Studying the Mathematical Knowledge Needed for Teaching:

The Case of Teachers' Knowledge of Reasoning and Proof

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Abstract

This paper focuses on the problem of how the mathematical knowledge needed for teaching can be studied. We identify and describe six plausible approaches to research into this topic: analyzing policy documents, teachers' mathematics curricula, teachers' mathematical knowledge, students' mathematics curricula, students' mathematical knowledge, and school mathematics practice. These approaches roughly correspond to different objects of analysis and methods of inquiry into teachers' mathematical knowledge for teaching. We also explore the extent to which the six approaches can or might be integrated by proposing a framework that aims to organize the approaches into a coherent program of research. As a context for our analysis we use the domain of teachers' knowledge of reasoning and proof for teaching. We use this domain both to exemplify the different approaches to research that comprise the framework and to provide evidence of the utility of the framework in organizing research related to the mathematical knowledge for teaching.

STUDYING THE MATHEMATICAL KNOWLEDGE NEEDED FOR TEACHING: THE CASE OF TEACHERS' KNOWLEDGE OF REASONING AND PROOF

Introduction

The mathematical knowledge important for the work of teaching is a significant issue in mathematics education. An extensive research focus on teachers' knowledge has emerged over the last two decades (e.g., Ball, 1988; Fennema & Franke, 1992; Ma, 1999; Shulman, 1986, 1987), in addition to a continuing public and policy concern about improving teachers' knowledge of mathematics as a means to improve instruction and maximize student learning (e.g., NBPTS, 1991; NCTAF, 1996). Despite the focus on understanding what teachers need to know about mathematics, the field of mathematics education still knows too little about the mathematical knowledge needed for teaching (Ball, Lubienski, & Mewborn, 2001; RAND Mathematics Study Panel, 2003).

One possible reason for which the problem of what teachers need to know remains unsolved is that people have been looking at it from different angles, having available only limited research that concentrated on connecting the knowledge produced by different perspectives. In this paper, we acknowledge the potential contributions of different perspectives to produce understanding about the mathematical knowledge needed for teaching and we explore the extent to which these can or might be integrated. In particular, we suggest a framework that brings together the contributions of six approaches to research in studying teachers' mathematical knowledge for teaching: analyzing policy documents, teachers' mathematics curricula, teachers' mathematical knowledge, students' mathematics curricula, students' mathematical knowledge, and school mathematics practice. This framework is intended to provide a helpful tool in thinking about what teachers need to know about mathematics, and also to initiate an effort for the development of a systematic program of approaches to research for studying different areas of teachers' knowledge for teaching.

To enhance appreciation of the complexity and subtlety of the mathematical knowledge needed for teaching, as well as the need for developing a framework for studying systematically this kind of knowledge, we peek in on a third-grade classroom and look more closely at what mathematical issues might arise for teachers as they teach.

The third graders in Deborah Ball's classroom have been working on Betsy's conjecture that "an odd number plus an odd number equals an even number." After the class has considered and ultimately ruled out a revision of the conjecture proposed by one of the students, Ball invites further comments from the students:

1. Ball: Other comments about the conjecture? […] Did anybody come up with an odd number plus an odd number that *didn't* equal an even number? Jeannie, you found one?
2. Jeannie: Me and Sheena were working together, but we didn't find one that didn't work. We were trying to *prove* that, um, Betsy's conjecture, um, that you *can't* prove that Betsy's conjecture always works. [*murmurs from other children*]
3. Ball: Go on, Jeannie. Say more about why you think that.
4. Jeannie: Because um there's um like numbers go on and on forever and that means odd numbers and even numbers, um, go on forever and, um, so you couldn't prove that *all* of them work.
5. Ball: What are people's reactions to what Jeannie and Sheena thought? [*pause*] They said they didn't find one that didn't work, but they don't think we can prove it *always* works because numbers go on forever and ever.

Ofala is the first to react to Jeannie and Sheena's point:

6. Ofala: I think it can always work because I um tried almost, um, [*pause while she counts examples in her notebook*] 18 of them, and I also tried a Sean number [i.e., an even number with an odd number of groups of 2] so I think, I think it can always work.

Mei also disagrees with what Jeannie said, justifying her objection on the fact that Jeannie accepted so far numerous other conjectures formulated by the class without insisting that these were shown to be true for *all* possible cases:

7. Mei: […] Because with those conjectures [*she motions to several previously discussed and widely-agreed upon conjectures, posted above the chalkboard*] we haven't even tried them with all the numbers that there is, so why do you [*referring to Jeannie*] say *those* work? Well, we haven't even tried all those numbers that there ever could *be*. […] So you can't really say that *those* are true if you're saying that you want to try every number there ever was.

