it is also reasonable since, addition (which usually is taught right before subtraction) <u>is</u> commutative. Children who are learning algorithmically often simply generalize from addition to subtraction.

The secondary candidates also examined a student paper — this one on slope and graphing. The book page gives the y-intercept and m (the slope). Students are supposed to use these values to write the equation for the line in two different forms and graph the line. The student — "Lynn" — has transformed the equations into ax + by = c form and used a for the value of the slope, reflecting, like Susan, an algorithmic approach to the mathematics.

Examining a student paper was a foreign task for the prospective teachers. One superficial measure of the novelty of the task was the sheer time it took them to analyze the paper. Their responses to other tasks, such as examining the entire textbook section, were much quicker. Often they spent over two minutes looking at the paper before they even started to comment and then, once they had started to talk, they paused to look at it some more and to reconsider what they were saying.

Most of the elementary candidates determined that Susan was subtracting the top number from the bottom ("switching them around"). Pam drew on her own experience to interpret the paper a little further. She decided that Susan got problems wrong "for the same reasons I get some of mine wrong" — because she felt a little anxious and confused when doing math. She thought that Susan <u>did</u> basically know "how to do it." Several others, like Pam, thought that Susan knew what she was doing, but that she was probably just "forgetting when to borrow." "Switching the numbers around' was a sign of forgetting the rules, not a signal of conceptual confusion.

One implication of taking an egocentric perspective on learners, as many of them did, is that the teacher candidates will recognize students' difficulties as conceptual (rather than application or memory) difficulties if, and only if, they have knowingly had the same difficulties themselves. For example, Barb, who remembered having a lot of trouble understanding slope when she first encountered it in ninth grade (and claimed to have only recently figured out what it was really about) was especially eager to focus on students' thinking and comprehension of the concept. Andy, on the other hand, thought slope was easy and that anyone should be able to learn it "if they have any desire at all." His egocentric view justified abandoning any responsibility for worrying about students' understanding.

The prospective secondary teachers, slowly, were able to determine that Lynn's graphs did not represent the given information correctly. This took them time. Like the elementary candidates, they tended to think the student was on the right track. As Terrell observed, "It seems like she knows what she's doing, but she's just misunderstanding something." They explained her difficulties in terms of minor confusions or forgetfulness and would remind her to use the $y = mx + b$ form to graph lines.

One of the secondary candidates was quite jolted by this experience. Accustomed to thinking about mathematics from her own perspective only, Cindy struggled to make sense of a ninth grader's graphs. First she examined the paper closely, commenting softly as she noticed things: ". . .if she put in 1 there, it would be -3. If you moved that over, that would be +5. Put 1 there . . . that's 5. She might have her x's and y's mixed up. That might be it. No, that's not . . . " Cindy spent over three minutes poring over the paper. In talking later about her difficulties with the task, she reflected,

> I never really thought about . . . that you are going to have to think about mistakes. You are going to have to try to guess where they came from. I can see that this one is wrong. But I'm going like, "How could she get l and -l as points on that line?"

For Cindy, and for several of the others, the idea that teachers need to know where mistakes "come from" was in itself a revelation.[75] Moreover, they had little knowledge of students or of common student confusions. Most of their interpretations were little more than restatements of the evidence. Saying that Susan is switching the top and bottom number around is not a deep explanation of what she is doing, nor is deciding that Lynn is using the wrong coefficient as the slope.

I compared the responses of the elementary teacher candidates on this task to the responses of a group of experienced elementary teachers who were asked the same question.[76] All the experienced teachers were able to examine the paper in just a few seconds.[77] They typically offered an interpretation of what they thought was going on with the student, as well as one or two possible alternative interpretations. In contrast, some of the prospective teachers found it difficult to interpret the papers at all. Others did come up with explanations, but

[75] On this task, Cindy has not begun the necessary "shift from commonsense to pedagogical ways of thinking" that Feiman-Nemser & Buchmann (1986) argue is central to learning to teach.

[76] These data are from the Teacher Education and Learning to Teach Study currently being conducted by the National Center for Research on Teacher Education at Michigan State University.

[77] Berliner & Carter (1986) obtained similar results in a study comparing the ways in which novice and expert teachers examined, interpreted, and used information about pupils.

generally seemed to be constructing their interpretations on the spot; they had no alternatives in storage.

Although the student papers represented common confusions that students have, both elementary and secondary prospective teachers struggled to make sense of them. Neither their understanding of students nor their knowledge of subject matter was adequate for analyzing the student's work — viewing the subject matter from the perspective of the pupil — and the teacher candidates lacked knowledge of patterns of student errors. This is not surprising, for knowledge of how students think is unlikely to be acquired through the apprenticeship of observation. Moreover, it is a central task of teaching that is not necessarily apparent to students as they watch their teachers correct papers (Dewey, 1904/1965; Feiman-Nemser & Buchmann, 1986). The fact that the behavior alone is what is learned is easily observed in children "playing school," checking the correctness of their "pupils'" work and just handing the papers back to them. One teacher candidate commented about the effect that the interview had had on her: "It was one aspect I never had thought of, was that students are going to make mistakes and I'm going to have to address those." Like many of the others, she felt she "could probably present it okay," but thought she would "need a lot more instruction on evaluating the mistakes that they make."[78]

What "Should" Kids *Know*?

While they had ideas about what might be interesting to students, most of the prospective teachers felt at a loss as to what students ought to know or be able to do at particular ages. From a perspective that was once again egocentric, they relied on some vague memories about what they learned when, but most felt understandably unsure about the accuracy of those memories. Their uncertainties about what students know stood out in two specific areas: curriculum and expectations of what students know or can do.

Curriculum: Images of what students need to learn in mathematics. I asked the prospective teachers to pretend they were a beginning teacher in a grade or course of their choice and were meeting with their building principal at the beginning of the year to discuss their broad goals in mathematics for the year. What would they say were the major ideas they were going to emphasize? How would they describe their goals for their students? One purpose of the question was to learn what teacher candidates considered central to learning mathematics. Would they emphasize computational skills? Would they talk about problem solving? What goals would they articulate? Perhaps the most critical finding from the results of this question, however, was how little the prospective teachers could say.

The responses to this questions were overall flat and vague. Everyone said that it was hard because they had no idea of what was taught at each level. Five (out of 19) of the teacher candidates felt so unsure that they didn't even identify a grade. Among the seven secondary candidates who did identify a grade or course as context for the question, five named Algebra I and the other two chose eighth grade. Five of the elementary candidates named a particular

[78] Most of the experienced teachers' interpretations, like those of the teacher candidates, emphasized the pupil's difficulties with the steps of the algorithm (i.e., "remembering not to subtract up") rather than the conceptual underpinnings of the procedure (i.e., place value and regrouping). This reinforces the suspicion that teachers do not necessarily learn content from teaching it.

grade: one named fourth, two chose first, and two selected third grade. After identifying a grade they talked vaguely about getting pupils to "understand" the material. A couple of the secondary candidates who chose a course were able to identify a few key substantive topics that they would emphasize but expressed these at a general level — i.e., "work with basic algebraic equations." For the secondary candidates who thought in terms of courses, "algebra" or "logic" triggered what they could say about the key substance of what they would teach. The elementary candidates had no such anchor, saying "fourth grade" does not convey what the content might include. Furthermore, memories of their own schooling served them less well than it did the secondary candidates, since elementary school was longer ago. They seemed to know little about what students would be learning at any particular grade and said they would have to see the curriculum for that grade. With few ideas about what pupils would know or be ready to learn, coupled with dim subject matter knowledge, and little idea of the general course of school mathematics, they were at a loss to discuss what they would emphasize in teaching math. This question highlighted how unprepared they were to reason pedagogically, the issue I take up in Chapter 6.

Although they knew little about what pupils of particular ages should learn, their assumptions about students' feelings about mathematics did figure in their ideas about what they would emphasize. In place of subject matter content, most of the teacher candidates, both elementary and secondary, talked about how they wanted students to feel about mathematics and about math class. Janet spoke of wanting to "motivate students who aren't really into mathematics" and Andy would try to build his students' confidence and get them "not to be afraid of seeing numbers and letters" in algebra. Terrell wanted students to enjoy math and to do that, his goal was that they would like his class. Others emphasized helping students feel comfortable with math. These goals fit with their assumption that most students don't like mathematics and took the place, momentarily at least, of considerations about students' substantive learning.

Expectations: Images of what students know and can do. Uncertain about what they could expect children to understand, the prospective elementary teachers tended toward more conservative assumptions about elementary pupils. The prospective secondary teachers tended to be more generous in their assumptions about what high school students might know; their expectations were visible in their ideas about how to respond to a novel student idea. I presented them with a scenario in which a student brings up an unusual suggestion or question in class. For the elementary candidates, the scenario involved a second grader with an invented algorithm for subtraction with regrouping:

$$\begin{array}{r} 36 \\ \underline{-\ 19} \\ -3 \\ \underline{+\ 20} \\ 17 \end{array}$$

For the secondary candidates, the scenario involved a student asking a question about slope that intuitively foreshadowed the derivative in calculus.

Over half the elementary candidates were completely amazed, even disbelieving, that a second grader would ever think of something like that. Teri said, flatly, "They wouldn't do that." Marsha exclaimed, "Oh, my God! This is a second grader who did that?" and then hastened to say that she wouldn't want the child to show the subtraction method to other kids because it would really confuse them. Most of the others echoed this concern. Implicit was the notion that if a second grader did really come up with this, the child would be an unusual and advanced student — that, in any case, the other (regular) second graders wouldn't be able to understand it, yet another echo of the "trait view" of learning mathematics. Linda did an imitation of the "kind of little kid he'd be." Giggling, she wrinkled up her nose and bragged in a squeaky voice: "I did this at home and it works out better."

Another critical factor underlying the elementary teacher candidates' conservative perspective on what second graders would know was the fact that two-thirds of them were not sure whether the student's algorithm even made mathematical sense in all cases. With their egocentric perspective on pupils, they were dubious that a seven-year-old would say or think something like this, since they themselves weren't sure about the mathematics of the pupil's proposal.

The secondary candidates, in contrast, were not at all amazed that a student would ask a question about curves when first learning about the slopes of straight lines. They were all quite matter-of-fact in their responses to the situation. One or two said they would be excited if a student asked this, because it would show that the student was "really into it." Given their worry that students do not like math class, such a question stood for the hope that students would be interested. Also, all the secondary teacher candidates thought that they understood the question. Even though some of their answers showed that they understood the slope of a straight line as a concept unrelated to the slope of a curve (see Chapter 3), none of the secondary candidates felt uncertain or anxious about the subject matter.

Knowledge about what students know or what they should learn at particular ages is again an example of something that is unlikely to be acquired through the apprenticeship of observation. Watching teachers and doing one's own math assignments in ordinary math classrooms does not typically provide the opportunity to learn how other students think nor to generalize about children of different ages. One's personal experiences as a student are mostly of limited usefulness in figuring out what pupils know in general or, especially, what they need to learn at different points in school.

Commentary: Prospective Teachers' Images of Teaching

The Oxford English Dictionary defines an **image** as "a mental representation of something by memory or imagination; a mental picture or impression; an idea, a conception." My interviews with prospective elementary and secondary teachers show that they come to teacher education with strong images of teaching mathematics. These have developed from their precollege experiences in schools and from living in our culture, in which teaching is viewed as commonsense (Buchmann, 1987b; Jackson, 1986). They have ideas about what to do, explanations and strategies, and a picture of themselves in the role.

The word "image" derives from the Latin imago which means to imitate, an interesting connection for this analysis. The prospective teacher's apprenticeship of observation, unplanned, is an apprenticeship in the most limited sense, one that only provides practices to imitate and a limited point of view from which to consider pedagogical questions and dilemmas. The apprenticeship of observation tends to supply prospective teachers with the outward signs of teaching only: things to say and show, methods and tools.[79] Prospective teachers' images seem to center on what they would do as teachers because that is what they have been watching as students. They usually know little about the thinking behind and rationale for the teacher actions they have been observing (Buchmann, 1987b; Dewey, 1904/1964; Feiman-Nemser, 1974; Lortie, 1975). What they know about students is also thin and vague and often based largely on what they know about themselves. Although they try to remember, they cannot recall what they learned in fifth grade nor what they or any of their classmates found difficult.

Prospective teachers' ideas are clearly based on the potency of firsthand experience; however, as I argued at the beginning of this chapter, these ideas are not wholly idiosyncratic. Although they grow out of prospective teachers' own experiences in and out of school, they are warranted also by both common sense and American traditions. Given this, it is not surprising that they use "explaining" and "figuring things out on one's own" in a weak sense, nor that their views about mathematical ability are mixed between effort and ability, nor that they think good teachers make mathematics "interesting." The teacher candidates' ideas, however, pose some difficult challenges for teacher education.

One critical feature of these ideas is that they, in some ways <u>sound</u> reasonable. Of course, teachers should explain mathematics. Of course, the emphasis should be on helping kids figure things out for themselves. Of course, teachers should make math class interesting. When teacher education students talk like this, they may sound like they are developing reasonable pedagogical judgment and ideas. My results point to the importance of digging below the rhetorical surface of the teacher candidates' talk to discover what terms like "concepts" or "explanation" really mean to them. Teacher educators and prospective teachers may easily talk right past one another — about the importance of stressing "concepts," for example — without even realizing it.

A second critical feature of the prospective teachers' ideas about teaching and learning mathematics is that the ideas are not only interwoven but also <u>tightly</u> connected to their subject matter knowledge — what they understand about particular topics and about mathematics more broadly. Over and over again, this came through in the interviews — in their efforts to explain, to come up with examples, and to respond to pupils' questions and comments.

The results discussed in this chapter must also be considered in light of the fact that many of the prospective teachers feel ready to teach. One of the math majors represented this point of view when he said,

> I think I could, I think I could step into a high school classroom and teach. I enjoy talking and I enjoy things like that. I think, yeah, I might be nervous, it

[79] John Dewey (1904) warned of the effects of prospective teachers acquiring "outward form of method without capacity to put it to genuinely educative use" (p. 148). While he was arguing for an appropriate emphasis during student teaching, his argument applies just as well to a phase of learning to teach that he never talked about: the phase prior to formal teacher education.

> might be difficult, just nerves and getting comfortable but once, you know, I get into the groove, I don't think I'll have any problem.

This quote is rich with the layers of this chapter: the idea that teaching, with its central task talking, is basically familiar work into which prospective teachers are ready to "step." This 21-year-old feels he that he might be a little nervous at first, but he expects to have little problem. Considering some of the things discussed in this chapter, ought teacher candidates to feel this comfortable?

American prospective teachers' familiarity with teaching is at once a boon and a hazard for professional teacher education. A boon because teacher-students already know a lot, this familiarity means that teacher educators do not have to start from scratch. In other professions, trainees arrive with far fewer ideas about and conceptions of the work. They must begin their studies and training from the ground up.

At the same time, the apprenticeship of observation in teaching "promotes conservative, commonsense orientations in teacher thinking" (Buchmann, 1987b, p. 161; see also, Feiman-Nemser, McDiarmid, Melnick, & Parker, 1987; Lortie, 1975;). Because teaching is familiar, and because their images are vivid, prospective teachers, like the one quoted above, may think that they already know "how to teach." Buchmann (1987b) argues that prospective teachers' knowledge is held with considerable certainty, validated as it is through firsthand observation, and that, therefore, teacher candidates are not disposed to question their assumptions.

Teri, one of the elementary candidates, explained that the main point of teacher education was to "get other people's input" — to learn what other people do when they "present." Yet, the apprenticeship of observation, while it may supply pedagogical strategies and ways of acting that can be imitated, is wholly inadequate preparation for pedagogical reasoning in teaching mathematics, the subject of the next chapter.

CHAPTER 6

RELATIONSHIPS OF KNOWLEDGE AND BELIEF IN TEACHING MATHEMATICS: WARRANTS FOR PEDAGOGICAL REASONING[80]

Understanding How Knowledge and Beliefs Interact in Teaching: Beyond Humpty Dumpty

In the last three chapters I have examined prospective teachers' knowledge in three domains central to teaching mathematics: subject matter, teaching and learning, students. One at a time I made each domain the figure, using the others as ground. Analyzing different components of the knowledge required for teaching in this way is a useful exercise for it sheds light on different parts of the whole. But the whole, in the case of teaching mathematics, is not simply the static sum of component knowledge bases.

Dewey (1931/1960) once argued that, while analysis is helpful in understanding a phenomenon, the results are meaningless

> . . . whenever the distinctions or elements that are discriminated are treated as if they were final and self-sufficient. The result is invariably some desiccation and atomizing of the world in which we live or of ourselves. (p.93)

If treated as complete in themselves apart from any context, the results may ultimately misrepresent the phenomenon by ignoring the connections and continuity among the analytic elements (Dewey, 1931/1960). And so it is with the domains of knowledge and belief for mathematics teaching that I analyzed: The interactions among those domains, the ways in which they shape and are shaped by one another, are equally critical to understanding the resources upon which teachers draw in teaching mathematics.

The significance of the integration and interaction of different kinds of knowledge and belief came through clearly over and over again in my interviews with the prospective teachers. For example, teacher candidates whose understanding of the mathematics at hand was similar chose different ways of responding to the hypothetical students in the scenarios. It was also plain that their assumptions about students shaped the ways in which they would represent particular mathematical ideas, and that their notions about the nature of mathematics influenced the role they described themselves taking in students' learning. These influences were dynamic and shifted across contexts.

The dynamic is of course inherent in the nature of teaching as a purposeful activity (Dewey, 1916/1944). Teaching requires, by necessity, the consideration of different kinds of "facts" in the situation and their relationships to one another. In light of judgments about probable consequences, teachers integrate different kinds of knowledge and belief to choose

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[80] I gratefully acknowledge Suzanne Wilson for her significant help and encouragement in developing the ideas in this chapter. Her work on "patterns of pedagogical reasoning" in the teaching of American history (Wilson, 1988) complements the framework presented here.

courses of action that take into account the obstacles as well as the opportunities. In other words, teaching entails selective use and integration of resources based on judgments relative to goals (Dewey, 1916/1944).

The purpose of this chapter is to propose a way of putting the pieces of teacher knowledge back together, in a way that goes beyond summing the parts, to capture the process by which these kinds of knowledge come together and influence one another in teaching. While the work of this chapter draws on insights gained from analyzing interviews, it also goes beyond the data. My intent here is to bridge where I have been with where I would like to move: It is a chapter as much about starting points as about results.

Teaching Involves Integrating Different Kinds of Knowledge

In teaching mathematics, teachers' understandings and beliefs in one domain interact with their understandings and beliefs in others to shape their interpretations, decisions, and actions (Ball, in press). For example, a teacher cannot explain the principles underlying the multiplication algorithm to his students if he does not explicitly understand them himself. However, another teacher who does understand the role of place value and the distributive property in multiplying large numbers will not necessarily draw upon this understanding in her teaching for her ideas about learners or about learning may intervene. If she thinks, for example, that fourth graders will not profit from such knowledge or that procedural competence should precede conceptual understanding in learning mathematics, she may choose to emphasize memorization of the algorithm. No one type of knowledge — not even knowledge of subject matter — plays a singular role in teaching. Thus, even when teachers have similar ideas and understandings, they may nevertheless teach in very different ways (Wineburg & Wilson, in press). Two teachers who have similar understandings of place value and numeration may teach differently based on differences in their assumptions about the teacher's role. One may talk directly about place value and explicitly show pupils what the digits in each place of a numeral represent. The other may engage students in a counting task which is designed to help them discover the power of grouping. These differences are, in part, a function of different assumptions about the teaching and learning of mathematics.

Trying to Capture the Dynamic

Teaching is a dynamic process in which, tempered by their personal and professional dispositions, teachers integrate different kinds of knowledge and skill to achieve particular goals in particular contexts over time. As Feiman-Nemser and Buchmann (1986) state, teaching "is a moral activity that requires thought about ends, means, and their consequences" (p. 239). They argue that "pedagogical thinking" entails identifying "worthwhile things" to teach and choosing "worthwhile learning activities" to promote student learning. This kind of thinking is what distinguishes everyday, commonsense notions of teaching from legitimate professional activity (Feiman-Nemser & Buchmann, in press).[81] Within this integration of knowledge in the

[81] Worth noting is that Feiman-Nemser's & Buchmann's (1986) use of "pedagogical thinking" is normative, outlining how teachers should think, not describing how they do. In this way, Feiman-Nemser's and Buchmann's work stands apart from the descriptive research on teacher thinking (see Clark and Yinger, 1979).

service of a goal lies a key to understanding mathematics teaching in a way that goes beyond the sum of its parts.

While Feiman-Nemser and Buchmann (1986) focus on the difference between commonsense and pedagogical orientations to thinking in teaching, Shulman (1987) approaches the question of teacher thinking from another angle. Based on his recent work on teacher knowledge, especially subject matter knowledge, he focuses on how teachers transform subject matter knowledge in order to help students learn. He presses for attention to the "features of pedagogical reasoning" involved in choosing and enacting particular goals and the means to attain them.

Is this talk of "pedagogical thinking" and "pedagogical reasoning" any different from what Tyler (1949) outlined forty years ago in Basic Principles of Curriculum and Instruction? In this classic book, Tyler (1949) explains that curricular and instructional goals lead to criteria for selecting and developing learning activities for students, a perspective that sounds quite like both "ends-means thinking" and "knowledge transformation."

However, a closer look reveals that Tyler's (1949) criteria for specifying goals or activities are remarkably subject matter free as well as behaviorist in nature, an orientation that does not match either Feiman-Nemser's and Buchmann's (1986) or Shulman's (1987) visions of teaching and learning. According to Tyler (1949), learning goals should be grounded in what students "need" rather than in a liberal view of what is worth knowing in particular disciplines. Learning activities should be designed to give students practice in and satisfaction with the task or skill at hand; Tyler does not mention developing or changing students' ways of thinking. Although his "basic principles" specify components of instructional planning, he does not address the knowledge or ethical demands inherent in such reasoning. Curricular planning, according to Tyler, is portrayed as linear and straightforward.

In contrast, Feiman-Nemser and Buchmann (in press) and Shulman (1987) are calling for attention to the juggling and reconstructing that teachers must do. Their work makes curricular thinking more problematic than does Tyler's model. Shulman articulates a cyclical process of comprehension, transformation, and representation; Feiman-Nemser and Buchmann describe orientations and skills central to that process. With the recognition that pedagogical thinking or reasoning cannot be specified algorithmically, their arguments point to the need for a way to conceptualize what would count as legitimate and professional ways to engage in the process of pedagogical thinking or reasoning.

The purpose of this chapter is to integrate the separate domains of teacher knowledge discussed in the last three chapters into a model of **warrants for justifying pedagogical reasoning** in teaching mathematics. Using examples from my interviews with elementary and secondary teacher candidates, I will argue that this model offers yet another critical perspective from which to view what prospective teachers bring to teacher education.

Representing Knowledge in Teaching

Teaching involves a wide range of activities that relate more or less closely to its essential purpose: helping others learn. In previous chapters I have argued that being a mathematics teacher demands an understanding of and caring for mathematics and an understanding of and caring for students. Teaching mathematics means helping to build bridges between the subject matter and learners in ways that respect the integrity of both.

To do this, teachers explain, ask questions, respond to students, develop and select tasks, and assess what students understand. These activities emerge from a bifocal consideration of mathematics and students, framed by the teacher's own understandings and beliefs about each, and shaped further by her ideas about learning and her role in promoting learning.

Melding different domains of knowledge is inevitably at the heart of teaching. Wilson and Shulman and their colleagues have labelled the <u>product</u> of this melding "pedagogical content knowledge" and study it as a domain of knowledge in its own right (Shulman, 1986; Wilson, 1988; Wilson, Shulman, & Richert, 1987). According to these scholars, pedagogical content knowledge consists of topic-level knowledge of learners, of learning, and of

> the most useful forms of representation of [particular] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations — in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 6).

Pedagogical content knowledge has, like other domains of knowledge, a body of accumulated ideas that have been constructed and used — in this case, pedagogical representations of the subject matter. Pizzas for fractions, money for decimals, number lines for addition and subtraction of negative numbers, are but a few examples of the vast accumulation of pedagogical representations of specific mathematical topics.

In this chapter, I focus exclusively on these "forms of representation," assuming them to be the crucial <u>substance</u> of pedagogical content knowledge[82], and examine their nature, sources, and warrants: the <u>syntax</u> of pedagogical content knowledge, or the standards that guide the pedagogical reasoning entailed in representing subject matter in teaching.[83]

What is a Representation?

According to Webster's <u>Unabridged Dictionary</u>, **to represent** something is to "bring it clearly before the mind; cause to be known, felt, or apprehended." A **representation** is "a likeness, picture,or model or other reproduction; a statement or account especially made to convey a particular view or impression of something with the intention of influence opinion or action." By representations in teaching, I mean models of a wide range that convey something

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[82] In her work on the teaching of history, Wilson also focuses on pedagogical representations as the critical element of pedagogical content knowledge. See Chapter 6 of Wilson (1988).

[83] Beyond the substance (the "stuff") of knowledge in a domain, Schwab (1961/1978) presents two kinds of disciplinary <u>structure</u>, or ways to analyze that knowledge: substantive and syntactic. The **substantive structure** of a discipline is a function of the basic conceptual frames guiding its work. **Syntactic structure** refers to the doing of the work — methods of experimentation, verification, analysis, and justification, for example. Arguing that both structures are critical to understanding the nature of knowing within a particular discipline, Schwab reminds us that each discipline has multiple structures because scientists do not agree on the key questions nor on the ways of pursuing those questions. Neither do they necessarily share key concepts, given different perspectives on the phenomenon under investigation. My argument in this chapter draws on Schwab's (1961/1978) framework for understanding knowledge. I take pedagogical content knowledge, like other domains of knowledge, to have both <u>substance</u> (e.g., representations) and <u>syntax</u> (e.g. ways of reasoning).

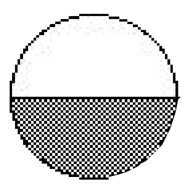
about the subject matter to the learner. For example, a story about hungry people sharing pizzas is a representation of division of fractions constructed to help students comprehend the mathematical concept. This representation is woven from knowledge of mathematics (e.g., the meaning of division, circular models for fractions), ideas about learning (e.g., people will understand things better that relate to their lives) and about learners (e.g., kids love food examples, and, in particular, pizza).

Whether they do so intentionally or not, teachers are constantly engaged in a process of constructing and using representations of subject matter knowledge, for all tasks, all explanations, all analogies that teachers use represent the subject (Doyle, 1986). Some representations are provided to teachers in textbooks, worksheets, or other teaching materials. Other representations teachers construct themselves. Some exist by default only; that is, they are unintentional, for representation is a fact of life in teaching, not an inherent good.

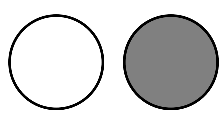
The Metaphoric Nature of Representations

Representations are likenesses, metaphoric devices. Metaphors involve the transfer of meaning from one system or idea to another (Black, 1962; Lakoff & Johnson, 1980). For this reason, metaphor is a valuable pedagogical tool, bridging the known and the unknown. It is a likeness borrowed from one domain to be used to clarify or illuminate something in another. Just as metaphors carry intended meanings only if people know and understand the metaphorical vehicle (Ortony, 1975), so it is with pedagogical representations.

Furthermore, in metaphor, an object is never isomorphic with its comparative referent. Mathematical ideas will be, therefore, by definition broader than any specific illustration or example. For example, a circle model of $\frac{1}{2}$ does not represent all there is to understand about the idea of "one-half" or its symbolic notation. The circle model is one representation of a continuous area divided into parts. The standard illustration:



is only one representation of the area meaning of $\frac{1}{2}$. Moreover, $\frac{1}{2}$ can also mean one out of a group of two discrete parts, another key model for fractions:



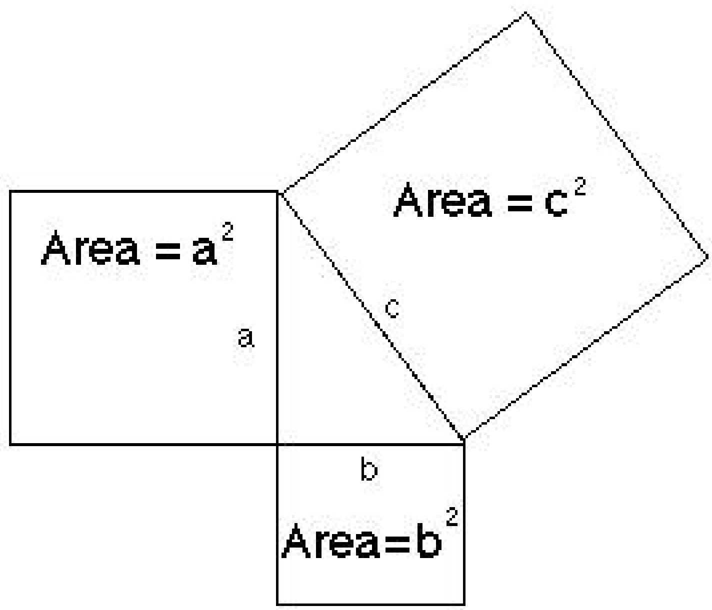
Types of Representation

Representations can be analyzed logically according to the **content, form, mode**, **vehicle** and **relation** of the representation.

A critical feature of any representation is its **content**, both explicit and implicit. What is being represented about mathematics? What specific idea or procedure? What is being conveyed about mathematics?

Form refers to what a particular representation is: questions, responses, explanations, analogies and metaphors, examples, tasks, and activities. A question, "What do the rest of you think?" posed to the entire class when one pupil proposes a solution, helps to represent the justification of knowledge in the discipline as subject to community appraisal.

Bruner (1970) identifies three general **modes** of representation: concrete, ikonic, and symbolic.[84] In mathematics, the distinctions can be made more helpful by being more discriminating: concrete, verbal, pictorial, graphical, or symbolic. The formula $\mathbf{a^2 + b^2 = c^2}$ is a symbolic representation of the Pythagorean theorem which expresses the relationship among sides of a right triangle. The same idea can be represented pictorially:



Representations, as metaphors, use **vehicles**. These vehicles are the stuff of the representation: M&Ms for statistical sampling, pizzas for fractions, class discussions for presenting and proving conjectures within the mathematical community. These vehicles provide the means for bridging the familiar with the unfamiliar in helping students learn mathematics and, in order to be useful, must be themselves understood by students.

Finally, representations, as metaphors, are **relational**. They may connect one mathematical idea with another, link a concept or procedure with an idea in another subject, or connect to something in the world around us. Some mathematics is itself a representation or modeling of the physical world; in this way, mathematical representations are sometimes

[84] Some would argue that pedagogical representations in mathematics should be catalogued into finer-grained mathematical modes: concrete, verbal, pictorial, numeric, algebraic, geometric, for instance (Lanier, personal communication).

bidirectional in their relation (see the discussion of slope in Chapter 3). Other concepts and procedures may have little connection with the world around us.

These logical features of representation provide multiple perspectives from which to examine particular representations. The features are neither entirely discrete nor universally applicable. For example, a class discussion in which students present and defend conjectures is a representation of mathematical argument within a mathematical community. "Class discussion" is both its form and its vehicle; yet it is not relational in nature or structure. The usefulness of this framework is to provide a means of analyzing the components of representations, absent analyses of their generation, justification or value.

What Do Representations Represent?

Through the representations that they select and the ways that they use them, teachers convey messages — sometimes implicitly, sometimes explicitly, intentionally or unintentionally — about both the substance and the nature of mathematics to their students. In mathematics teaching, the enacted curriculum — the "school subject" — is itself a representation of the discipline. In the view of mathematics teaching underlying this study, the aim is to represent mathematics appropriately in terms of the discipline. Thus, the tasks, explanations, and models teachers use must adequately portray the conceptual and epistemological underpinnings of mathematical knowledge.

Even when the messages are unplanned, however, the representations used in any approach to teaching mathematics form a specialized hidden curriculum and thereby shape students' opportunities to learn about mathematics. For example, a worksheet of 52 long division exercises may communicate to students that division is no more than a procedure to be memorized. Students repeat the steps "divide, multiply, subtract, bring down" until they runs out of digits and finish with results such as 578R5, answers with little meaning for fifth graders[85], although this depends on the context in which the instruction takes place. Such assignments suggest that doing mathematics consists of following steps and rules. Furthermore, the fact that the teacher takes charge of verifying the correctness of their answers obscures from students the means for reasoned justification of knowledge in mathematics. Consequently, the understandings of mathematics and of particular mathematical ideas that students develop may be inappropriate from a disciplinary perspective.

Where Do Representations Come From?

Teachers' representations for teaching derive from two primary sources, one outside themselves and the other within. Outside sources include curricular materials, colleagues, courses or workshops for teachers as well as their own academic studies, reading, and the culture around them, among others. Mathematics teaching materials — textbooks, kits,

[85] Analysis of the most recent round of results from the National Assessment of Educational Progress (Carpenter et al., 1983) revealed that students could not select the reasonable response to a division problem even when they could calculate the answer correctly. For example, they would select an answer with a remainder when the context of the problem required a whole number solution, such as: A bus holds 66 people. How many buses are needed to carry 478 volleyball players? While the calculated answer may be 7 with a remainder of 16, the answer demanded by the problem is 8.

computer software — are themselves representations of the discipline. Both the topics or ideas included, and the way in which they are set out represent the substance and the nature of the subject matter to pupils. In many math textbooks, for instance, doing mathematics seems to mean performing computations and knowing the right answers (i.e., matching the answer key).

Besides tacitly representing the subject, accompanying teachers' guides also explicitly suggest activities, questions, and ways of explaining the content. For example, one high school algebra text offers the following:

> There are several models that may help illustrate the definition of slope: the pitch of a roof, the grade of a hill; the rise / run of a stairway or a ramp. (Dolciani, Wooton,& Beckenbach, 1980, p. T56)

Teachers may acquire ways of representing the subject matter from other teachers: A methods instructor may advocate the use of base 10 blocks for teaching place value; a colleague may pass along an analogy for explaining negative numbers; a college professor may have used a particular approach to engaging students in making and defending conjectures; an elementary school teacher may have impressed them with a statistics project. Through a variety of sources, teachers collect both tangible and intangible "stuff" for representing mathematics.

Over time, teachers develop a repertoire of subject matter representations. As these representational repertoires develop, teachers have more options for connecting students with subject matter (McDiarmid, Ball, & Anderson, in press; Putnam, 1987; Putnam & Leinhardt, 1986; Shulman, 1987; Wilson, Shulman, & Richert, 1987).[86]

<u>Beyond the Substance of Pedagogical Content Knowledge: Pedagogical Reasoning</u>

An extensive repertoire of pedagogical representations is not sufficient for teaching mathematics, however. Teachers must be able to <u>appraise</u> the pedagogical potential of an available representation and how well it fits the context. This includes evaluating the approach used by the school's textbook, assessing activities passed on by fellow teachers, considering the relative value of particular questions, and analyzing models or pictorial representations. Is money a good model for helping children understand decimals? Is a particular textbook's explanation of division by zero appropriate for eighth graders? Would investigating probability by examining the chances of winning the lottery be a problematic activity in a small New England town? Is this fraction activity something with which <u>I</u> am comfortable?

In appraising representations, teachers must also ask themselves: What else might my students learn from this particular activity or model? Might they learn something I <u>don't</u> want to convey? For example, if I use money to represent something again, am I implicitly communicating a utilitarian, consumer perspective on mathematics? Because every representation falls short in some way, teachers must also ask themselves how they will

[86] Although teacher accumulate and use tasks, questions, examples, and ways of responding, they do not necessarily think about them as representations of knowledge, but, many times, simply as pedagogy — activities, neat things to do or say.

compensate for the inadequacies, the undesired aspects, and the missing dimensions.[87] Modifying representations to better suit the content, the learners, learning, or the context, is therefore another important part of the process of pedagogical reasoning.

In addition to appraising and using available representations, teachers also invent their own, fashioning representations from their understandings, knowledge, values, and experience. They create units of instruction, make worksheets, design activities, develop explanations, think of questions, and respond to students. Often these are modifications of ideas gathered elsewhere; sometimes they are original inventions. The construction and revision of pedagogical content knowledge is an essential part of teaching, for no repertoire of representations could possibly suit all possible teaching contexts.

What Makes a Representation *Good*?

I have presented a number of dimensions along which pedagogical representations can be examined; however, issues of value have not been addressed. The criteria of worth form a framework of pedagogical considerations that can be used to evaluate representations as well as guide their generation. How may teachers think about choosing, modifying, or constructing representations? These are important questions about the warrants for pedagogical reasoning, questions about standards for the use, generation, and justification of knowledge in teaching mathematics. As I will show, the **warrants** for justifying pedagogical content knowledge also provide a basis for examining pedagogical reasoning.

In order to better understand the relationship among different domains of knowledge in teaching and to critically appraise the pedagogical reasoning entailed in choosing, using, and constructing representations, I propose below an analysis of mathematical pedagogy from an epistemological perspective. I assume, for the sake of argument, a conception of pedagogical content knowledge as a domain of knowledge in its own right, and then draw on Schwab's (1961/1978) ideas about disciplinary structures as conceptual levers to construct a model of the justification of knowledge and reasoning in the domain. This analysis offers a way to describe and appraise the ways in which teachers integrate different kinds of knowledge and belief in teaching mathematics.

Warrants for Pedagogical Content Knowledge: Justifying Pedagogical Reasoning in Mathematical Pedagogy

How can the representations — questions, tasks, responses, models, examples, and activities — that teachers select, modify, and construct be analyzed? How can they be appraised? I propose below a normative framework of **warrants** for judging the products and process of pedagogical reasoning that underlie representing mathematics to students.[88]

In any approach to teaching mathematics, the sources of justification derive from the cornerstone domains of subject matter, learners, learning, and context. What is central,

[87] The accumulation of individual representations a teacher uses to represent mathematics produce global-level representations of the subject, of a particular idea (e.g., fractions) or activity (e.g., mathematical problem solving). In this way, "representation" exists in different conceptual sizes.

[88] Pedagogical reasoning is, of course, informed by and subject to other considerations — moral , ethical, and social, for instance. See Scheffler (1958).

however, about the development and use of pedagogical content knowledge depends on one's view of the goals of mathematics teaching. With a view in which the goal is to help students develop proficiency with mathematical calculations, for example, representations that support students' memorization, skill, and speed — such as mnemonic devices or games — would be of central importance. Representations that highlight the key steps of an algorithm (e.g., first, outside, inside, last = foil, as a mnemonic for factoring in algebra), for example, or that portray mathematics as a domain of linear step-by-step procedures would be appropriate.[89]

Because mathematical pedagogy is fundamentally concerned with engaging students in mathematical thinking and activity, however, the subject matter warrants are different and take on more weight in considering pedagogical representations.[90] Figure 6.1 shows the key warrants and criteria for pedagogical reasoning in mathematical pedagogy.

[89] Different goals suggest different criteria for what constitute "worthwhile" learning activities (Feiman-Nemser & Buchmann, in press). This parallels Schwab's (1961/1978) substantive structure — that different substantive orientations or perspectives lead to different standards and foci, different questions and priorities. A view of teaching and its associated goals determine, to some extent, the substantive structure of that view of mathematics teaching. Similarly, Dewey's (1914/1944) argument about aims suggests that, in teaching, goals are instantiated in the selection and enactment of means.

[90] Questions about the balancing and weighing of competing warrants are critical to pursue in developing a framework of pedagogical reasoning. I discuss this more in Chapter 7.

MATHEMATICS

conceptual essence
epistemological appropriateness
appreciation of and propensity toward mathematics

LEARNING

focus
differentiation
multifacetedness

LEARNERS

accessibility
interest

CONTEXT

feasibility
sensitivity

Figure 6.1
Warrants for pedagogical content knowledge and reasoning in mathematical pedagogy.

Mathematical warrants. The substance and nature of mathematics is a critical dimension of any pedagogical representation in mathematical pedagogy. Representations must distill and simplify ideas without distorting them. Substantively, a representation should feature the **conceptual essence** of the content at hand, not just its surface or procedural characteristics. For example, to divide $18\overline{)459}$ the mnemonic "divide, subtract, multiply, bring down" for long division represents nothing more than the steps in a conventional algorithm. In contrast, the question, "If we have $459 dollars, how many hundred-dollar bills can each of 18 children have if they are trying to share the money equally? How many ten-dollar bills?" represents the meaning that underlies those steps, and connects the written symbolic form to a real situation.[91]

Beyond the substance of mathematical knowledge, representations should be **epistemologically appropriate —** they should appropriately portray a disciplinary view of what it means to do and know mathematics. On one hand, using the answer key as the means for arbitrating among competing solutions, while common practice, conveys a misleading

[91] The answer is 0 hundred-dollar bills, as $100 x 18 = $1800, but 2 ten-dollar bills since $20 x 18 = $360. This represents why, in the conventional algorithm, we write a 2 above the 5 in 457 — there are 2 tens in the answer.

image of the nature of mathematical knowledge and of what it means to do mathematics. On the other hand, expecting students to explain and justify their claims represents the reasonable and constructive nature of mathematics, and illustrates the interaction between accepted mathematical knowledge and the generation of new ideas. When students have to try to persuade one another of the reasonableness of their suggestions, they experience the purpose and value of proof.

Mathematical pedagogy aims to engage students in mathematics in ways that help them see that they can be both competent and confident in this domain. Therefore, teachers may want to verify that the representation is likely to support the development of **appreciation of and propensity toward** mathematics.

Warrants of learning theory. Pedagogical representations should help pupils learn particular mathematical topics or ideas. Three criteria emerge from this purpose: **focus**, **differentiation**, and **multifacetedness**.[92] All three, while fundamentally about mathematics, are also necessarily about learning. This shows, once again, how pedagogical reasoning is a specialized kind of thinking that integrates relevant domains of teacher knowledge.

In order to learn mathematics, students' encounters with substance and nature of the subject should focus their attention on key elements. Having a sense of the "big" ideas helps the learner to sensibly organize what he or she knows, affording greater understanding and control. Thus, the criterion of **focus** addresses the extent to which the form and relation of the representation call attention to the conceptual essence of the content. That is, representations should spotlight the central components.

A nonexample of focus illustrates the point best: Allen's mango model of subtraction with regrouping (see below, as well as in Chapter 5) focused on regrouping "twentyfourness" and on mangoes as an exciting medium. Yet regrouping twenty-four objects is not the only key conceptual issue in "borrowing." Additionally, because of their novelty, mangoes as a model are likely to distract from rather than highlight the content. The decimal numeration system and its relationship to grouping, the mathematical heart of the matter, are not in focus in the mango model. As such, this representation seems unpromising as a means of helping students understand regrouping in subtraction. I return to this in the next section, analyzing Allen's representation more closely.

Mathematics is a logical system, a system of ideas and ways of thinking. To help students learn mathematics, they must understand the parts and how they fit together as well as knowing the "big" ideas. **Differentiation**[93] is the key factor here. Does the representation make the parts plainer, the thinking apprehendable? For example, using graph paper to investigate the area of different rectangles makes visible the formula for area. This model also accentuates the difference between perimeter and area. In comparing the relative sizes of disparately shaped rectangles, students can make conjectures and try to substantiate them both empirically and logically. Graph paper, as a representational material, reveals several key components of content and process in measurement.

[92] Teachers' use of these criteria is obviously determined by their own understanding of the content, as I will illustrate below.

[93] In analyzing people's knowledge of history, Wilson (1988) uses "differentiation" to refer to an understanding of the major parts of a historical phenomenon.

Differentiation reveals that most content has multiple conceptual dimensions; any single representation is unlikely to carry all of them. Thus, a third warrant for learning is **multifacetedness**. Over time, the pedagogical representations encountered by students should be multiple and should round out the conceptual essence of a particular idea. Looking at the division example again helps to illustrate this point. Taking 459 bingo chips and counting out them out into 18 equal piles provides a concrete representation of one meaning of division; taking the 459 bottles and packing them into cartons that hold 18 bottles each shows another. Furthermore, each representation leads to a different interpretation of the answer. With money, each child gets $25.50. With bingo chips, each pile has 25 chips and 9 are left over. And with the bottles, one needs 26 cartons to pack them all, but 25 cartons are completely full. In each case, the computed answer, 25.5, takes on a different meaning, showing students that mathematics must be interpreted in context.

The three warrants of learning theory are interdependent although not necessarily coincident. Differentiation contributes to focus through a topographical mapping of a topic. What are its components? How do they relate to one another and to other, outside, ideas? Multifacetedness, too, grows out of differentiation and focus, for knowing the key and contributing components of a topic permits the teacher to ensure polyfocal representation over time. Still, a representation may have focus — that is, it may emphasize a key component — but lack differentiation. A representation of multiplication that dealt only with repeated addition would have focus, yet lack attention to other aspects of the concept which would need to be compensated for through other, multifaceted, representations.

Warrants based on knowledge of students. The questions, responses, and tasks that mathematics teachers give are effective only if they connect with the students. The selection and creation of good representations is therefore critically dependent on their match for learners. Two criteria are central: **accessibility** and **interest**.

For representations to be helpful to students in learning mathematics, they must be accessible to students; this **accessibility** is primarily a function of their comprehensibility. Prior knowledge is key here. On one hand, explanations, examples, and activities employed to help students learn mathematics are more likely to carry the intended messages and meanings if their content is familiar to students. For example, although electrical charges may provide a promising model for the multiplication of negative numbers, sixth graders are as unfamiliar with the behavior of electricity as they are with the behavior of negative numbers. As such, this vehicle will not contribute to an accessible representation for teaching about negative numbers.

On the other hand, analogies may also play into or foster students' misconceptions about a topic. For example, the everyday idea of "borrowing" may distract students from regrouping and place value two-digit subtraction, and may encourage them to think of numbers in the righthand column "borrowing" equal-sized numbers from the next column.[94]

To attract students' focus, **interest** is a critical criterion. Interest includes both those

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[94] For instance, in

$$\cancel{3}^{1}6$$

the 6 borrows a 1 from the 3 — when it is actually 36 being regrouped and expressed as 2 tens and 16 ones.

things about which pupils care, as well as those things which might stimulate students' engagement in the subject. In other words, interests can be sparked as well as used. Furthermore, choosing representations that interest students per se or that expand their horizons of interest helps to convey that mathematics is itself interesting.

Warrants of context. Valuable pedagogical representations must suit the contexts of teaching by being both **feasible** and **sensitive**. To be **feasible**, they must be reasonably possible to accomplish within the situation. For example, collecting a million bottle caps as a means of helping second graders develop a sense of very large numbers is not a feasible activity, but engaging eighth graders in proving that it is not feasible is. Eighth graders can figure out that collecting a million bottle caps requires that the class come up with 5,556 bottle caps every single day of the school year. They can see that this means that each child must bring in, on the average, 186 bottle caps per day. Playing around with different compositions of 1,000,000 across 180 days and 30 children is a feasible task for 13-year-olds in a classroom. Another aspect of feasibility is the teacher's comfort and skill with the representation — is it something that he feels he can pull off?[95]

Sensitivity is a second contextual criterion. Representations, as borrowed likenesses, necessarily carry with them other features beyond those which connect them to the content. Certain features may be unacceptable or unhelpful whereas others may mesh particularly well within a given cultural or social context. For example, in some Eskimo cultures, the notion of a "fair share" implies that each person gets what he or she needs, not necessarily a share that is equal to others' in quantity or quality. To represent the concept of division (which implies equal portions) using situations in which people are trying to share food fairly may not make a lot of sense to children from this culture (McDiarmid, Ball, & Anderson, in press). Yet, the cultural or social contexts may range in receptiveness to certain representations. Teachers need to factor these into their pedagogical reasoning.

Pedagogical Reasoning and Argument

Even with explicit standards for what makes a good representation in mathematical pedagogy, no representation is perfect. First, as borrowed likenesses, representations are never isomorphic with the object of representation. Buchmann (1988) reminds us that pedagogical reasoning as a process cannot produce necessary logical or unambiguous conclusions: "The arguments of teachers will, of necessity, involve contents, persons, conflicts, and interpretations; together with criticism and regrets, these elements account for the continuity and inconclusiveness of teacher thinking" (p. 10). As such, the warrants for pedagogical content knowledge in mathematics that guide the development and use of pedagogical representations are signposts only. These signposts mark the essential considerations for pedagogical reasoning about the teaching of mathematics. However, as signposts, the warrants neither bound nor prescribe the products of that process, for, in teaching, competing considerations come into play. Teachers cannot completely resolve all these competing considerations at once (Berlak & Berlak, 1981; Buchmann, 1988; Lampert, 1985a). Furthermore, Lampert (1985a) reminds us that

[95] The latter dimension of feasibility is different in that teachers' comfort and skill are things that can change.

this juggling is not abstract — teachers' pedagogical reasoning takes place in concrete, dynamic situations, inherently fraught with dilemmas. The point here is that justified action and logically necessary conclusion are not the same thing.

The standards for good pedagogical representations are thus anything but straightforward and easy to apply. Students are diverse, even mathematicians disagree about the nature and substance of mathematics (Davis & Hersh, 1981; Kline, 1981), and representations that work well for one purpose or context may be inadequate or misleading in other respects.

A brief example illustrates the conflicted nature of pedagogical reasoning and justification in mathematical pedagogy. Money is often used to model decimal fractions. Students can explore the base ten relationships among pennies, dimes, dollars, and so on. Taking advantage of their familiarity with coins, the model relates mathematics and the everyday world.

This is a common and reasonable representation. Yet, like all representations, it is unevenly warranted. From the mathematical perspective, our money system does model base ten relationships well. Yet, except for gas prices (i.e., $.97^9$ per gallon), money does not model the number system beyond hundredths, a limitation. Considering how well it might help students learn, money does focus on some critical components of the concept of decimals: part-whole relationships and base ten. However, since nickels and quarters are equally valid components of our monetary system, students may be distracted from the critical base ten concept. Moreover, analysis points up a shortcoming: Nothing inherently requires you to "trade up" when you have 10 of any coin: "What's `wrong,' for example, with having 16 dollars, 22 dimes, and 14 pennies?" (Schoenfeld, 1986, p. 235)

What about learners? Ten-year-olds are both familiar and competent with money, and so money may serve as a useful link to understanding decimal numeration. That they already know that 100 pennies are equal to a dollar, and that "$.01" represents a penny may help pupils to understand the number "one hundredth" and its notation. Children of this age also usually like money; using money may enhance their attention to a lesson. Is it necessary to use real money to exploit this representation with fifth graders? If so, it may be unfeasible.

This example reveals the multiple considerations teachers face in trying to represent mathematics in appropriate and helpful ways for their students. Still, the standards discussed above do provide an initial framework for evaluating both the products — pedagogical representations — and the processes — pedagogical reasoning — of teaching mathematics. The following extended example, taken from Lampert (1986), illustrates the ways in which an experienced teacher might reason about and justify her decisions about representation in teaching mathematics when she holds mathematical pedagogy as a view of good teaching. In the following discussion, I rely on Lampert's voice from her writing about her own teaching. This example provides an image of warranted pedagogical reasoning against which to compare the reasoning of the prospective teachers.

The Pedagogical Reasoning of a Fourth Grade Teacher

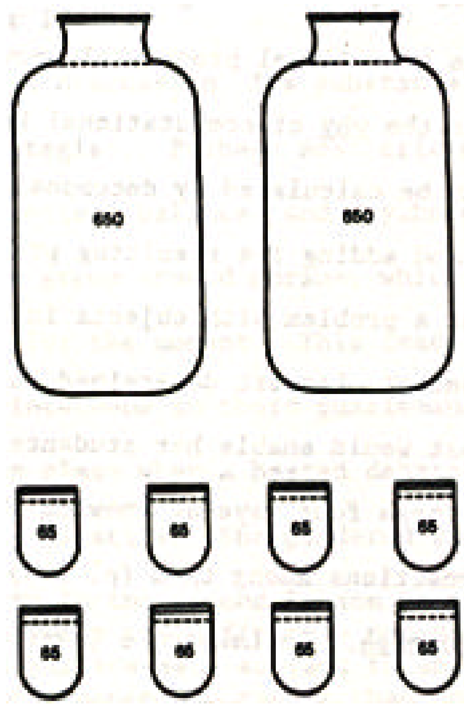
Early in a unit on multi-digit multiplication, Lampert engaged her students in a coin problem: "Using only two kinds of coins, make $1.00 with 19 coins." She explains that the procedures needed to solve this problem entail many of the same mathematical principles as those that underlie multiplication computation: additive composition, distributivity, place value, among others. Lampert notes that, although money does not map perfectly onto place value (e.g., nickels and quarters are reasonable in money, but not part of the base ten system), the students' familiarity with money gave them "the opportunity to do mathematics confidently in an area where they would later be introduced to more abstract forms" (p. 318) — i.e. multiplication computation. As the class did other coin problems, Lampert guided them to record their solutions in ways that helped to make transparent the multiplication and addition composition procedures involved. For example, in trying to make a dollar out of combinations of dimes and / or nickels, children would record both their attempts and their successful solutions as follows:

d	n	total amount of money

Lampert argued that, although this recording system was quite abstract, the fact that it was built on the students' familiarity with exchanges of coins meant that it could help to prepare the students for the unfamiliar procedures of multidigit multiplication.

Next, Lampert launched a different representation. She decided to engage her students in telling and illustrating multiplication stories. For example, she asked the pupils to come up with a story for 28 x 65. One girl suggested 28 glasses with 65 drops of water in each glass. Lampert accepted this proposal, although she was aware that drops of water were problematic as a representation of quantity (p. 327).

She told the class that she did not want to draw 28 glasses on the board so she would draw big jugs that held the equivalent of 10 glasses. She asked the class how many jugs and how many glasses she would need in order to represent the 28 glasses. They told her: 2 jugs and 8 glasses. As she drew big jugs and glasses on the chalkboard, she queried again: How many drops of water in each glass and in each jug? Each time a student answered, Lampert asked the student to explain his or her answer. The chalkboard drawing looked like this:



Next Lampert asked the class how they could find out how many drops of water there are altogether. They said that they should add the jugs and glasses together. The pupils understood readily that the two jugs contained a total of 1,300 drops. Lampert then proceeded to teach them a "trick" that made it easier to add the 8 glasses together: She suggested that they could take 5 drops of water out of each glass and put them in another container, leaving 60 drops in each glass. She asked the class how many drops would there be in all the glasses then. Someone explained that it would be 480 with just 40 drops in the other container. Combining those yields 520 drops, and adding those to the 1,300 equals 1,820. Lampert pointed out that by using "clever groupings" they had figured out 28 x 65 without doing any paper-pencil computation.

At one point, one of her girls said she had come up with "another way of thinking about it." Lampert listened intently and wrote the student's explanation on the board "so as to give it equal weight in the eyes of the class" (p. 329). Together the class explored whether and why the student's idea made sense mathematically.

Lampert spent a few more days using students' stories to draw pictures and examine the ways in which the numbers could be decomposed, multiplied in parts, and recombined. Next she constructed assignments which required the students to make up and illustrate stories, as well as write the numerical representations. Sometimes she asked them to decompose and recombine the quantities in more than one way. They presented and defended their solutions to other class members. Lampert moved on from this to work with her class on the meanings of the steps in paper-pencil computation, using alternative algorithms (i.e., "no-carry" method) as well as the traditional one.

In writing about this work, Lampert (1986) reflected on the contributions of this series of lessons to her overall goals in teaching multiplication:

> They were using the language and drawings we had practiced to build a bridge between their intuitive knowledge about how concrete knowledge can be grouped for counting and the meaning of arithmetic procedures using arithmetic symbols. By rewarding them for inventing reasonable procedures rather than for simply finding the correct answer, I was able to communicate a broader view of what it means to know mathematics and learn something from what they were doing about how they would use mathematical procedures in a concrete context. (p. 330)

She observed also that her students were gaining in their ability to substantiate their claims using reasons "that came very close to the steps of a mathematical proof as well as inventing "legitimate variations on both concrete and computational procedures" (Lampert, 1986, p. 337).

Lampert's Warrants

In reasoning about and justifying her decisions about how to represent mathematics to her students, Lampert explicitly considered what she knew about students, about learning and about mathematics into account (see Figure 6.2).

MATHEMATICS
conceptual essence
epistemological appropriateness

LEARNING
focus
analysis
multifacetedness

LEARNERS
accessibility

Figure 6.2
Lampert's warrants for representations in teaching multidigit multiplication

Mathematics. Lampert's goal was to represent the conceptual essence of the mathematics she was teaching. Analyzing multiplication and place value, she isolated several key principles that formed the basis for both the concept and the procedure, and she chose tasks, focused her questions, and offered suggestions that she thought would contribute to students' engagement with those key ideas. For example, when she gave her students a system for recording their solutions to the coin problems, it was a system intended to feature the conceptual basis of composition. Lampert also carefully considered the appropriateness of the epistemological messages she conveyed by expecting students to justify their ideas, by encouraging them to record false starts as well as successful solutions to problems, and by explicitly valuing alternative approaches.

Learners. While Lampert recognized some problems with money as a model for the mathematics she was teaching, she decided that using coin problems would take advantage of students' familiarity with and interest in money. They were thereby able to use the familiar to access the unknown. Because half her students are from countries outside the United States, Lampert also spent some time teaching about American coins so that her students could understand the model she was using.

Learning. Lampert used multiple representations — coins, charts, pictures, the conventional algorithm — in order to focus on different features of the content. These representations were verbal, pictorial, and symbolic in nature. Her analysis of multiplication helped her to see the component underlying principles; she used this insight to make sure she highlighted different aspects of multidigit multiplication over the course of her work with her students — place value, composition, and distributivity, for example.

Context. Lampert does not mention context explicitly in her explanations of what she chose to do. Still, the representations she selected seemed to be feasible, for she managed to do the things she chose. Sensitivity is difficult to evaluate without more information about the context; however, no signs of insensitivity to context are apparent.[96]

Pedagogical Reasoning: Fusing Knowledge and Disposition

Lampert's choices and her justifications for those decisions reflect both what she knows and believes as well what she holds to be important — her goals. Seriously committed to engaging students in genuine mathematical discourse and activity, she is inclined to seek representations that facilitate those goals. She is disposed to see and thoughtfully weigh competing considerations in teaching mathematics. Without her understandings of mathematics and her knowledge about fourth graders, however, her capacity to generate good representations — tasks, questions, and so forth — would be severely hampered. Neither knowledge nor disposition alone can account for teacher reasoning, a fact that becomes still more clear when we turn to examine the reasoning of the prospective teachers.

Prospective Teachers' Representations and Ways of Reasoning[97]

How do prospective teachers' representations and ways of reasoning compare with this model? What warrants do they consider? What criteria do they hold to be important? These questions offer another critical perspective on the question of what prospective teachers bring with them to teacher education related to the teaching of mathematics. In other words, we ask not only what prospective teachers know and believe, but also how they think and reason.

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[96] Her failure to mention context in her written account may be because context bounds her pedagogical reasoning; it may precede the specific considerations involved in a particular lesson. Questions about the priority or place of these warrants in relation to one another is an issue I have not yet unpacked (see Chapter 7).

[97] See Chapter 6 of Wilson (1988), in which she analyzes the patterns of pedagogical reasoning of four different categories of people — expert history teachers, expert English teachers, novice social studies teachers, and historians — in teaching U.S. history.

This, the substance and syntax of their knowledge, is what they bring with them to their professional preparation.

None of the prospective teachers took the representation of mathematics as a discipline as their guiding principle. Instead, they were most concerned either with engaging students' interests or being direct and clear about the specific mathematical content. In this section, I illustrate these two patterns of reasoning, and show how they differ from the model of pedagogical reasoning in mathematical pedagogy. I do not claim that these three are the only patterns of reasoning used by the teacher candidates, but only that these patterns were prevalent. I introduce and discuss them here primarily to support the argument for paying attention to teacher candidates' ways of reasoning about the teaching of mathematics, and to further illustrate the interaction between knowledge and disposition in pedagogical reasoning.

Making Mathematics Fun or "Relate It to What Kids Know"

In reasoning about tasks, explanations, and examples, many of the teacher candidates relied heavily, if not exclusively, on the learner warrant: What will students find fun or interesting? What will they be able to relate to? The prospective teachers' focus on the learner warrant reflected their underlying assumption about learning: that if children are having fun or are able to "relate" to the material, they will learn. To illustrate and analyze this pattern, I return to Allen's use of mangoes as an instance of this way of thinking.

Allen generated an original representation for teaching subtraction with regrouping, using mangoes as the model. He described how he would "get a lot of mangoes" and place a pile of them in the middle of the floor. Then, he explained,

> You would arrange 'em in columns of tens and ones. You have 1 group of 10 mangoes and 1 group of 4 mangoes — or 4 groups of 1 mango, I guess you would have to call them. And, uh, then, then you would, uh, you know, write the number 14 with that to go along and, and you'd want to separate 'em like they have here to show the borrowing part and, uh, then actually go through the borrowing process showing that if you take, you know, the 10, 10 mangoes from. . . . (pause) oh, 24 mangoes would be there! If you had some left on the left hand side when you took 10 away, that, uh, it doesn't change the number of mangoes any. Which is really what borrowing is about. It's the manipulation of numbers without changing them and I would have them just rearrange the mangoes in different orders and have them write the number that goes along with how many mangoes there are and, uh, have them borrow.
>
> Right, and then move 10 mangoes over to the right hand side and you would have 1 and 14.

When asked why he thought of using mangoes, Allen explained that "it sounds fun! If you're juggling mangoes all around, they're gonna enjoy it and they're gonna learn it."

What did Allen seem to take into account in generating and justifying his representation? His primary justification seemed to be that mangoes would be fun for the pupils; he assumed that students would therefore learn the content. He also considered what he thought was the conceptual essence of the content: the process of "manipulating the numbers without changing them." After arranging some number of mangoes in piles of tens and ones, Allen thought that the mangoes would show how, even when one rearranges the piles, the same number of mangoes remain. This is, however, closer to a demonstration of conservation of number than it is of regrouping and place value.

Comparing Allen's reasoning to the model of warrants for pedagogical reasoning in mathematical pedagogy (shown in Figure 6.1) reveals that fun (or interest), not mathematics, is his primary criterion, and serves as his learning warrant — i.e., children will learn if they are having fun. He justifies his model by his sense of the conceptual essence of "borrowing" (see Figure 6.3). He does not appear to consider context, and mangoes are probably not a feasible pedagogical representation, due both to expense and size. Allen's learning warrant — fun — is distinct from the warrants in the model. Using 64 (or even 24) mangoes with a group of children is unlikely to focus them on key features of the content; instead, the mangoes are likely to distract the pupils' attention, something Allen does not seem to consider. Furthermore, as a model of place value, piles of mangoes are not especially effective at highlighting key components of the concept. The "tens piles" of mangoes look barely different from a pile of 7 mangoes, and do not necessarily make plain why we write seventeen as "17." That we have one pile of ten mangoes is a weak model of the meaning of the 1 in 17. Allen's emphasis, however, when he describes the conceptual essence of "borrowing," is just on physical regrouping. He does not seem to emphasize the relationship between number and numeration: that 1 in the tens place means 1 ten. In this way, although he does consider the content in justifying his representation, his own understanding determines the result.[98] Finally, Allen's representation does not seem to model the doing and knowing of mathematics from a disciplinary perspective. In all, Allen's representation is shaped by his understanding of the content and his aim of making mathematics fun, which, he assumes, will lead to learning.

MATHEMATICS "conceptual" essence (partial)
LEARNING (fun)[99]
LEARNERS interest

Figure 6.3
Allen's warrants for mangoes as a representation of borrowing

Emphasizing representations that would interest students was the frame evident in many of the prospective teachers decisions about examples, explanations, and activities. On this

[98] Many of the elementary candidates focused on subtraction and ignored the place value and regrouping issues; a few focused on "borrowing" as the concept and related it to borrowing in an everyday sense (see Chapter 3).

[99] In the sketches of prospective teachers' reasoning, I put in parentheses criteria used by teacher candidates that are not part of the model.

basis, food was often a popular representational material. This overarching emphasis on interest was also clearly the basis for the elementary candidates' appraisal of the textbook section on subtraction. The prospective teachers liked the colorful pictures and the school fair story line, explaining that "it gives the kids something to relate to" and would attract and keep pupils' attention.

Sometimes "interest" was interpreted more as "fun," other times as "something kids can relate to." Andy's efforts to teach slope illustrate the latter emphasis, and underscore once more the interdependency of knowledge and disposition in pedagogical reasoning. Andy said he wanted to think of some examples to use with students in teaching about slope "something that someone in ninth grade could grasp," but couldn't think of any that he thought would be appropriate:

> You know, the slope of a graph is equal to the mean of the density of a material or something, in physics — which some ninth grader's not gonna grasp. . .I mean you could say, well, you use this in physics to figure out density and things like that.

Andy was dissatisfied with this. Although he wanted to use example in teaching slope, he was unable to think of anything that ninth grade students would be able to understand.[100] Part of this was the result of a lack of knowledge about high schoolers, but, mostly it was a consequent of his understanding of slope, which was procedural and rule-bound. Despite his disposition to relate mathematics to real situations that his students would find interesting, his own mathematical knowledge limited Andy's capacity to act on his commitment.

"Going Over"

I dubbed "going over" the pattern of representing mathematics by saying steps. The teacher candidates who thought about representation in this way believed that the critical issue was to be clear about exactly how to do the mathematics. They were much less concerned with making mathematics interesting; in fact, some commented that teachers should be careful not to distract students with engaging material or stories. Instead, they sought terminology and ideas about "how to word things."

Teri thought that subtraction with regrouping was a straightforward procedure, that second graders would have little trouble learning it. To Teri, "learning it" meant learning to do it: "switching the numbers over a column and subtracting." What she wanted to do was to explain the steps clearly:

> I would go through, maybe, do a few examples. Go step by step through and show them how to do it. And then have them try a few and if they don't understand it, maybe do it a couple of more times slowly so that they can see exactly what they are doing. Or maybe even take in an example, like use pencils

[100] Most of the other secondary teacher candidates focused on the everyday meaning of slope as "steepness," and concentrated their efforts on representing the idea of steepness. This emphasis revealed what they thought was the essential concept underlying slope as a mathematical idea. That they thought ninth graders would need help to understand the idea of "steepness" was also an indication of their limited knowledge of ninth graders.

> or something, I don't know how many you have, but, like, maybe take twenty pencils and show them, alright, I have this many, I'm taking away this many, how many does that leave? Just so they have a visual example. I would say: These numbers, you can't subtract them in your head. Alright, you have to cross out one of the tens from the top. And put it over in the ones column on the top so you are able to subtract the two numbers. And then when you cross out that tens number, change it, like subtract one from it. So you change, like if it was a sixty-four, change it to a, you know, a six to a five, and the four to a fourteen. And maybe I would show them, like sixty-four, like, maybe I would write sixty-four on the board. And then put it, that it equals fifty plus fourteen, so they see that it is still the same amount.

Teri's representation of subtraction with regrouping was primarily verbal. She was trying to represent the essence of the content, but, for Teri, the content was steps of a procedure. Her goal was to get students to be able to do it. When asked what she would use as a sign that her pupils were "getting it," she said she would look to see if they were getting the right answers. "This is sort of a basic thing — so they either know it or they don't." Teri's approach also seemed to be undergirded by the commonsense sensory epistemology which links seeing and hearing to learning (see Chapter 5). She would demonstrate the steps on the board, she would use pencils as a "visual example," and she would write 64 = 50 + 14 "so that they could see that it is still the same amount."

This way of thinking about what it means to represent mathematics was evident in the teacher candidates' explanations of multiplication, of solving algebraic equations, and of slope. Their representations consistently reflected their understanding of mathematics as a collection of rules and procedures. Unlike those who searched for interesting examples to represent the content, these teacher candidates were concerned with terminology and language. They assumed, like Teri, that by saying and showing how to "do" the math, students would learn.

MATHEMATICS "conceptual" essence (procedure)
LEARNING[101] (seeing and hearing) (repetition)

Figure 6.4
Teri's warrants for showing steps as a representation of subtraction

Skewed, Incomplete, and Other Rationales

What is yielded by examining patterns in prospective teachers' reasoning using a framework of the syntactic structure of pedagogical content knowledge? Figure 6.5 places the two portrayals of the warrants used by teacher candidates (Figures 6.3 and 6.4) side by side

[101] In the sketches of prospective teachers' reasoning, I put in parentheses criteria used by teacher candidates that are not part of the model.

with the model of pedagogical reasoning in mathematical pedagogy (Figure 6.1). This highlights some significant differences between teacher candidates' reasoning about teaching mathematics and the normative framework I have proposed. Their rationales, as revealed in these three patterns, are skewed in the direction of particular warrants other than mathematics, omit critical warrants, and include bases that are outside of the central thrust of mathematical pedagogy.[102]

[102] These findings fit with the evidence that the teacher candidates held a view of mathematics teaching that was significantly different from mathematical pedagogy. In other words, the substantive and syntactic structures of their knowledge necessarily differs from that proposed here; the bases for judgement fit <u>their</u> ideas about what counts as good mathematics teaching, a view that is constructed out of common sense, everyday experiences, as well as out of their understandings of mathematics.

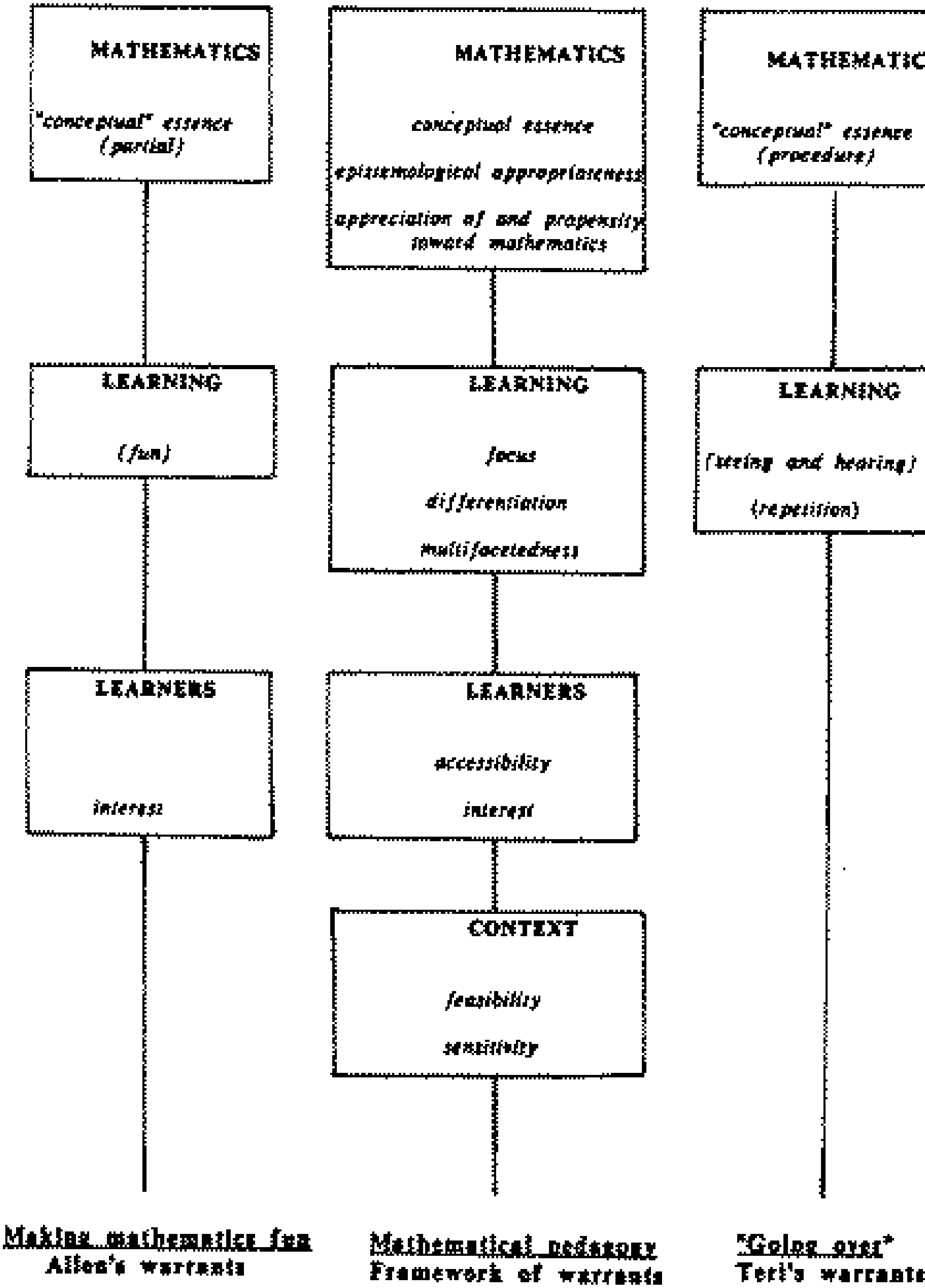

Figure 6.5
Comparing Allen's and Teri's warrants with the warrants for mathematical pedagogy

Skewed rationales. In these two patterns the teacher candidates' ways of reasoning about teaching mathematics seemed to draw most heavily on their ideas and assumptions about learning and learners. They emphasized making mathematics fun or meaningful, and seemed to assume that if students were having fun they would learn. Others were more focused on giving clear directions about how to do the math, and assumed that verbal explanations combined with showing would produce learning. The teacher candidates' explicit goals and assumptions about learning and learners were influenced by their understandings of mathematics (e.g., that mathematics is a body of rules and procedures, or that pupils do not like learning math[103]) but mathematics as a discipline was absent from their deliberations and choices.

Incomplete rationales. Many of them did focus on what they thought was the essence of the content, although it often was not conceptual. In constructing or evaluating representations, however, none of the teacher candidates explicitly considered the portrayal of what it means to know or do mathematics from a disciplinary perspective. Although, as I have argued, their explanations, tasks, questions, and responses to students all did implicitly represent the nature of mathematics, the teacher candidates never discussed any considerations of the epistemological appropriateness of their representations.

Neither did the prospective teachers seem to consider contextual factors. Although this may be a consequent of the fact that interviews conducted in an office tend not to provoke consideration of a real teaching context, it is also possible that prospective teachers are not yet inclined to think about questions of feasibility and sensitivity to context.

Other criteria. Across these two patterns of reasoning, not only did the prospective teachers ignore certain warrants critical for mathematical pedagogy, they also used criteria that were not part of the framework I have outlined. For example, "fun" as a theory of learning provided the basis for justifying the use of novel or entertaining vehicles, such as mangoes. The assumption that students learn mathematics by listening helped to justify teacher candidates' focus on using clearly worded step-by-step instructions.

Knowledge Effects

The teacher candidates did use bases included in the model to ground their reasoning about teaching mathematics, but because of their understandings in those domains, what it meant to take that kind of knowledge into consideration diverged in some cases from the focus of the model of pedagogical reasoning in mathematical pedagogy.

Nowhere was this more evident than in mathematics. The prospective teachers did, in most cases, consider the particular content they were trying to explain or teach. They did try to figure out a way to help students learn the essence of the specific concept or procedure. For example, Allen was thinking about "borrowing," and Teri was considering subtraction. Other prospective teachers focused on the content as well. However, in many cases, their own understanding of that content meant that their focus was not on the essential underlying ideas nor on the key components. For example, most of the teacher candidates emphasized the rules for "lining up" numbers in the multiplication algorithm; for them, this was the key component that the errant eighth graders were missing. Only a few teacher candidates had explicit

[103] I discuss this in Chapter 5.

understanding of the centrality of place value and the distributive property, the conceptual keys to the procedure. As such, although they did consider the content specifically and directly, what they focused on was not consistent with the aims of mathematical pedagogy.

Similarly, although they did think about what would help students learn mathematics, their assumptions about learning math were different from those underlying this approach. They tended to view learning as a product of repetition or as a direct consequent of having fun.

This point brings me back to the beginning of this chapter. Dissatisfied with an analytic approach that fragments domains of knowledge and separates knowledge and reasoning, I set out to argue that pedagogical reasoning is an identifiable concept that can be examined in its own right. I claimed that, as the syntax of pedagogical content knowledge, it reveals the warrants for knowledge and standards for reasoning in teaching mathematics. It is therefore unsurprising to discover that the process is shaped not just by teacher candidates' dispositions to take certain kinds of things into account, but also by what they know and believe in each of the relevant domains.

The argument presented in this chapter, that examining prospective teachers' ways of thinking and reasoning is a critical part of exploring what they bring to teacher education, offers a framework that pulls together the pieces of this study. In the next, and last, chapter, I discuss the results and growth points of this work.

CHAPTER 7

ARGUMENTS AND REFUTATIONS: A DISCUSSION AMONG INTERESTED PARTIES[104]

I have invited several people to join me in this last chapter. Some have agreed to come because they share my concerns about the preparation of mathematics teachers and want to reinforce certain points that they think are especially clear in this work. Others are here to question the validity, significance, or implications of my claims. My guests — an educational philosopher, a philosopher of mathematics, a teacher educator, a mathematics educator, and an educational researcher — have all read my dissertation and are eager to talk.

I initiate our conversation by summarizing what I have learned, beginning with a comparison between the framework of teacher knowledge that I outlined in Chapter 1 and the knowledge and beliefs of the prospective teachers in the study. I move from that comparison to a discussion of warrants for the products and processes of pedagogical reasoning, at which point my guests will break in, for they have a number of challenges to pose, to one another as well as to me.

What Do Prospective Teachers Know and Believe About Mathematics, the Teaching and Learning of Mathematics, and Students as Learners of Mathematics?

I originally set out to provide a way of examining what prospective mathematics teachers bring with them to their professional preparation, as well as to contribute to knowledge about what they do bring. I argued that prospective teachers bring many things to teacher education and that what is specifically of interest depends on the goal — what one wants to prepare teachers to be able to do in teaching mathematics. This study was grounded in a particular vision of good mathematics teaching that I referred to as a mathematical pedagogy; thus, the appropriate framework for investigating what prospective teachers know and believe came from the logical requirements of this approach to teaching. Constructed around the four domains of subject matter, teaching and learning, students, and context, my speculative framework specified categories of knowledge, belief, and disposition needed to teach mathematics.

Where did the prospective teachers in this study stand relative to the knowledge and beliefs essential to mathematical pedagogy?

[104] D.C. Phillips's (1988) article, The Dialogue: An Author's Perspective, was helpful in writing this chapter.

Subject Matter Knowledge

Knowledge about mathematics as a discipline. If a central goal of mathematical pedagogy is to help pupils engage in authentic disciplinary activity and to help them learn about the nature of mathematical knowledge, teachers must themselves be grounded in knowledge about mathematics. This kind of understanding is critical to the capacity to go on learning mathematics oneself as well as to foster the learning of others.

The prospective teachers I studied did not, on the whole, think about mathematics from a disciplinary perspective; instead, they tended to think about math as a school subject: a body of rules and procedures, more or less connected. They appeared to have thought little about questions concerning the nature of mathematical knowledge or about what it means to "do" mathematics. Moreover, they tended to assume that "doing mathematics" meant following standard procedures to arrive at the right answers. They were resistant to invented algorithms — due, in part, to not understanding the inventions themselves, and, in part, to a firm belief that using standard procedures was "better" — or even necessary. Few teacher candidates conceived of mathematics as a domain in which there could be argument or alternative interpretations, nor as a field in which there may be uncertainty, unsolved problems, or change.

Substantive knowledge of mathematics. In order to help students develop meaningful understanding of mathematics, teachers themselves need to have explicit and conceptual connected understandings of mathematical concepts and procedures. This includes being able to explain why a procedure works, to be able to connect one mathematical concept to another, or to make links between mathematics and other domains. The 19 prospective teachers varied — by person and across topics — in the nature of their knowledge. On the whole, however, few were able to explicitly articulate underlying meanings or principles. They were unsure — "rusty," they said — and, frequently, incorrect. The prospective teachers tended to overlook critical conceptual links — place value in the multiplication algorithm, for example. Moreover, their knowledge tended to be compartmentalized, not connected: division with fractions a separate issue from division by zero, slope isolated from the derivative.

Dispositions toward mathematics. Although teachers should enjoy their subjects and feel as confident as possible about the content they teach, the prospective teachers in my study varied greatly — ranging from highly anxious to passionate — in how they felt about mathematics. A few were highly anxious — "petrified" — while others had resigned themselves and accepted the fact that they didn't like or feel very comfortable with mathematics. Noteworthy was the calm confidence felt by some of the teacher candidates who believed that they knew math well. Worrying far less than the others that their answers might be incorrect or inadequate, these teacher candidates responded smoothly to questions and were quite embarrassed when they couldn't remember. Their confidence notwithstanding, they often revealed misconceptions and fragmented knowledge, but they just didn't realize it. They felt generally secure about their mastery of mathematics. While these teacher candidates liked mathematics, only three prospective teachers in the sample were strikingly enthusiastic about mathematics.

Teaching and Learning Mathematics

The teacher candidates varied less in their ideas about teaching and learning and about the teacher's role in helping students learn mathematics and, in general, their notions were substantially different from those that underlie mathematical pedagogy. Although their language for talking about teaching and learning was, at times, quite like the language embedded in discussion of mathematical pedagogy — terms like understanding, concepts, explaining, figuring out — the referents were quite different.

Goals for teaching mathematics. The teacher candidates generally thought the goal of teaching mathematics was to enable students to be able to get right answers on their own, with an emphasis on learning standard procedures. In service of that goal, they wanted to make mathematics class fun and engaging for students, relating it to the "real world." There was little sign, however, that they took a disciplinary perspective on the goals of school mathematics — that is, to engage students in using and doing mathematics mathematically rather than in conventional, schoolish ways. Instead, they assumed a school subject view of mathematics.

Views of learning mathematics. The assumption that underlies this mathematical pedagogy is that learning is largely a constructive and social process that often involves the learner in extending and revising prior knowledge, as well as debating and justifying it within the learning community, although some things in mathematics demand rote memorization or practice in order to incorporate them as part of the intellectual armatoire. Yet the teacher candidates tended to think of learning mathematics as entirely an individualistic process of acquiring information and technique. With a "bucket" theory of learning, several talked about "embedding" and "drilling" facts and procedures into pupils' heads. Sometimes the learner was described in more active terms, "grasping" or "grabbing" ideas, but the predominant implication was that learning entailed getting certain "stuff" into the learner's head. Toward that end, the prospective teachers stressed the need for repetition and practice and if students didn't understand, the teacher candidates spoke of "going over" or "re-explaining" the material and, indeed, that is what they demonstrated in responding during the interviews.

Views of teaching mathematics. Overwhelmingly, the prospective teachers construed the central task of teaching as seeing to it that pupils "get it" — with the major avenue to doing that being to make the mathematics fun. "Getting it" generally meant being able to do the procedures correctly. The teacher candidates did not think of teaching as connecting students to the discipline of mathematics in some way. This was not an idea they explicitly rejected; instead, it was something that simply did not figure in their thinking about teaching mathematics.

Seeing to it that pupils learn is indisputably a responsibility of teachers. Still, what the teacher candidates thought this entailed was "wording" things clearly to students and making class fun, strategies quite different from the discussion and carefully planned representations used to help students "get it" in mathematical pedagogy. These differences in what it means to see to it that students "get it" were a function of the teacher candidates' notion of what the "it" was that students need to "get": mostly rules and procedures.

Teacher's role. In their descriptions of what they would do in response to a variety of classroom situations, the teacher candidates tended to hold themselves responsible for imparting knowledge. They used a wide range of "telling" and "showing" verbs — explain, show, point out, present, give, demonstrate — to refer to one basic teaching activity: telling students definitions and "how to do it." This view of teaching and of the teacher's role, is logically connected to a view of learning as "filling up" with knowledge. Yet it was in tension with their concomitant goal of getting pupils to be able to do the procedures <u>independently</u>, an interest that may have been grounded in the dominant cultural value placed on doing things on one's own. Thus, although their responses emphasized <u>telling</u>, they also said they were interested in "helping kids figure things out on their own." What this meant in most cases, however, was asking questions that <u>led</u> the learner through the steps of an algorithm.

Students As Learners of Mathematics

The mathematical pedagogy that I described is founded on a serious respect for and attention to students and their thinking. This was a domain in which teacher candidates had strikingly little knowledge. Certainly they had some scattered <u>assumptions</u> — about what pupils would find interesting or difficult — but they knew little of what students of different ages might know or be able to understand.

One area in which they did express views about learners was on the question of ability to learn math. Some thought that the ability to do math was essentially an innate characteristic and relatively immutable. More wrestled with the issue by focusing on temperamental and personal qualities it "takes" to learn mathematics. However, even those who focused on effort tended to think of the necessary traits as, to a certain extent, given. For example, teacher candidates would talk about the need to have the "persistence" to be able to stick with it. Persistence, and its cousins — patience, tolerance for frustration, and desire — were all discussed as fixed personality traits. Their idea that some people are just "naturally" good at math reflects some dominant cultural beliefs about mathematics and about ability; the teacher effects, by implication, are limited.

Grounding success in natural ability or disposition, a culturally embedded belief, flies in the face of another dominant cultural dogma, however: that anyone can succeed if they try hard enough. The prospective teachers appeared to be caught between these two central cultural beliefs, vacillating in their explanations and attributions, their ideas about the roles of teachers and students and their assumptions about the effects of teaching.

Context

In mathematical pedagogy, the teacher tries to connect the students with mathematics. The key to doing this requires that the teacher take the context — of the classroom, the school, the community — into careful account. It also suggests the usefulness of constructing the classroom as a microcosmic mathematical community, in which students posit conjectures, offer arguments, revise their hunches, and engage in really "doing" mathematics. The classroom itself becomes a representation of the discipline — its activity and growth as well as its substance.

Critical to acknowledge is that the interviews did not necessarily present rich opportunities for teacher candidates to think about contextual factors. Such thoughts seemed, however, to be far from the teacher candidates' minds. They dealt with one student at a time or with the whole class as a unitary whole. They showed no tendencies to take other contexts — community or school, for instance — into account.

Warrants for Pedagogical Reasoning

Analyzing what prospective teachers know and believe in each of these domains was useful and, yet, teaching mathematics is more than the sum of these parts. It is dynamic and entails juggling and integrating, balancing and compromising. In this dynamic, as teachers ask questions, give examples, choose tasks, and assess pupils' work — in other words, represent mathematics, they are influenced by what they know and believe across different domains of knowledge. I knew this before I began this study. Then, in the interviews, I repeatedly saw instances of teacher candidates who seemed to know similar things and yet arrived at different conclusions, choosing to represent mathematics to students in different ways. Analyzing the data from the interviews with prospective teachers helped me to advance a theoretical framework for thinking about the integration and interaction of knowledge in teaching mathematics.

This process of integrating knowledge, which I refer to as *pedagogical reasoning,*[105] varies across teachers. For some teachers, subject matter considerations may be figure, while for others, a conviction about how pupils learn best may be most prominent in their considerations. For example, in teaching mathematics, teachers whose knowledge of mathematics is grounded in the discipline may consider the lessons they want students to learn about the nature of mathematical knowledge, while those whose own understanding consists of a school subject, or body of rules and procedures, may be more likely to emphasize algorithms and computational performance. While there are differences across teachers, an individual teacher's patterns of reasoning probably also varies across contexts.

I proposed the idea that teachers' <u>warrants</u> for their decisions and judgments reflect how their knowledge and beliefs come together in their pedagogical reasoning. What they take into account influences their planning, their judgments, their responses — in short, all the things they do in representing mathematics to their pupils, as well as how they judge the value of specific pedagogical representations. What do I look at in order to decide whether something is a good idea in teaching mathematics? Is this a good activity for my pupils? Should I ask this student to work on this problem some more? How should I respond to this complicated idea that one of my first graders has just bumped into?

Depending on teachers' views about the goals of mathematics teaching, pedagogical reasoning and its products will be warranted differently. Teachers whose aim is to get students to become proficient calculators will be likely to judge an activity in terms of its potential for helping students practice the procedures, while those who want students to become comfortable with mathematics and see its usefulness are likely to discard such tasks and seek "meaningful" or "fun" activities instead.

I proposed a set of specific warrants consistent with the goals of a mathematical pedagogy that provides a normative standard for legitimate pedagogical justification, given this perspective on the teaching of mathematics. Within the cornerstone domains of teaching —

[105] This term was first used by Shulman (1987). See discussions in Chapter 6.

subject matter, learners, learning, and context — I identified particular warrants that take priority in justifying good representations of the subject matter for teaching.

The idea that pedagogical reasoning can be understood in terms of how teachers integrate what they know and believe — the warrants they use either tacitly or explicitly to justify the ways in which they represent the subject to students — offers another critical perspective from which to view what prospective teachers bring to teacher education. Not only does it bring the pieces of teacher knowledge back together, but it affords a means of assessing the patterns of reasoning and justification that they assume as they enter formal teacher education.

* * *

Now I invite my guests to begin their questions. The Educational Researcher, who has recently re-read Lortie's (1975) book Schoolteacher, opens the discussion.

Educational Researcher: *I find your work interesting, Deborah, but Lortie already pointed out in his book 13 years ago that prospective teachers come to their professional preparation with a host of ideas about teaching and learning, based on their experience as students in schools. He noted that we often forget to pay attention to this "apprenticeship of observation" despite the fact that it is a powerful influence that tends to perpetuate the conservatism of teaching practices in this country. I am puzzled as to what you think is new or different in your work.*

Deborah: In short, I have expanded on his idea. While his argument, and the idea of an "apprenticeship of observation," was (and continues to be) a helpful lever in understanding what it means to become a teacher in this country, he essentially identified a process and argued that what is learned through this process is profoundly influential on prospective teachers' ideas and expectations about teaching. He did not explore what the content of this process was — that is, what people actually learn from the apprenticeship of observation. My work focuses on ways of examining what is actually learned from the apprenticeship of observation, as well as from out-of-school experiences, within the context of a particular subject — mathematics. In doing this, I have extended and elaborated the concept of "apprenticeship of observation" — in effect, by exploring the specific outcomes of one such specialized apprenticeship. This work emphasizes what people may be learning specifically about the teaching and learning of mathematics.

Educational Researcher: *I assumed that because students' prior school experiences would be so diverse, there would be little reason to expect any commonality in prospective teachers' ideas and images about teaching.*

Deborah: I think you jumped to that conclusion a little too quickly. These things may vary by subject matter. The prospective teachers in this study held a great deal in common in terms of their images of teaching and learning mathematics. This actually isn't surprising in the case of math, when you consider how consistent the patterns of instruction in American math classrooms are. Students who spend thousands of hours in these classrooms are likely to hold some ideas and images in common, don't you think? It would be reasonable to predict that prospective teachers, like the ones I interviewed, might assume that teaching math means telling students "how to do it," that learning math requires considerable repetition and drill, and that when pupils don't "get it," teachers should "go over" the steps again. This, after all, is what they have been through. Other school subjects may not reflect this degree of consistency, but math teaching, unfortunately, tends to follow a similar pattern in a tremendous number of classrooms across this country. While I wouldn't want to claim that what I found is generalizable, I don't believe that it was highly atypical, either.

Educational Researcher: *Well, I'm sure that we'd both agree that that's an empirical question. Perhaps someone will use the conceptual frameworks that you have developed and will examine the ideas and assumptions of a larger, random sample of entering teacher education students. Is that the main thing you're hypothesizing — that the apprenticeship of observation produces some predictable outcomes with respect to mathematics teaching?*

Deborah: No, there is something else worth noticing about this specialized apprenticeship of observation. Being a student in math classes does more than develop prospective teachers' images of math <u>teaching</u>; it also affects the way in which they understand mathematics — the subject matter — itself. If they have been subjected over and over to authoritarian and rule-bound presentations of the content, they only memorize procedures and definitions. If their only experience with math is following algorithms to arrive at right answers to pointless problems, they are unlikely to be substantively well-prepared in mathematics. Moreover, considering the ordinary mathematics class, it is likely that prospective teachers will believe that mathematical knowledge is fixed and a matter of absolute truth, that doing mathematics means doing exercises and getting right answers. My point is that prospective teachers acquire most of their subject matter knowledge for teaching while they themselves are still in elementary and secondary school, going through the apprenticeship of observation. And yet we persist in talking about subject matter preparation for beginning teachers as though it were just a matter of what goes on in liberal arts courses. In the case of mathematics teaching at least, I don't think that's quite right.

If you think about these two points about <u>what</u> is learned from this specialized apprenticeship of observation together, one more point is worth highlighting. In the apprenticeship of observation, as in teaching, subject matter knowledge and pedagogy <u>interact</u>, for prospective teachers develop their ideas about teaching and learning mathematics in the context of learning the subject matter itself. What they think about good math teaching, for instance, is probably intertwined with their own conceptions of what they think it <u>means</u> to learn or to know math. Prospective teachers' judgments about particular tasks or activities are likely to be influenced both by what they believe about teaching and learning as well as their own understanding of the particular content that is embodied in those activities. Moreover, watching their own teachers teach math may have given them not only the idea that teachers should explain, but may have also provided them with specific explanations: little scripts or examples for teaching multiplication, "borrowing," or slope. Ideas acquired in school about

<u>teaching</u> and about the <u>subject matter</u> are likely to be interconnected. I think that recognizing these interconnections may be key in unpacking and changing what prospective teachers bring.

Educational Philosopher: *Deborah, can I interrupt here a moment? I'm not sure I understand what you are implying with that last statement and, indeed, with the tone that runs through a lot of your interpretation. You make these students sound like they are <u>wrong</u>. I got this impression most, by the way, in Chapter 5. In that chapter, you describe the prospective teachers' ideas as "weak" — which implies that they can be strengthened — and then go on to intimate that their ideas need to be <u>changed</u>. And just now you said it explicitly. It seems to me that the students' own direct experiences with teaching and learning mathematics are a tremendous practical resource in learning to teach math. They already know a lot about teaching and learning — it's just that their ideas are partial and under development. There is a lot to build on in what they know and believe — to try to break with those ideas and, in effect, start over, would be quite misdirected, I fear.*

Deborah: I know that you have been influenced by the writing of John Dewey and that you tend to think of learning as a continuous, growing process. I agree with you. But what troubles me is that you seem to fail to take into account the possibility that the knowledge and skills a person acquires through one set of experiences may be <u>inappropriate</u> or <u>undesirable</u>. There has been considerable research in recent years that indicates that people's ideas or ways of seeing may misguide what they see and learn from experience.

I think that the point is that when prospective teachers get to formal teacher education, they come from more than twelve years of experience in classrooms. They are likely to have developed ideas about techniques, about the teacher's role, about the nature of classroom activity, which, although vivid and validated through personal experience, are not critically analyzed. Their ways of looking at classrooms may be superficial and focused on what teachers <u>do</u>; they probably know little about teachers' thinking or the dilemmas with which they grapple. The continuity of experience in preparing to teach — from this apprenticeship of observation through formal teacher education — is problematic in a way that you seem to overlook.

Educational Researcher: Let me just interject a point here, too, Deborah, for I have something to say to my colleague, the educational philosopher. This apprenticeship of observation may, in a curious way, affect prospective teachers' receptivity to professional education — that is, they may already think they "know it." Furthermore, the continuity of their experience in schools — from student to teacher — which you see as something good, can be an <u>inhibitor</u> of change. We all know how conservative teaching practice is in this country — traditions passed down from one generation of teachers to another. It is interesting that the term "continuity" connotes something positive to you; for me, "continuation" stands for the heavy weight of tradition on teaching in this country.

Educational Philosopher: *Of course I'm not interested in perpetuating many prevailing educational practices, but, still, you provide no evidence for your claim that students' prior experiences in school inhibit their growth as teachers. Just because some of us don't like what we see when we really take a look at what prospective teachers bring, do we really know that student teachers cannot simply change their* *minds and acquire pedagogical ways of looking and thinking from good experiences during formal teacher education? For example, Deborah's work shows that the prospective mathematics teachers she interviewed were not "saturated with their subject matter," as they need to be. If this is the case more broadly, shouldn't we just provide more coursework in mathematics for teachers?*

Deborah: I think you are raising an important and interesting question. When we look at the continuity in teaching practices, though, I'd argue that there is abundant evidence that what people bring to teacher education from the apprenticeship of observation has some lasting power. Many people have described the negligible impact of preservice teacher education by saying that the university effects are "washed out" by the power of the school culture after teachers leave the university. I tend to think that an equally plausible explanation has to do with the power of the school culture before teachers enter teacher education — that certain ideas are "washed in."

Whatever the explanation, though, I think we don't really know what would make a difference in helping to move prospective teachers along, helping them to acquire those things they need to know and be able to do in order to begin teaching. Whether what prospective teachers bring is simply partial and under development, or whether it is discontinuous and requires reconstruction, is an empirical question, a question about teacher learning. In other words, we need to follow people who are becoming teachers, focusing on whether and how their entering knowledge and beliefs change, given different approaches to teacher preparation.[106] Furthermore, whether you consider something "partial" or "discontinuous" depends on the goal itself — what teachers need to know, given a particular view of math teaching.

***Teacher Educator**: This gives me an opportunity to jump in. The reason I was troubled when I read your work, Deborah, was that you assume a view of teaching that necessarily shows up the teacher candidates as weak and holding views quite different from the ones you are promoting with this mathematical pedagogy. My feeling as a teacher educator is that I am preparing my students to enter real classrooms, where they will have to use textbooks and be accountable to tests — which means emphasizing computational skills and covering the district's curriculum. They are not going to be meandering around, worrying about their pupils constructing theories and proving claims. So when I think about what they will be doing, I am actually somewhat pleased with your report on what they bring. They have a sense of their role, they realize that learning math requires considerable drill, and they are clearly inclined to make math fun for the students. I believe that there are things they need to learn more about — error patterns in students' work, how to evaluate students, writing lesson plans — and I think they need, as several of them said, to brush up on basic math facts and definitions. But that's no big deal.*

Deborah: I understand your argument. Certainly from your perspective, these prospective teachers are in better shape than they are from mine. But I think there are some good reasons why you should reconsider your apparent complacency with current practices in the teaching of

[106] This is the central research question of the Teacher Education and Learning to Teach Study currently underway at the National Center for Research on Teacher Education at Michigan State University.

mathematics and your willingness to prepare teachers who will teach in these ways. Have math classes? I mean, are you content to discover that students who can calculate correctly cannot interpret the meaning of their answers in context? Doesn't it matter to you that students tend to develop the idea that mathematics consists of a body of rules and procedures, that doing mathematics means filling out exercises, or that problem solving means finding answers to word problems? And what do you think of the fact that so many adults in our country feel incompetent in situations where quantitative reasoning is entailed — like interpreting percents, understanding statistical information, or estimating?

Mathematics Educator: *Just a minute, Deborah. Even though I am a mathematics educator, I think the goals of mathematical pedagogy, as you describe them, are questionable. While I agree that mathematics teaching as we now know it is in serious need of improvement, it sounds like what you are trying to do is to create little mathematicians, when what is needed is just a much greater and more direct emphasis on concepts — more use of manipulatives, for example, not just paper-pencil tasks. And more problem solving. I like that part of what you describe, but all this stuff about making and validating conjectures doesn't sound like something that anyone really needs to know unless they are going to be a mathematician. Research on student learning, as you yourself point out, provides ample justification for your constructivist orientation to learning, but I think that your heavy overlay of disciplinary knowledge places too much emphasis on the discipline of mathematics, and too little on just helping students to explore concepts.*

Deborah: Would you make the same argument about reading? That children don't need to interpret literature unless they are going to be literary critics? Would you argue that, if not, they should just learn to decode and extract the literal meaning of text? This is the analogy to your argument about learning mathematics. I argue that learning mathematics with meaning, as you would like, necessarily entails doing mathematics. To be able to use and learn mathematics independently, which is just what we want in terms of the reading and English curricula as well, students must learn to define, approach, and solve problems and to justify their strategies and solutions and to understand what counts as a legitimate solution. (The analog in reading is to learn how to construct and substantiate alternative interpretations of text.) I think you mean something a little different by "problem solving" than I do, too. I am not just talking about students' ability to find the answers to word problems or puzzles. Problem solving, from a disciplinary point of view, entails as much about finding and posing good questions as it does about figuring out ways to approach and answer questions. (In reading, we want students to figure out ways to analyze and interpret text independently, not just answer questions that someone else poses.) Being able to use mathematics in a variety of formal and informal contexts entails understanding different ways in which it can model situations or problems, and includes learning to invent legitimate alternatives in search of a solution. Mathematical pedagogy is based on the premise that understanding any subject in a way that is "empowering" entails learning about the nature of the knowledge — how it is generated, its certainty, its sources of justification — as well as learning some of its substance. This claim is not unique to mathematics, but holds for history, English, biology, and so on. I'm not sure that it's such an esoteric perspective at all and it is absolutely connected to your interest in having students learn the meanings of concepts.

Mathematics Educator: *I hadn't thought about it in quite that way before. I'll have to think about it some more. But, I still have another question. Even if we were to agree that mathematical pedagogy is a desirable goal, it might still be questionable for preservice teachers, don't you think? However one chooses to enact it, it is not an easy way to teach. There are almost no curriculum materials available that reflect this perspective on teaching and learning, and it is unreasonable that beginners should invent their own curriculum materials. If it is an undesirable goal for beginning teachers, then why did you construct your entire study around it? As the Teacher Educator said, this perspective only makes the prospective teachers look bad. Why not try to get them underway first? Learning to teach, after all, takes time. Let them begin by learning to teach pretty traditionally, but critically. As they get that under control, help them begin emphasizing concepts and meaning more and more. Still later, they can <u>add</u> this piece, this mathematical pedagogy.*

Deborah: I'm not at all sure that mathematical pedagogy is so unrealistic. First, of all, this approach does not necessarily entail alternative curriculum materials, but, rather, critical modifications of existing ones, supplemented with ideas that are rarely included in most textbooks. Beginning teachers can start with some aspects of mathematical pedagogy and gradually learn more; I am not convinced that mathematical pedagogy can be layered <u>on top of</u> an algorithmic or even a conceptual approach. The differences among these three approaches do not lie only in <u>what</u> gets taught. Each of those approaches has embedded within it an epistemology and social structure that profoundly shape the nature of knowing and learning.

Philosopher of Mathematics: *This brings up something that concerns <u>me</u>. The view of mathematics upon which you base mathematical pedagogy is far from a mainstream view of mathematics. I mean, let's face it, Lakatos hardly represents a dominant perspective on the nature of mathematical knowledge. Basically, he turns proof, that sacred tool, and reduces it to something much less significant by claiming that there can <u>be</u> no truth in mathematics. Lakatos asserts that we can propose something and back it up in a way that makes it seem plausible, but that the best we can ever do is to be unable to <u>dis</u>prove our claim — for the time being, anyway. Most mathematicians still think of proof as establishing the truth of a claim, even though they <u>are</u> aware that, from time to time, new ideas will challenge those they had previously held to be true. And besides that, I think that most mathematicians would put a much higher premium on learning the principles, procedures, and theorems that are part of the accepted system of mathematical knowledge than you seem to. What's the point? Why found a pedagogy on a such a radical view of the discipline? Shouldn't pupils encounter a more dominant perspective instead?*

Deborah: I certainly understand why you would raise this question and, in fact, I anticipated that you might. I think there are two good reasons for grounding a mathematical pedagogy in a fallibilist epistemology which emphasizes a social, constructive perspective on disciplinary knowledge. First, if one believes that learning is largely a constructive process, that when children learn concepts, they build on or reorganize things they already know, then this view of mathematical knowledge is compatible with that. Mathematics is not presented as a finished body of knowledge, but rather as something that changes and grows over time through a process of working from what you know to what you don't. But, and this is my second reason, learning is also not just a matter of knowledge growth over time, but also that what you believe to be true at one point in time may later be refuted with additional knowledge. Thus, when second graders think that the next number after 2 is 3, they are right — given what they know at that point. As members of a mathematical community, everyone in the class would agree. Once students encounter the set of rational numbers, perhaps through division, then the question of what the next number after 2 is becomes debatable, and the old answer — 3 — is clearly refuted. In this way, pupils' encounters with mathematics represent Lakatos's conception of the discipline. I would also hunch, by the way, that many working mathematicians' implicit conceptions of the nature of mathematical knowledge and activity are actually much closer to a Lakatosian perspective than they might <u>say</u> if asked. Their work, after all, like <u>children's</u> learning, proceeds much more in this vein.

Now, I know that you still want to know how I reconcile the fact that students are learning a radical view of mathematics as a discipline. Shouldn't they learn a formalist perspective — or at least encounter <u>multiple</u> perspectives? I think this is an important and good curricular question. My answer is to say that I would help students become more familiar with alternative conceptions of mathematics later. I think this philosophical perspective makes sense pedagogically and is more likely to empower students mathematically than a formalist or multiple perspectives can.

Teacher Educator: *I've been thinking about what you said about starting points for beginning teachers. Teacher educators face a difficult dilemma. Should we prepare beginners to teach in relatively conventional ways — even if we think those are not particularly good ways to teach — since the more innovative approaches are too complicated for beginners to pull off? Or should we think of our work with beginning teachers as an opportunity to effect some change in public schools? I am inclined to say that, in the case of mathematics, we should try to prepare beginning teachers to learn to teach math <u>differently</u> since conventional practice is so often failing to help students learn mathematics meaningfully.*

I do think this is a tall order. I have thought of some possible avenues, though, based on your work. For example, couldn't beginning teachers learn ways of reasoning pedagogically about representations — using the framework of warrants, for instance — that would enable them to critically use and modify a textbook? This would make it possible for them to <u>begin</u> teaching in a real classroom setting without having to construct the entire curriculum themselves —

Mathematics Educator: *That makes sense to me, but without subject matter knowledge, the modifications they make may be necessarily conceptually or epistemologically <u>in</u>appropriate. And I am particularly struck in this study by how weak the teacher candidates' understanding of mathematics was. The results on the secondary teacher candidates, who were math majors, was especially disheartening.*

Educational Researcher: *This brings us to a point I wanted to raise. You make it seem as though there was little difference between the elementary and secondary teacher candidates in terms of their subject matter knowledge. I suspect that this reflects a bias, that you were actually looking for a lack of difference. But I think you have flattened the data in a misleading way. In fact, if you were to run chi-square tests on the data by level, my guess is that you would find some significant differences between the two groups.*

Deborah: Undoubtedly you are right — I would get some statistically significant differences. For example, 5 out of 9 of the math majors were able to generate an appropriate representation of $1\frac{3}{4} \div \frac{1}{2}$, while none of the elementary candidates were able to do it. Statistically, that would be a whopping difference. But, if we think about this reasonably, I think I have to ask, is statistical significance really significant here? When only slightly more than 50% of a group of people who have majored in mathematics are able to make meaning out of a simple division statement? And the division with fractions task was one where the differences were greatest between the two groups. What about the division in algebra task? While one elementary candidate gave a conceptual explanation, none of the secondary candidates did. And in evaluating the false student conjecture about perimeter and area, 5 out of 9 secondary candidates thought it was true; only one recognized that it was, indeed, false. Among elementary teacher candidates, 2 out of 10 knew it was false, and half were at least skeptical and said they weren't sure. With results like these, statistical significance does not seem to be the issue.

Most of the secondary candidates did not have a more disciplinary orientation to the subject either. They tended to conceive of mathematics as a body of rules and procedures, and equated explanations with statements of these rules.

Educational Researcher: *But this just reflects the problem of who goes into teaching — it is the weaker students in the math department who get funnelled into math education, into teaching.*

Deborah: Wait a minute. You're jumping to conclusions. I was about to remind you that the teacher candidates in this study were earning high grades — mostly A's and B's — in their mathematics classes. They had strong college entrance examination scores. I interpret the results to be an indictment of precollege mathematics teaching, of what goes on in many elementary and secondary math classrooms.

Educational Researcher: *I still think you were biased — that, for some reason, you were out to show that majoring in mathematics made little difference in terms of subject matter knowledge — a fairly preposterous thing to assume. Your methodology reflects this bias: You ask only about elementary and high school content in your interviews. You are asking math majors about division with fractions and then you throw up your hands when they cannot make up a little story about* $1\frac{3}{4} \div \frac{1}{2}$. *Had you asked them about abstract algebra or calculus, you would have uncovered some real differences between these groups of prospective teachers.*

Deborah: I think it was reasonable to focus on pieces of mathematical content that appear in the K-12 curriculum, since this is, after all, a study within the context of teacher education. I was interested in the nature of the subject matter understandings that prospective teachers brought to teacher education about the topics they might actually teach or need to refer to in teaching: solving algebraic equations, division in several contexts, perimeter and area, the nature of theory and proof. Given limited time, it doesn't make a lot of sense to interview prospective teachers about topics or concepts that they will not teach, nor about things they have never encountered before, either. For example, if I interviewed them about permutations or about statistical sampling, they probably wouldn't have had much to say.

Another thing to keep in mind is that I did explore and analyze the prospective teachers' ideas about the nature of mathematics — about the warrants for mathematical knowledge, about what it means to do math, and about relationships of mathematics to the world. I also investigated teacher candidates' feelings about mathematics and about themselves in relation to the subject. The second two dimensions of subject matter knowledge — knowledge and feelings about mathematics — were not focused at any particular level, and one might reasonably predict that math majors, who have been steeped in mathematics, would have a more robust disciplinary perspective. One might anticipate that they would understand the role of proof in knowing mathematics, that they would have a sense of what knowledge is logical and what is arbitrary, and that they would be able to draw connections between mathematics and the real world. This is not, however, what I found. The secondary teacher candidates' ideas about mathematics were, on the whole, no more sophisticated than the elementary candidates'. The two groups differed strikingly only in their enjoyment of and feelings of confidence about mathematics.

Educational Researcher: *Well, perhaps it's your conception of substantive knowledge of mathematics that bothers me. You expect prospective teachers, who have never taught, to explain division by zero, to make up a representation for* $1\frac{3}{4} \div \frac{1}{2}$*, and to think about what it means that division by zero is undefined. You say their knowledge will need to be both conceptual and explicit, and that they need to understand the substantive connections among mathematical ideas and topics. That all seems pedagogical to me — you're not really tapping their own understanding of the subject. That may be where the bias lies.*

Deborah: Are you saying that without those features you would be comfortable saying someone knows mathematics?

Educational Researcher: *I'm not sure.*

Deborah: Well, it sounded to me as though you were saying that if a person remembers to invert and multiply when they have to divide fractions that you are satisfied that they know division with fractions, even if this person cannot describe a concrete situation for which the mathematical statement is a model; in other words, they cannot come up with a story or a picture to go with it. And do you believe that saying that "you can't divide by zero" and that that is "a rule that you just have to remember" is sufficient to be called mathematical knowledge? What about when people see no connection between these two questions — does that not make you question their understanding of the concept of division?

I don't think it is too controversial to define knowledge in terms of meaning. On that count I do stand guilty. But there is no special reason, in any case, that adopting this conception of substantive knowledge would bias me against the secondary teacher candidates. Why would you expect the elementary candidates to be more readily explicit and conceptual, to have more connections among mathematical topics?

Educational Researcher: *Hmm. I hadn't really thought of that. That's true — there's no reason to assume that the elementary candidates would be favored by your conception of substantive knowledge. On the contrary, I would have expected it to favor the math majors.*

Teacher Educator: *Are you two done? I have to say, on this topic, that I am quite struck with what you found about the prospective teachers' subject matter knowledge. I had expected that the elementary majors would be somewhat weak, but I hadn't anticipated this problem with the secondary candidates.*

Mathematics Educator: *Me either. I would never have predicted the relative lack of differences in their understandings. It makes me think that we have to, once again, confront the question of what the major requirements in our department ought to be — or, even more difficult, what the elementary candidates need in mathematics. Have you given that any thought?*

Teacher Educator: *But while you're thinking about that, I'd like to hear your ideas about teacher education more generally. Based on what you have learned through this study, what directions do you suggest for teacher education?*

Educational philosopher: *Before Deborah tackles this, I have a question, too. I've been quiet for a while now, listening to all this talk about subject matter knowledge. I do have some opinions about this. Do you know Dewey's essay from Democracy and Education entitled "The Nature of Method"? In it he argued that method and subject matter are not separate, that method is embodied in the way the subject is used. Deborah, don't you think that this way of thinking about teacher education as broken up into subject matter, pedagogical method, practical experience is something worth questioning? I see them as all interconnected. Yet our two colleagues here seem to be reifying this traditional structure by each separately asking you your thoughts about their part of teacher education.*

Mathematics Educator and Teacher Educator: *(together) Not necessarily.*

Deborah: I have been thinking about this and I would like to try out some of my ideas with you. A few things seem clear to me.

My initial comment just brings us back to subject matter. All these other things aside, we have to turn more attention to helping teacher candidates develop conceptual, explicit, and connected understandings of mathematics. Teacher education has to stop taking for granted the mathematical knowledge of its students. This alone is a tall order, given the nature of most existing college math classes.

I agree, nonetheless, that subject matter and method are inextricably intertwined. Schwab, taking it further, said that the methods we use to teach subject matter shape what our students learn. He argued that if we use a teaching approach that doesn't fit what we really want students to learn, we will inevitably misrepresent the content. When I think hard about what these ideas mean, I see two levels of significance for mathematics teacher education.

First, prospective teachers learn about teaching and learning through all their courses, but their ideas about mathematics teaching and learning come most from their mathematics classes. There they may learn mathematics and methods for helping others learn it, at one time. This is exactly what I argued that prospective teachers had gotten from their math classes during the apprenticeship of observation.

Second, though, mathematics classes provide prime opportunities for prospective teachers to acquire both substantive knowledge and knowledge about mathematics. In this sense, students may learn the subject and methods of inquiry in that subject.

The mathematics courses that prospective teacher take represent critical learning opportunities for all the domains of knowledge relevant to teaching math. Teacher educators need to engage in experimental approaches to taking advantage of the complicated layered setting of the college math classroom. By that I mean that, for prospective teachers, these classes (whether instructors plan to convey particular messages or not) are occasions for learning both about the substance and nature of the discipline, as well as about the teaching and learning of the subject.

Another promising contribution to teacher education lies in the framework for justifying the products and processes of pedagogical reasoning. Using warrants for representations, prospective teachers can learn to consider a broader set of criteria as they evaluate textbooks, generate examples, and select activities. They can also be helped to begin recognizing and learning to manage the dilemmas teachers face in trying to take such a wide range of factors into account.

Educational Philosopher: *I'd like to discuss your idea about warrants. You say that this framework provides a set of warrants for justifying and analyzing the products and the processes of pedagogical reasoning. Do you see any difference in justifying a process as opposed to a product?*

Deborah: Well, the products, as I have defined them, are the representations of mathematics found throughout a teacher's math teaching. They are the questions she decides to ask, the activities she uses, her responses to students' questions, the tasks she sets, and the ways in which she deals with written homework, for example. Since teaching involves multiple and competing considerations, any particular representation is necessarily incomplete for it cannot possibly satisfy all such considerations. Warranting these representations, these products of pedagogical reasoning, therefore, involves decisions about what to emphasize and what to ignore for the moment. Pedagogical reasoning, though, can be evaluated by whether it includes the consideration of all relevant warrants. One might say, therefore, that teachers who neglect to consider whether a task will be accessible to their students have omitted a critical consideration. This does not mean, however, that a teacher who selects a task that is less immediately accessible to his students is necessarily wrong, if he can justify that decision by showing how another warrant took priority in this case (for example, if he argues that the representation has sufficient mathematical value and he thinks he can make it more accessible to his students).

Educational Philosopher: *That makes sense, but I think you are overlooking something. If you want* *to think about warranting the* <u>*process*</u> *of pedagogical reasoning, you will need to talk about a second level of warrants to specify the criteria for* <u>*appropriate*</u> *integration and balancing. For example, I know you wouldn't consider it appropriate if a teacher considered all the warrants, but consistently let considerations of student interest take priority over the subject matter warrants. How will you deal with the justification of decisions made when considerations seem to conflict? You surely don't want these warrants to stand for nothing more than a list of touchstones to check off after each has been considered, right?*

Deborah: That's certainly true. It's not something that I am yet prepared to answer, although I would say that <u>cumulativeness</u> must somehow be a factor in this second level of warrants. This is one reason that I included multifacetedness as a warrant — what I was trying to mark was a recognition that representations will necessarily be incomplete, and, that therefore, the <u>additive</u> message of a teacher's representations is a critical piece of any judgment. The framework of warrants does not yet address this issue of what counts as an legitimate weighing and balancing of the warrants.

Teacher Educator and Mathematics Educator*: We'd like to turn the conversation to another topic. You began this dissertation arguing that although we teach our students about constructivist theories of learning, we fail to find out or take into account what* <u>*they*</u> *bring. We just "deliver" content. We are quite impressed with what we learned here about ways to think about what prospective teachers bring, but we are a little unsure about what you are recommending to us now. In our math classes and our math methods courses, should we try to find out about our students every term? Or, although we understand that you didn't make this claim, can we just go on what you reported here? After all, you did find a lot of commonality among the students in the study.*

Deborah: I'm glad you brought that up, because it is an important point. I do <u>not</u> mean to imply that what I found can generalize to all prospective teachers. I think the usefulness of my results lies in the frameworks and categories that I used, elaborated, and generated through this inquiry. They offer some ways of exploring teacher candidates' knowledge and beliefs. For instance, these frameworks may guide you to investigate what your students believe about the sources of success in mathematics, their ideas about the teacher's role, or their notions about learning mathematics. The analyses of this study can also help you listen more closely and critically to things that they say — when they talk blithely about "concepts" or about "explaining," for example, you will be alert to the different senses in which they may use these terms. You might even consider conducting some interviews with your own entering students, analyzing what they know and believe, and writing an article about what you find. If we could accumulate more knowledge about beginning teacher candidates — in other subjects as well as in mathematics — it would be a big help to the field. Of course, that still leaves the big question about whether these things they bring need to be changed and, if so, how and when they can best be challenged.

Educational Researcher*: This reminds me. Before we all leave, I'm curious about whether you would make any specific recommendations about the directions future research should take. Are there things* *that you think ought to be pursued while perhaps others that you believe are fruitless to explore?*

Deborah: As a matter of fact, I have been thinking about this. Two lines of inquiry stand out to me as especially critical: research on how prospective teachers' knowledge changes, and research on pedagogical thinking. I can say a little about each one.

One of the questions I hear most frequently is how difficult it would be to get prospective teachers to modify or adapt their ideas — about mathematics, about mathematics teaching and learning, about students as learners of mathematics. This is really not a question I can answer on the basis of my data: I don't know. We need more research of the sort currently underway at the National Center for Research on Teacher Education: studies that track the relative effects of different approaches to teacher education on teachers' knowledge, skills, and dispositions. It would be especially useful to study efforts with different structural and content dimensions.

We need to understand better what kinds of pedagogy, what kinds of experiences, contribute to the development of subject matter knowledge, for example. One current controversy centers on what mathematical content is of most worth for prospective teachers to take — content from the school math curriculum (operations with whole numbers, fractions, and decimals, algebra and so forth) or non-traditional content (probability, number theory, and so on). Research should examine the differential effects of each approach in terms of prospective teachers' specific and general understandings of mathematics, their ideas about mathematics, their feelings, as well as their notions about mathematics teaching and learning.

The framework of warrants presented in Chapter 6 offers another fruitful avenue of research. What warrants do very good experienced teachers do to guide and justify their representations? To what extent are teachers' warrants tacit? How might teachers' pedagogical reasoning inform the development of a normative framework? What are some ways of justifying the integration and balancing of competing knowledge and beliefs? I think there is also interesting work to be done on whether there might be helpful ways to use a normative framework in teacher education as a means for helping students begin to reason pedagogically — without turning it into yet another prescription for planning? Where are the gaps in the framework and how does the particular framework I proposed differ across views of teaching mathematics, as well as across subjects?

One final point: Empirical work on these questions as well as on other related issues, will continue to demand sophisticated data collection strategies. On one hand, better interview questions and tasks, observation guides and interviews, are needed. On the other hand, the field also needs simpler but nevertheless sharper strategies for assessing what prospective teachers know. This agenda is critical to advancing this work on teacher knowledge and teaching. The strategies that I and others have been developing in our research provide places from which to work and to develop improved methods of inquiry.

* * *

I see that my guests are tiring. I thank them for joining me in this last chapter of my dissertation. Their questions have pushed me to think and helped me to highlight some of the key issues embedded in my study. No doubt my readers will have other questions.

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APPENDIX A

Summary of study participants' academic records

Teacher Candidate	Rank in high school graduating class	College entrance exam score - mathematics		College GPA	College GPA mathematics	Remedial mathematics placement	*Credits in college math courses	Math for elementary teachers course taken	Advanced placement credit - calculus
		ACT*	SAT*						
Elementary									
Allen	8/71	27	-	2.88	2.50	-	27	Yes	-
Anne	7/230	23	-	3.74	3.61	-	18	Yes	-
Cathy	99/263	22	-	3.07	-	X	0	No	-
Linda	14/288	15	-	2.86	2.00	X	4	Yes	-
Marsha	19/135	13	-	2.95	-	X	0	No	-
Mei Ling	174/525	26	590	2.81	3.00	-	4	Yes	-
* Pam	N/A	-	-	3.03	-	-	0	No	-
* Rachel	N/A	-	-	3.59	3.88	X	10	Yes	-
Sandi	1/150	28	-	3.87	4.00	-	4	Yes	-
Teri	106/366	25	520	3.01	2.96	-	14	Yes	-
Secondary									
* Andy	N/A	-	580	2.46	2.76	-	22	o	-
* Barb	N/A	-	-	3.00	2.45	-	43	o	-
* Carol	N/A	-	-	3.54	3.23	-	21	o	Yes
Cindy	1/191	24	-	3.74	3.64	-	34	o	Yes
* Janice	N/A	36	770	3.62	3.50	-	19	o	Yes
Jon	1/53	36	740	4.00	4.00	-	21	o	Yes
* Mark	N/A	26	-	3.42	3.27	-	43	o	
* Terrell	N/A	28	-	2.49	2.48	-	51	o	
Tim	5/87	25	-	3.34	2.79	-	43	o	
		*36 point scale	*800 point scale				*Term credits		

* - High school records are incomplete because student transferred from another college.

N/A - Not available

o - Not applicable

Appendix B

INTERVIEW

This interview is designed to take two sessions — labeled here as Part 1 and Part 2.

Part 1

A. PERSONAL HISTORY

The point of this section is to find out what the prospective teacher remembers about being a mathematics student, to begin to develop a sense of the person, and, to develop a context for the interview.

The questions that ask the participants to tell a little more about their high school and college math courses provide a means of seeing how they understand what they have learned, what kind of perspective they have of the substance of their courses, the way they organize that knowledge, and the articulateness with which they can talk about it.

It offers a second checkpoint (besides section C) for exploring what prospective teachers "bring" with respect to knowledge of and about mathematics.

Reasons for teaching

A1. ***I'd like to start out by learning a little about what brings you to teaching. When did you first start thinking you might want to teach? Why are you interested in teaching?***

Probe informant's intellectual interests and his/her perspective as a student. For instance, many of the elementary candidates mention their love of reading. Try also to discover what the person especially enjoys about school or about learning.

Try to learn what experiences the informant has had that may bear on learning to teach — e.g., is a parent, has worked as a counselor, etc.

School experience

A2. ***Let's shift the focus now. I'm interested in your own past experience in school, with mathematics in particular.***

a. ***You're planning to teach (elementary, high) school, is that right? When you think back to your own experience with math when you were in _________school, what stands out to you?***

Probe for specificity:

What do you mean? Can you give me an example of that? Is there anything else you remember?

If the teacher candidate does not mention one of the following, ask:

You haven't mentioned (much about) ________. Do you remember anything in particular about that?

o what you learned
o your teachers
o how you felt about math class
o how you felt about yourself in relation to others in your class

b. ***What about at the ________(other) level? What stands out to you about your experience with mathematics when you were in ______ school?***

If the teacher candidate does not mention one of the following, ask:

You haven't mentioned (much about) ________. Do you remember anything in particular about that?

o what you learned
o what your teachers
o how you felt about math class
o how you felt about yourself in relation to others in your class

Note: For the high school level, wherever that comes up — (a) or (b), ask:

What did you take in high school?

(Write down courses named.)

I'm interested in what these courses were about at your high school. Can we take just a few minutes and go through this list (list of courses identified by prospective teacher), and could you tell me briefly what each course was about?

If the student reports having stopped taking mathematics sometime in high school, probe to find out why and what response he/she got from others.

Probe also how the student's parents (mother, father separately) affected his/her interest in/participation in mathematics.

c. ***What have you taken at the college level?***

(Write down courses named.)

As you know, these courses can vary a lot from one university or college to the next. Can you tell me what each of these was primarily about?

What stands out to you about your experience with mathematics at this level?

If the teacher candidate does not mention one of the following, ask:

You haven't mentioned (much about) ________. Do you remember anything in particular about that?

o what you learned
o your teachers
o how you felt about math class
o how you felt about yourself in relation to others in your class

B. IDEAS ABOUT MATHEMATICS AS A SUBJECT AND AS A DISCIPLINE, TEACHING AND LEARNING MATHEMATICS, ABOUT PUPILS, ABOUT SELF, AND LEARNING TO TEACH MATHEMATICS

This section focuses on the teacher candidate's beliefs, both articulated and assumed.

Learning to teach

B1. ***You know that I am interested in learning about how people who are learning to teach math think about becoming teachers. I'm curious: as you think ahead to learning to teach mathematics at the ________ level, are there things that you feel you especially need in order to be able to teach a math class?***

Why does that seem important?

What do you mean by that?

How/where do you think you could learn that?

Being good at mathematics

B2. ***Now I'd like you to think of someone you know who's good at mathematics . . . . Who is it?***

a. ***Why do you think of ________ as good at math? What does he/she do?***

Note: What does "good at math" mean to the informant?

b. ***What's your hunch about why this person is good at math?***

Note: What's the source of success in mathematics?

What do you mean? Can you give me an example? What does ________ have to do with being good at mathematics?

B3. ***What about the opposite? Do you know anyone who you would say is really not good at mathematics? Who?***

a. ***Why do you think of ______ as not good at math? What does he/she do?***

b. ***What's your explanation for why this person isn't good at math?***

What do you mean? Can you give me an example? What does ________ have to do with being good at mathematics?

Note: Many students identify themselves for this question. Ask:

What's the explanation you give yourself about why you aren't so good at (or don't do so well at) math?

Card sort task

B4. ***These past few questions were somewhat open-ended. Now I'd like to switch to another kind of task. Here are some cards with statements about mathematics and about teaching and learning mathematics. I'm interested in learning what you think about them.***

Take a few minutes to read them, and sort them into three piles — the things you agree with, the ones you don't, and the ones about which you are not sure. You might not be sure about some because you don't quite understand the card, and others may be about things which you just aren't sure of.

When you're finished, we'll talk about the cards and how you sorted them. Take as long as you like. I'll turn off the tape recorder and you can just tell me when you're ready to talk.

Constructs: knowledge and ideas about —

- o mathematics [K]
- o teaching mathematics [T]
- o learning mathematics [L]
- o learners [P]
- o self in relation to mathematics and learning to teach mathematics [S]

1K There are unsolved problems in mathematics.

2K There are quite a lot of things in mathematics that must simply be accepted as true and remembered; there aren't explanations for them.

3K Reading and writing are the most important elementary school subjects.

4K Just like physicists or medical researchers, mathematicians uncover or create new knowledge.

5K Most math is quite tedious and boring.

6K Proofs are a means for making arguments in mathematics.

7T As a teacher, I'd feel embarrassed if a student asked me a question and I didn't know the answer.

8T The most important thing is not whether the answer to any math problem matches the answer key, but whether students can explain their answers.

9T When teachers give math homework, the homework should be very similar to what was done that day in class.

10T I would encourage students to develop and use unconventional approaches to solving mathematics problems.

11T It's important to be willing to go over material until everyone is getting it.

12T It is not a good idea for students to work together on math assignments in class.

13T As a teacher, I would like to try to avoid telling. Instead, I would try to lead my students to the answers by asking pointed questions.

14L In learning math, it is important to master topics and skills at one level before going on.

15L Students learn better when they can work at their own pace on individual math assignments.

16L It is a bad idea to give students "rules of thumb" to remember in mathematics (e.g., "invert and multiply").

17L Students learn best if they have to figure things out for themselves instead of getting explanations from the teacher.

18L Students often need rewards or bribes in order to do their math.

19L If students have unanswered questions or confusions when they leave class, they will likely be frustrated by the homework.

20L Mathematics helps you learn to think better.

21L To learn mathematics, many basic principles must be drilled into the learner's head.

22P Males are naturally somewhat more mathematically inclined than females.

23P While actual mathematical ability can't change, effort and motivation can increase.

24P There are basically two kinds of minds, and some people are good at logic and mathematics while others are better at English and more creative things.

25P Older students do not really need physical models in order to understand mathematical concepts.

26P Young pupils are not ready to understand ideas like intersection of sets, probability, and permutations.

27P People who are good at mathematics are able to think logically, in step-by-step manner.

28P There is no such thing as a "mathematical type" of person.

29S In order to begin teaching, I basically know enough math; what I need is to learn how to manage a classroom.

30S I can handle basic math, but I wouldn't do well at advanced mathematics.

31S I want to teach so that I can turn kids on to mathematics.

32S I would take a math course as an elective if I had the time and the content looked interesting to me.

33S I enjoy being challenged in mathematics.

34S Being good at math personally has very little to do with being a good teacher of math.

35S Getting experience in a real classroom is the most important part of learning to teach.

After the informant has had time to sort the cards as he/she wishes, ask:

a. ***Let's start with the ones you're not sure about. Could you go through these and tell me a little about why you put them in the "not sure" pile?***

Ask informant to read cards as he/she comments, so that statements are recorded. Try to let the informant talk as much as possible at this point, so that the task yields a commentary by the informant on the various statements. Probe generally as necessary to clarify terms or meanings, e.g.:

Why do you feel that way?

What does that mean to you?

What does that term mean to you?

b. ***Now, what about these other piles? As you read these and sorted them, were there some you felt especially strongly about? Or some that you wanted to qualify or comment on? Start with either pile (agree or disagree) — don't feel like you have to comment on every card; you decide which ones you want to comment on to help me understand your views.***

Repeat with other pile.

B5. ***Are there some things in mathematics or about mathematics that you especially like/enjoy?***

What do you mean? Why is that?

B6. ***What about the other side of the coin? Are there some things in mathematics or about mathematics that you especially dislike?***

What do you mean? Why is that?

B7. ***As a teacher, there'll be many things you'll be teaching, some of which you will probably understand better than others.***

a. ***Can you think of something in mathematics that you feel that you yourself understand really well? Take your time if you want to think about it for a bit.***

Can you tell me more about it?

What's the key that makes you feel that you really understand this? When/where did you learn it?

How do you think you came to understand it well?

Is this something you think you'll be teaching?

b. ***What about the flip side of this? Can you think of something in mathematics that you feel you really don't understand very well?***

What do you know about it?

What's the key that makes you feel that you don't really understand it?

When/where did you learn it? Why do you think you didn't "get it"?

Is this something you think you'll be teaching?

End of Part 1

Part 2

C. KNOWLEDGE OF MATHEMATICS AND IDEAS ABOUT TEACHING (& LEARNING) MATHEMATICS

The questions in this section have two basic functions: (1) to sample the prospective teacher's understanding of a range of mathematical concepts, and (2) to explore views of mathematics and of teaching and learning mathematics by asking what s/he thinks s/he would do or say in given situations.

Remember that one of the things I am trying to learn is how you think about teaching mathematics. In this next part of the interview, I'll be suggesting some things that a student might do or say, and that, as a teacher, you might have to respond to. I'm also going to bring up a couple of other tasks you might face in your teaching.

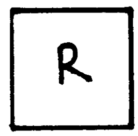
Some of these might seem trivial to you; others may be things you've never really had to think about before. I don't want you to worry if you are baffled by something — this is not a test. Teaching isn't straightforward, and I'm just as interested in how you think about things that puzzle you as I am in learning exactly what you'd do. But please feel free to take time if you need to work something out or need a few minutes to puzzle about something. For each one, I'd like to know what you think you'd do or say, and why that's what you'd do.

Probe for understanding of the content in each case and why they would do what they say they would. It may be useful to probe whether what they would ideally like to do is different from what they say they would be likely to do, and if so, what accounts for the difference. Some possible reasons might include, for instance, not understanding the concept well enough themselves, not knowing another way to explain something, considering the management of the whole class, etc.

Note: Examples are printed on separate cards to show informant.

C1. RESPONDING TO STUDENT IDEAS: GEOMETRY

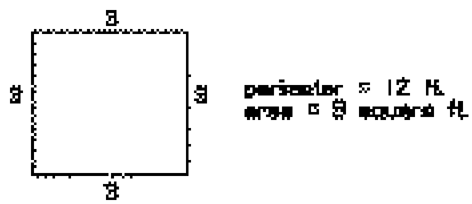
Something that is often taught in kindergarten and first grade is the names of geometric shapes. Suppose you are teaching first grade and you notice that one of your students has labeled a picture of a square with an R (for rectangle). What would you want to do or say?

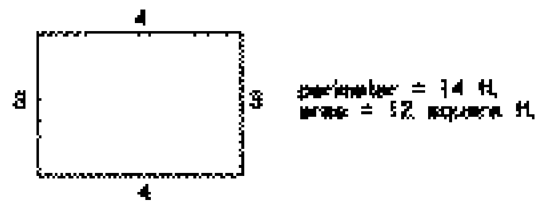


Why is that what you would do or say?

C2. RESPONDING TO STUDENTS' NOVEL IDEAS: PERIMETER/AREA, PROOF

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases (gets longer), the area also increases. She shows you this picture to prove that what she is saying is true:





How would you respond to this student?

(Note: Give grade level — 8th-9th grade — only if informant asks.)

<u>Probes</u>:

If informant is uncertain about whether this is actually true or not, say,

> *In teaching, this will happen that something may come up where you aren't sure yourself about whether the mathematics is correct or not. I'm interested in how you think you'd react to that. What would you do or say?*

If informant comments that this is not a <u>proof</u> or that they would be concerned that the student thinks this is a <u>theory</u>, try to learn why this is not a proof or a theory, and what s/he would do or say in response to the student.

If informant focuses on praising the student for doing some math outside of class, say:

> ***Is there anything else you'd want to do or say?***

If informant says that how s/he would respond would depend on who the student was, say:

> ***Could you give me a couple of examples?*** *and ask:* ***Why is that what you would do with such a student?***

If person doesn't mention the rest of the class, say:

> ***Is this something you would bring up with the rest of the class? Why or why not?***

C3. RESPONDING TO STUDENT DIFFICULTIES: PLACE VALUE

Suppose you are trying to help some of your students learn to multiply large numbers. You notice that when they try to calculate

```
  123
x 645
```

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They are doing this instead:

```
  123
x 645
-----
  615
  492
  738
-----
 1845
```

instead of this:

```
  123
 x645
-----
  615
 492
738
-----
79335
```

What would you do if you noticed that several of your students were doing this?

Probes:

If informant says, "I'd show them to put zeroes in," ask:

> ***What if some student asks, "How can we just add zeroes like that — it changes the numbers!"***

If informant says, "I'd tell them to just put X's in to hold the places lined up," ask:

> ***Where did you get that idea?***

If informant mentions "places," probe to find out how he/she talks about place value. Don't assume that this means that the informant is referring to the value of the places.

Probe comments about zero being a "placeholder" or "not a number."

C4. EVALUATING STUDENT RESPONSES: RATIO/PROPORTION, VARIABLES

Now suppose your students are learning to use variables to express mathematical relationships. Imagine that you have given them some statements to transform into mathematical statements. Here is one student's work on two of the exercises:

a) In a pound bag of M&Ms, there are five times as many brown candies as yellow ones.

Student answer:

b=the number of brown candies
y=the number of yellow candies
5b=y

b) In the same bag of candy, there are 50% more tan M&Ms than brown ones.

Student answer:

t=the number of tan candies
b=the number of brown candies
1.5b=t

Probes:

Are these expressed the way you'd do them?

Do you think this student is getting the idea? What is it that they are getting (or not getting)?

What would be your next step if this were your student?

Many people find this difficult. Why do you think that is?

If informant says that either or both of these answers is wrong, ask:
o How would you try to help the student understand this?

C5. RESPONDING TO STUDENT REQUESTS FOR HELP: SOLVING EQUATIONS

Suppose that one of your students asks you for help with the following exercise:

$$\textbf{If } \frac{x}{0.2} = 5\textbf{, then x=}$$

How would you respond?

(Try to pose this in such a way that the informant doesn't feel that you are assuming that they should <u>tell</u> the student what to do.)

Why is that what you'd do?

C6. RESPONDING TO STUDENTS: DIVISION, ZERO

Suppose that a student asks you what 7 divided by 0 is. How would you respond?

Why is that what you'd want to say?

If informant asks what age the students is, make a note of this request, and then let him / her select the age. Later ask:

You talked about a ________ grader. Would it make a difference if the student were younger (or older)?

Probe statements about zero (as a "placeholder," as "not a number.")

If informant says, "I'd say it's undefined," say:

What do you mean by "undefined"?

If informant says, "I'd say you can't divide by 0," say:

What if a student asks, "Why can't you divide by zero?"

If informant says they would show students how, as you divide by smaller and smaller numbers, the answer gets larger and larger, say:

What would I see or hear you doing?

C7. GENERATING REPRESENTATIONS: DIVISION, FRACTIONS

a. ***Division by fractions is a often a little confusing for students. People have different approaches to solving problems involving division with fractions. How would <u>you</u> solve this one:***

$$1\tfrac{3}{4} \div \tfrac{1}{2}$$

b. ***Something that many mathematics teachers try to do is to relate mathematics to other things.***

> (Informant may have already talked about this earlier in the interview. If so, refer to that.)

Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. Sometimes this is pretty challenging. What would you say would be a good situation or story for $1\tfrac{3}{4} \div \tfrac{1}{2}$? (Something real for which $1\tfrac{3}{4} \div \tfrac{1}{2}$ is the appropriate mathematical formulation?)

After informant has described a situation or story, ask:

> ***How does that fit with $1\tfrac{3}{4} \div \tfrac{1}{2}$?***
>
> ***Would this be a good way to help students learn about division by fractions?***

If the informant struggles with this, or cannot do it, say:

> ***Many people find this hard. In your view, what makes this difficult?***

C8. ORGANIZING CURRICULUM: "THE BIG IDEAS"

For this question, I'd like you to pick a grade you can imagine teaching. . . . What grade is that? Early in the fall, the principal of your school meets with each teacher to discuss the teacher's goals for their students. What would you say in describing what some of the most important things you'd be trying to accomplish in mathematics across the year with your __________ grade pupils?

(The point of this question is to explore the informant's sense of the important ideas in mathematics and the goals of school math instruction.)

Probes:

What do you mean by that? Why is that important to you?

If informant mentions "problem solving" or other fashionable terms, probe for what they mean by such terms: e.g.,

You just used the word ______ which is something many people are talking about these days. What do you mean when you use that term?"

If informant says he/she doesn't know enough about the school curriculum for that grade, ask:

Are there any important ideas that come to mind around that grade?

Are there any things you'd say regardless of the grade you were teaching?

D. PLANNING AND TEACHING MATHEMATICS: Textbook Exercise

For the last question, I'd like to spend a little more time thinking about one particular topic that you may work with when you teach.

Elementary candidates: subtraction with regrouping

Secondary candidates: slope

This question is intended to do several things: (1) to delve more deeply into the student's understanding of one topic, (2) to explore his/her thinking about how to help others learn it and what that would mean — can he/she see the concept through the eyes of a learner? (3) what tasks would he/she use? what explanations? how would he/she decide if the learners had learned it?

<u>Elementary Task: Subtraction with Regrouping</u>

Now we'll spend a little more time thinking about one particular topic that you may work with when you teach. We'll use this page from a second grade math textbook as the basis for the last part of the interview.

ED1. ***Do you remember learning this yourself? What do you remember?***

Listen for what the person considers the "this" here — e.g., subtraction, "borrowing," lining up numbers in columns. Don't impose "subtraction with regrouping."

Do you remember this being easy or difficult for you or for any of your classmates?

Do you remember anything your teacher did?

ED2. ***What do you think about this workbook page? I'm interested in what your impression of it is.***

Are there things you think are quite good in here? Some things you think are weaknesses or flaws? Why?

ED3. ***What would you say a pupil would need to understand or be able to do before they could work on this?***

Why is _______ important for this?

Is there anything here that you think might be especially hard for pupils?

ED4. ***Can you describe a little bit about how you would approach this if you were teaching second grade? Don't feel that you have to stick to either of these pages page if you have another way you'd want to work with your class, but you can use it if you choose.***

Why would you do that? How did you come up with this idea/approach? What do you mean by __________?

Can you give me an example of what you mean? Is there another way you can imagine doing this?

ED5. ***How could you tell if your students were "getting it"?***

Probe for what it means to "know" or "understand" — or whatever word they use — something in mathematics.

What would you look at or try to pay attention to?

ED6. ***Now, here's a copy of one student's work on this page. Take some time to look it over and then let's talk about what you make of Susan's work.***

I'm curious about what different people do when they check work like this. Can you describe what you did as you looked over her paper?

ED7. ***What do you think is going on here with Susan? What do you think she understands? Why do you think that?***

What's your hunch about why she got some of these wrong? Why do you think that?

What do you think Susan doesn't understand? Why do you think that?

ED8. ***Okay, imagine that Susan is your second grade pupil. How would you respond to this paper?***

What would you do next with Susan, or what would you have her do?

ED9. ***If you were working on this with your class and one pupil said, "Why are we learning this? I already have a calculator and I can do these problems on there," how would you respond?***

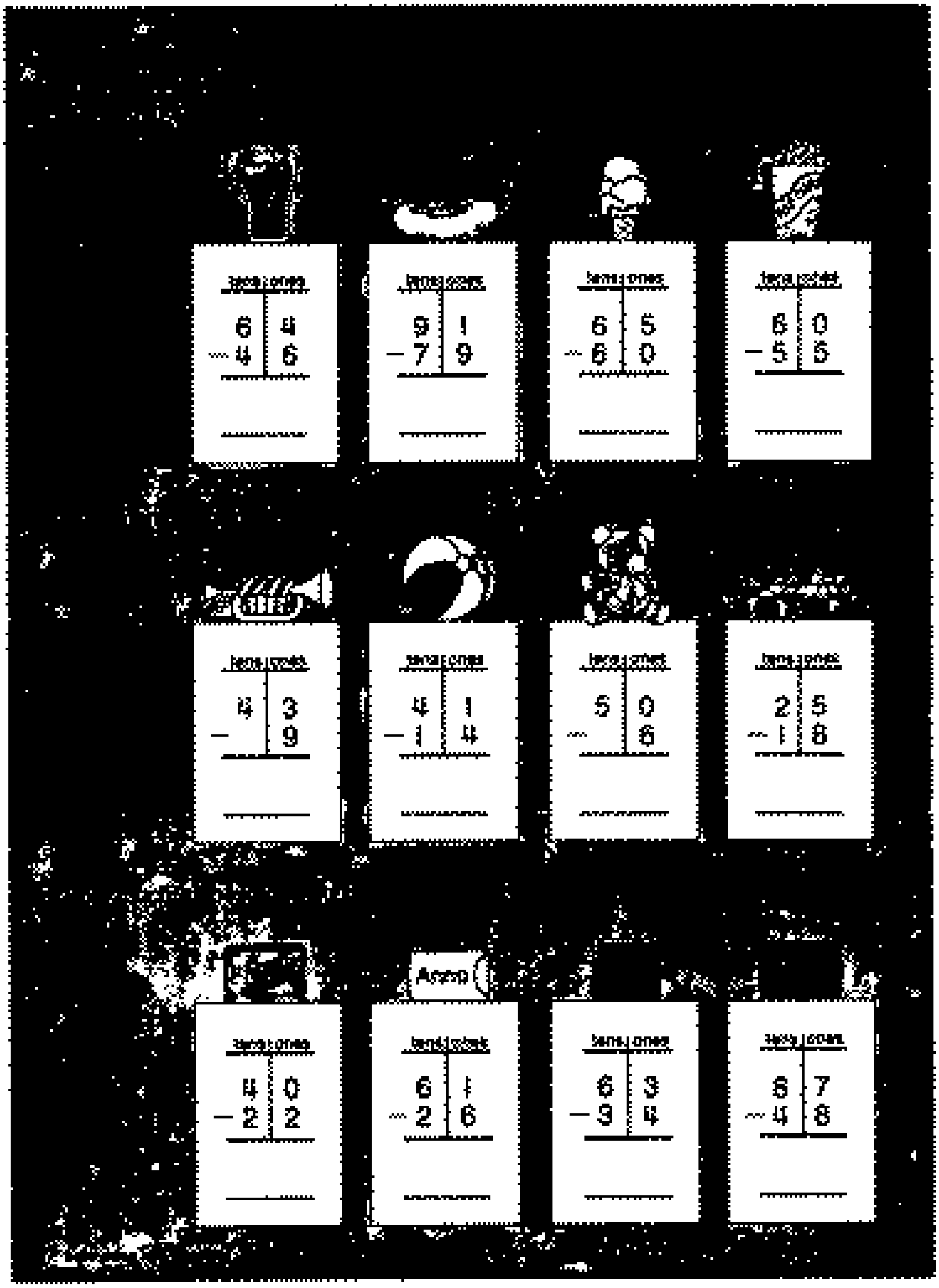
ED10. ***Suppose one of your pupils told you that he or she had come up with a new way to do this that didn't require "all that crossing out." The pupil came up and showed you the following:*** *(Explain)*

$$\begin{array}{r} \mathbf{36} \\ \underline{\mathbf{-\ 19}} \\ \mathbf{-3} \\ \underline{\mathbf{+\ 20}} \\ \mathbf{17} \end{array}$$

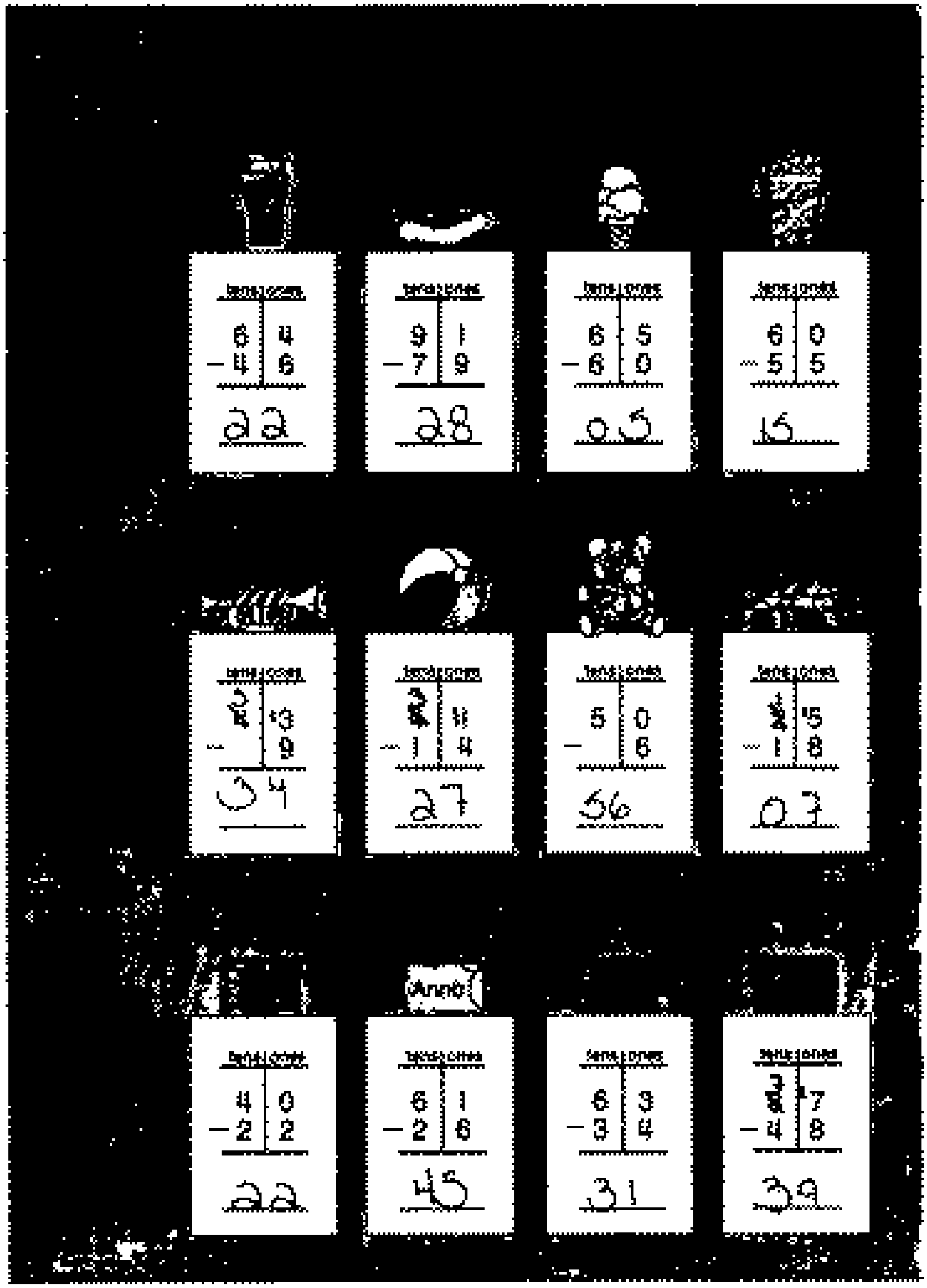
What would you make of this and what do you think you'd say?

ED11. ***Is there anything you wish you knew more about in order to teach this?***

How would you go about learning that?



This textbook section is taken from Bolster, Cobb, Gibb, Hansen, et al., Mathematics Around Us, 1975, p.217.



pages 217, 218

Subtraction with renaming

objectives

- Rename numbers so that there are more than 9 ones.
- Find the difference of two numbers less than 100.

pre-book activities

1. Put exercises similar to the following on the board.

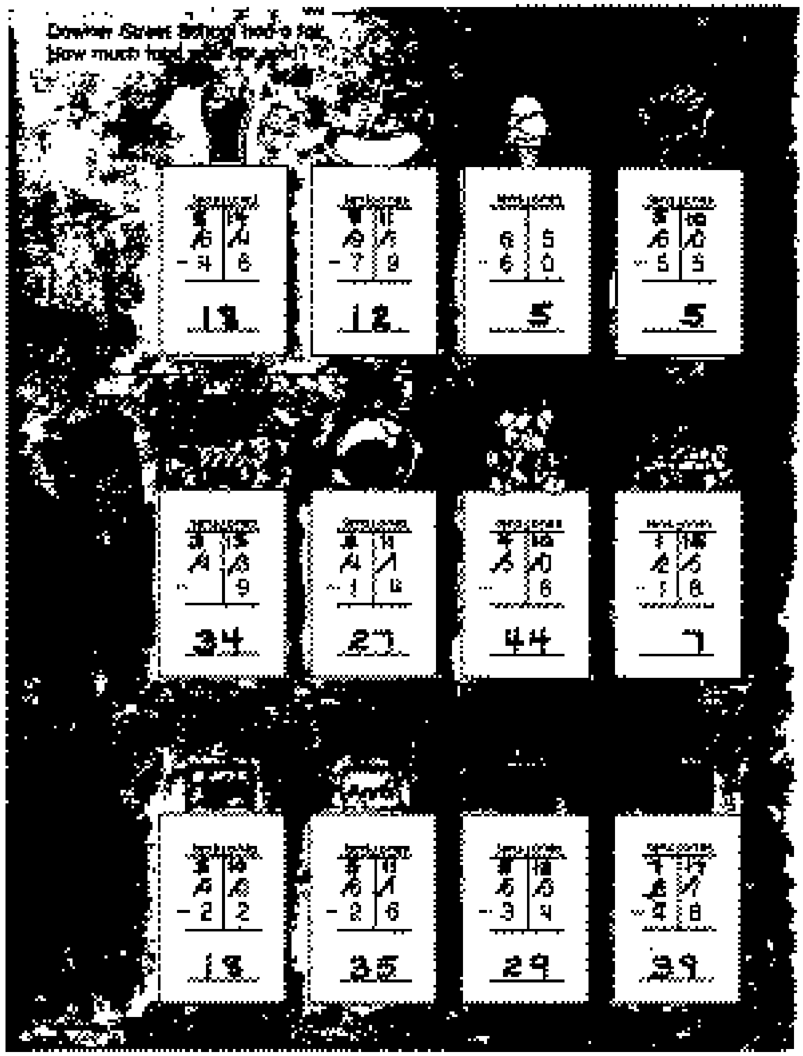
tens	ones		tens	ones
9	5		5	2
− 2	7		− 1	4

Have the children tell how they would find each difference. Then have them copy and complete each exercise. Ask various children to give the missing numbers for each exercise. Use the results of this activity to determine which children need more help before they do the work on pages 217-218.

2. For those children who need help, adapt and use the suggestions given for pages 215-216.

use of the pages

page 217 Before you have the children work the exercises independently, you may want to talk with the children about school fairs. Ask if any of them have been to a school fair, what kind of prizes they won, what kind of food they bought, and what other things they bought. After you read the directions, point out that the object at the top of each box shows the kind of object (or names the child), that the top number in each exercise tells how many things (or how much money) there were to begin with, and that the bottom number tells how many were sold (or how much was spent). After the children have found each difference, ask questions about each row. For example, for the bottom row, you might ask, "Who had the most money left? Who had the least? Who had more than 30¢ left? Who had less than 30¢?"



page 218 Have the children complete the exercises in the first row. When all are done, ask them what they noticed about the differences in this row (three of the differences are 23). Have the children circle, or mark in some other way, each exercise that has an answer of 23. Tell them that they are to work the exercises in each row and then circle the exercises that have the same difference.

post-book activities

1. Use any of the appropriate post-book activities suggested for pages 215-216.

2. Give each child the worksheet suggested in post-book activity 3 for pages 215-216, or use Teacher Aid 14. Tell a number story and have the children write the appropriate numerals in the box you designate and find the answer. Limit your stories to those in which the children must subtract to find the answers. The following are examples of stories you might use.

- 42 apples, 16 were sold. How many apples were left?
- 68 trucks, 35 buses. How many more trucks?
- 59 boys, 93 girls. How many more girls?
- 93 geese, 27 flew away. How many geese were left?

<u>Secondary Task: Slope</u>

Here's a section from an algebra textbook. I've included the pages from the book as well as the teacher's guide notes. I'd like to use this as the basis for this last part of the interview. Since it's kind of long, why don't you take a few minutes to look it over, and then we'll talk.

SC1. ***Do you remember learning this yourself? What do you remember?***

(Listen for what the person considers the "this" here — e.g., slope, graphing equations, learning the formulas for lines).

Do you remember this being easy or difficult for you or for any of your classmates?

Do you remember anything your teacher did?

SC2. **What do you think about this textbook section? I'm interested in what your impression of it is.**

Are there things you think are quite good in here? Some things you think are weaknesses or flaws? Why?

SC3. ***What would you say a pupil would need to understand or be able to do before they could work on this?***

Why is _______ important for this?

Is there anything about this that you think might be especially hard for students?

SC4. ***Can you describe a little bit about how you would approach this if you were teaching, say, ninth grade? Don't feel that you have to stick to any of this text material if you have another way you'd want to work with your class, but you can use it if you choose.***

Why would you do that? How did you come up with this idea/approach? What do you mean by __________?

Can you give me an example of what you mean?

SC5. ***How could you tell if your students were "getting it"?***

Probe for what it means to "know" or "understand" — or whatever word they use — something in mathematics.

What would you look at it or try to pay attention to?

SC6. ***Here is an incomplete homework paper turned in by Lynn. Take some time to look it over and then let's talk about what you make of Lynn's work.***

I'm curious about what different people do when they check work like this. Can you describe what you did as you looked over her paper?

SC7. ***What do you think is going on here with Lynn? What do you think she understands? Why do you think that?***

What's your hunch about why she got some of these wrong? Why do you think that?

What do you think Lynn doesn't understand? Why do you think that?

SC8. ***Okay, imagine that Lynn is a pupil of yours. How would you respond to this paper?***

What would you do next with Susan, or what would you have her do?

SC9. ***Something that students often have trouble understanding is the slope of a vertical line. Do you have any ideas about how you'd explain that to your class?***

SC10. ***If you were working on this with your class and one pupil said, "This is so boring!! Why are we learning this?", what would you say?***

SC11. ***Suppose a pupil says to you, "You said we can think of slope kind of like steepness. So I was thinking about climbing up a steep hill or something. But how do you tell the slope of a curvy line like a hill?"***

What do you think you'd say (or do)? Why would you say (or do) that?

SC12. ***Is there anything you wish you knew more about in order to teach this?***

o ***How would you go about learning that?***

Lines in a Coordinate Plane

***OBJECTIVES** for Sections 6-7 through 6-10:*

1. *Find the slope and y-intercept of a nonvertical line given its equation.*
2. *Graph, and find an equation of a nonvertical line given its slope and y-intercept.*
3. *Find an equation of a line given the slope of the line and the coordinates of a point on it.*
4. *Find the slope and an equation of a line given the coordinates of two points on the line.*
5. *Graph linear inequalities in two variables.*

6-7 Slope and y-Intercept of a Line

Like many other everyday terms, the familiar word "slope," as in "ski slope," has a special mathematical meaning. To discover that meaning, let us look at some examples.

The table below shows the coordinates of a few of the points (Figure 18) on the line

$$y = 2x + 3.$$

	x	y	
	-2	-1	
$+1$	-1	1	$+2$
$+1$	0	3	$+2$
$+1$	1	5	$+2$
$+1$	2	7	$+2$

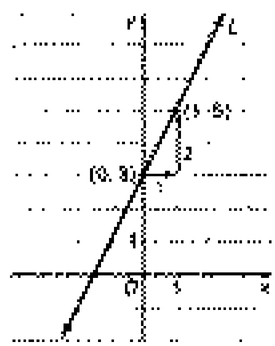


Figure 18

Notice that when the x-coordinates of two points on the line differ by 1, their y-coordinates differ by 2. We call the number 2 the *slope* of the line.

If you study the graph of the equation

$$y = 4x + 3$$

(Figure 19), you see that when the x-coordinates of two points on that

This textbook section is taken from Dolciani, Wooton, and Beckenbach , Algebra 1, 1980, pp. 218-221 line differ by 1, their y-coordinates differ by 4. The number 4 is the slope of the line.

x	y	
-2	-5	
-1	-1	$\} +4$
0	3	$\} +4$
1	7	$\} +4$
2	11	$\} +4$

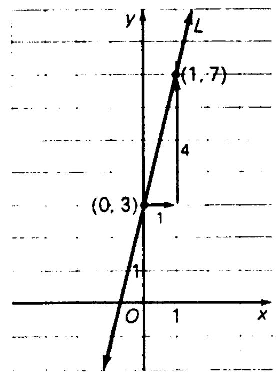


Figure 19

Notice that the numbers 2 and 4, the slopes of the lines in Figures 18 and 19, indicate the "steepness" of the graphs.

In general, if L is the graph of

$$y = mx + b,$$

then we call m the **slope** of the line. If you move from one point on L to another whose x-coordinate is 1 more (Figure 20), then the y-coordinate changes by m.

Notice that the coordinates of the point where L crosses the y-axis are $(0, b)$. The y-coordinate of this point, b, is called the ***y*-intercept** of the line.

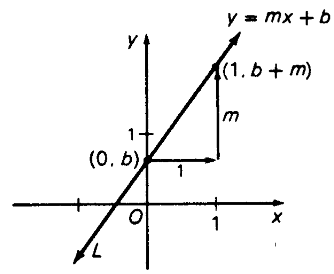


Figure 20

EXAMPLE 1 Find the slope and y-intercept of the graph of $3x + 4y = 8$.

SOLUTION Transform the equation into an equivalent one in **slope-intercept form**, $y = mx + b$, and then read the values of the slope m and the y-intercept b.

$$\begin{aligned} 3x + 4y &= 8 \\ 3x + 4y - 3x &= 8 - 3x \\ 4y &= 8 - 3x \\ \frac{4y}{4} &= \frac{8 - 3x}{4} \\ y &= 2 - \tfrac{3}{4}x \\ y &= -\tfrac{3}{4}x + 2 \end{aligned}$$

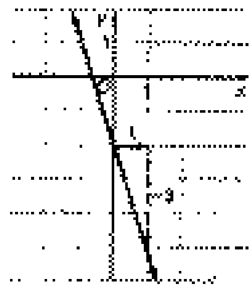
$\therefore$ the slope is $-\frac{3}{4}$ and the y-intercept is 2. **Answer.**

EXAMPLE 2 Draw the line with slope -3 and y-intercept -2; then find an equation for the line.

SOLUTION

1. The y-intercept is -2. Therefore, you plot the point $(0, -2)$.
2. Since the slope is -3, you move from $(0, -2)$ 1 unit to the right and 3 units down to locate a second point on the line.
3. Draw the line containing the two points.
4. $y = mx + b$

   $y = -3x + (-2)$, or $3x + y = -2$.



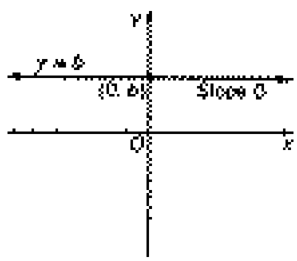
Figures 18 and 19 and the diagram accompanying Example 2 illustrate this fact: Lines that rise from left to right have positive slope, and lines that fall from left to right have negative slope.

The equation of any horizontal line (Figure 21) is equivalent to one of the form

$$y = b, \quad \text{or} \quad y = 0 \cdot x + b.$$

Thus, *the slope of every horizontal line is 0.*

Figure 21



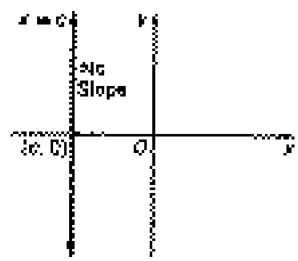


Figure 22

Any vertical line (Figure 22) has an equation of the form

$$x = c, \quad \text{or} \quad 0 \cdot y = x - c.$$

You cannot transform such an equation into slope-intercept form, because you cannot divide by 0. Thus, *vertical lines have no slope.*

Oral Exercises

State the slope and y-intercept (if any) of the line whose equation is given.

1. $y = 3x - 2$ 3; −2
2. $y = x + 3$ 1; 3
3. $y = -4x - 1$ −4; −1
4. $y = -\frac{2}{3}x$ $-\frac{2}{3}$; 0
5. $y + 2 = 0$ 0; −2
6. $x - 3 = 0$ none; none
7. $2y = 8x - 4$ 4; −2
8. $7 + 5x = y$ 5; 7

Written Exercises

In Exercises 1–6, P and Q are points on a line with slope m. Find the y-coordinate of Q.

EXAMPLE $P(4, 6)$, $Q(6, \_\_)$; $m = -4$

SOLUTION Note that: x-coordinate of Q = x-coordinate of $P + 2$

since $6 = 4 + 2$.

$\therefore$ y-coordinate of Q = y-coordinate of $P + 2m$.

$$6 + 2(-4) = -2. \quad \text{Answer.}$$

A

1. $P(0, 0)$, $Q(2, \_\_)$; $m = 4$ 8
2. $P(3, -4)$, $Q(5, \_\_)$; $m = -6$ −16
3. $P(3, 6)$, $Q(2, \_\_)$; $m = 9$ −3
4. $P(7, -4)$, $Q(-5, \_\_)$; $m = -\frac{1}{2}$ 2
5. $P(-2, -1)$, $Q(3, \_\_)$; $m = 0$ −1
6. $P(1, -3)$, $Q(7, \_\_)$; $m = \frac{5}{6}$ 2

a. Graph the line whose slope and y-intercept are given.
b. Write an equation of the line. Give your answer in the form $Ax + By = C$ where A, B, and C are integers.

7. $m = 3$, $b = 2$
8. $m = -4$, $b = -1$
9. $m = -2$, $b = -3$
10. $m = 8$, $b = -2$
11. $m = \frac{1}{4}$, $b = 1$
12. $m = -\frac{2}{3}$, $b = 0$
13. $m = 0$, $b = -3$
14. $m = -\frac{7}{4}$, $b = -\frac{1}{2}$

7. $3x - y = -2$ 8. $4x + y = -1$ 9. $2x + y = -3$ 10. $8x - y = 2$

11. $x - 4y = -4$ 12. $2x + 3y = 0$ 13. $y = -3$ 14. $7x + 4y = -2$

6-8 Determining an Equation of a Line: Slope and One Point

Given the slope of a line and the coordinates of any point through which it passes, you can use the slope-intercept form of a linear equation to find an equation of the line.

EXAMPLE 1 Find an equation of the line through the point (3, 1) with slope $-\frac{7}{4}$.

SOLUTION 1. The slope-intercept form of the equation of the line is

$$y = -\tfrac{7}{4}x + b.$$

2. Since the point (3, 1) lies on the line, its coordinates must satisfy this equation:

$$1 = -\tfrac{7}{4}(3) + b$$
$$1 = -\tfrac{21}{4} + b$$
$$\tfrac{25}{4} = b$$

3. $\therefore$ an equation of the line is

$$y = -\tfrac{7}{4}x + \tfrac{25}{4}. \quad \text{Answer.}$$

An equivalent form of this equation is $7x + 4y = 28$.

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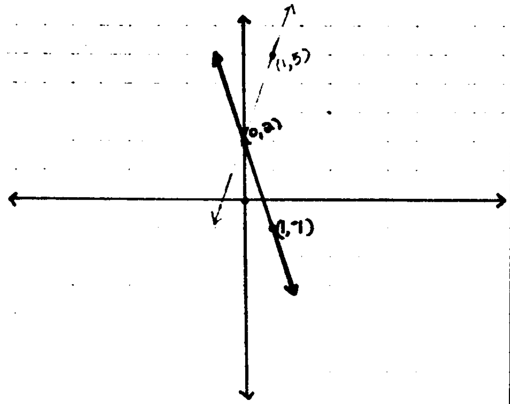
7. $m=3$, $b=2$

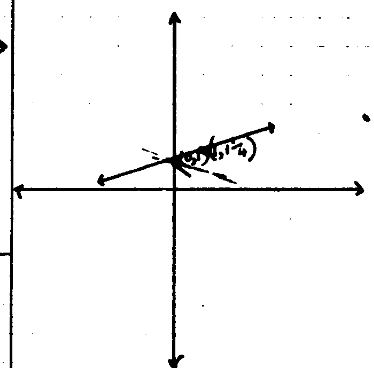
$y=3x+2$

$-3x+y=2$



11. $m=\frac{1}{4}$ $b=1$

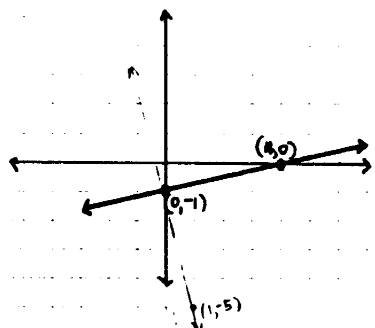
$y=\frac{1}{4}x+1$



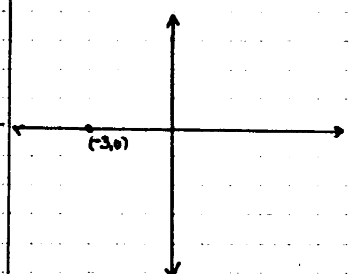
8. $m={}^-4$ $b={}^-1$

$y=-4x+{}^-1$

$4x+y=-1$

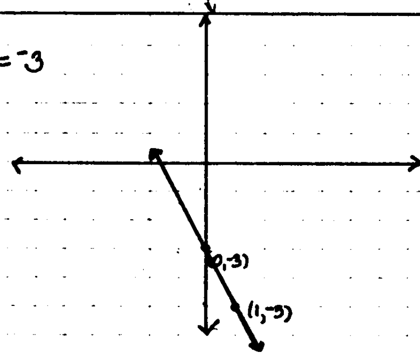


13. $m=0$, $b=-3$



9. $m={}^-2$ $b={}^-3$

$y={}^-2x+{}^-3$



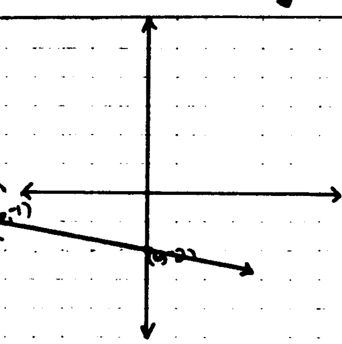
14. 7/4

10. $m=6$

$b={}^-2$

$y=6x+{}^-2$

$-6x+y={}^-2$



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Teaching Suggestions

In this section we develop the intuitive notion that slope means steepness. For this reason, we put off introducing the formula for slope until page 223, Section 6-9. There are several models that may help illustrate the definition of slope: the pitch of a roof; the grade of a hill; the rise/run of a stairway or ramp.

Ask students to graph on the same sets of axes lines with the equation $y = mx$ where $m = 1, 2, 3, 4, 5$. On a separate set of axes, have them graph $y = mx$ where $m = -1, -2, -3, -4, -5$. Have students describe how changes in slope affect changes in the graphs.

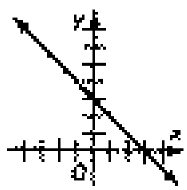
Chalkboard Examples

Find the y-coordinate of Q if P and Q are points on a line with slope m.

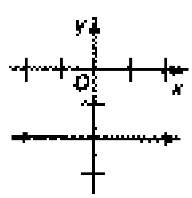
1. $P(-3, 4)$, $Q(-2, \underline{\ ?\ })$; $m = 7$ 11
2. $P(5, -6)$, $Q(2, \underline{\ ?\ })$; $m = -2$ 0

Graph the line with the given slope m and y-intercept b, then write an equation of the line in the form $Ax + By = C$ where A, B, and C are integers.

3. $m = -1$, $b = 3$ $x + y = 3$



4. $m = 0$, $b = -2$ $y = -2$



Note: The following statements are the ones that were used as the card sort in the first round of interviews. Based on the responses I got on these, as well as what I learned from listening to my ten winter term informants, I developed a new card sort (above). During the second round of interviews, I asked informants to respond on a written form to the following statements from the old card sort task — in case I might later want comparable data across all the people I interviewed.

[P]	Very young children are not capable of engaging in mathematical problem solving.
[K]	Even if you get an answer right, you may still not understand the mathematics or the problem.
[K]	Some problems in mathematics have no answers.
[K]	Even when you understand the mathematical concept, you may still get the problem wrong.
[P, L]	In exploring mathematical concepts, concrete experience is important for learners of all ages.
[K]	The basic point of teaching mathematics in school is to enable students to function in everyday life (e.g., balancing checkbook, counting out change, performing standard calculations, measuring).
[P]	Many people are simply not good at mathematics; one needs to be mathematically inclined in order to do well at it.
[L]	A very important thing to do in learning mathematics is to practice a lot in order to master the procedures.
[S, T]	In order to teach high school mathematics, teachers should really know a lot of math, way beyond calculus.
[K]	Good mathematics students develop their own ways of doing problems.
[K]	To be a liberally educated person, it is actually more important to read major literary works than to study big ideas in mathematics.
[K]	Being good at mathematics means being able to perform computations quickly and accurately.
[L]	Students should not leave math class feeling confused or stuck.
[K]	It is better to use an established procedure than to invent your own way of doing a problem.
[S]	The best part of mathematics is the challenge of solving difficult and puzzling problems.
[P]	Girls and boys are equally capable in mathematics.

[S] The best part of mathematics is getting right answers.

[T, L, K] If I were teaching, I would make sure all my students mastered basic computational skills before I spend a lot of time on more creative things, like problem-solving, logical reasoning exercises, or mathematical puzzles.

[T, L, K] The effective mathematics teacher plans thoroughly and sees to it that he/she sticks to it, minimizing inefficient digressions so that time is not wasted in the classroom.

[T, L, K] Teachers should not spend most of their time presenting mathematical content to students; students must be actively engaged in exploration and problem-solving in order to learn mathematics.

[P] For many of my pupils, math will probably be their least favorite subject.

[T, L, K] The teacher should encourage students to make guesses and conjectures and should allow them to reason on their own rather than show them how to reach a solution or answer.

[T] As a teacher I would like to avoid using the mathematics textbook, but be able to make up my own lessons and activities instead.

[T] If I were teaching mathematics, my major task would be telling — or showing — the students how to do the problems.

[T, L] Unlike a social studies class in which discussion is a central activity, an ideal mathematics classroom should be quiet so that people can really concentrate on their own work.

[T, L] Students learn math best if the teacher organizes the work clearly for them.

[P] Everyone can learn mathematics.

[S] If I heard that there was a good class being offered in the mathematics department, I would sign up for it as an elective.

[S] Elementary school teaching does not actually require much knowledge of mathematics — basic computational skill is sufficient math for teaching young children.

[S] I would be proud to be thought of as really outstanding in mathematics.

[P] Many people simply cannot "get" mathematical ideas.

[S] I have always been quite anxious about mathematics in school.

[S] I usually feel confident about my ability in mathematics.

[S, P] I'm not the type to do well in mathematics.

[T, L] It is a waste of time for students to "reinvent the wheel" by figuring out their own ways to attack and solve different kinds of problems; the teacher's role is to show them how to use the standard procedures accurately.

[P] For some reason, males are generally more capable in mathematics than females.

[S] I feel okay about mathematics. While I don't feel especially strong at mathematics, I am not fearful of it either.