The discussion continues with some animation for a few more minutes before the class ends. During the next couple of days, Ball made systematic efforts to help her students think how they could use their definitions of even and odd numbers to construct an argument that would treat adequately the general case. Ultimately, one of the students, Betsy, proposed the following argument:

8. Betsy: All odd numbers if you circle them by twos, there's one left over, so if you… plus one, um, or if you plus another odd number, then the two left over will group together, and it will make an even number.

What mathematics would the teacher need to know in this situation? We are not going to discuss this question in depth here. Briefly, we claim that Ball had to understand and evaluate students' *reasoning*. Jeannie (lines 2 and 4), Ofala (line 6), and Mei (line 7) advanced different arguments about whether or not the conjecture could be proved, and about the legitimacy of each other's ideas. Ball would have to validate these arguments, identify their mathematical basis, and decide how she could help her students resolve their disagreement on mathematical grounds. In addition, she would have to know and be able to decide – first for herself and then for her students – what would constitute an appropriate *proof* for Betsy's conjecture. Given that third graders are typically not proficient with symbolic notation, the standard algebraic proof of the conjecture would most likely be beyond their conceptual reach. What might be a mathematical argument that would be, at once, honest to mathematics as a discipline and honoring of students as mathematical thinkers? Does Betsy's argument (line 8) fulfill these conditions? Does it count as proof? What mathematical resources were necessary for the development of this argument? Ball considered that it was crucial for her students to recognize the power of definitions in mathematical argumentation.

The episode reveals the complexity and subtlety of the mathematical knowledge needed for teaching. It identifies in particular a specific area of teachers' *mathematical* knowledge: the set of understandings about *reasoning and proof* that are essential for effective teaching. We call this area of teachers' knowledge "knowledge of reasoning and proof needed for teaching" and we bring it to the center of our investigation for the development of a framework for studying teachers' mathematical knowledge for teaching.

Our focus on reasoning and proof is justified for several reasons. Reasoning and proof is recommended to be integral to *all* students' mathematical experiences across *all* grade levels (e.g., Ball, Hoyles, Jahnke, et al., 2002; Hanna, 2000; NCTM, 2000; Schoenfeld, 1994). It is also considered central to doing mathematics, communicating mathematical ideas, and learning mathematics with understanding (e.g., Ball & Bass, 2003a; Hanna & Jahnke, 1996; Steen, 1999; Thompson, 1996; Yackel & Hanna, 2003). While there is such an increased emphasis on reasoning and proof, students of all grades have difficulties with logical thinking (e.g., Chazan, 1993; Fischbein & Kedem, 1982; Porteous, 1990; Senk, 1985). These difficulties highlight how essential it is to clarify the knowledge of reasoning and proof needed for teaching as a first step toward improving students' competencies in this domain. If teachers have poor understanding of reasoning and proof or themselves harbor the same misconceptions as their students, student difficulties are likely to continue.

Because different kinds of teaching place on teachers' knowledge different demands (e.g., Schifter, 1998; Stein, Silver, & Smith, 1998), it is important that we specify the type of teaching whose knowledge demands we intend to study. This teaching targets the creation of mathematics classrooms modeled as communities of mathematical discourse in which the validity of ideas rests on reason and argument, mathematics are treated with integrity, and students' thinking and ideas are considered seriously (for examples of practices committed to this kind of goals, see Ball, 1993; Ball & Bass, 2000b, 2003a; Carpenter, Franke, & Levi, 2003; Lampert, 1990, 1992, 2001; Reid, 2002; Zack, 1997). Yackel and Hanna (2003) note that the creation of classroom environments that foster the view of mathematics as sense-making activity is "a highly complex undertaking that requires explicit effort[s] on the part of the teacher[s]" (p. 234). In order for teachers' efforts to be successful, teachers need to have some knowledge about reasoning and proof.

To recap, the primary goal of this paper is to provide insights into how the problem of teachers' mathematical knowledge for teaching can be studied in a systematic way. Toward this end, we identify six plausible approaches to research for producing understanding about the mathematical knowledge needed for teaching, and we propose an integrative framework that brings these approaches together in a coherent program of research. The framework is intended to be useful in the study of different areas of teachers' mathematical knowledge, such as mathematical practices (like reasoning and proof, and representation) and topics (like rational numbers, and functions). As a context for our analysis we use the domain of teachers' knowledge of reasoning and proof for teaching. We use this domain both to exemplify the different approaches to research that comprise the framework and to provide evidence of the utility of the framework in organizing research related to the mathematical knowledge for teaching. In so doing we sometimes draw on available mathematics education literature on reasoning and proof, and other times describe hypothetical studies. The hypothetical studies can also be seen as suggestions for future research in a domain that, although recognized as important, has so far received relatively limited attention.

The paper is organized into two parts. In the first part, we present separately the six approaches to research for studying the mathematical knowledge for teaching, explaining what their function is or could be in investigating teachers' knowledge of reasoning and proof. In the second part, we present the framework that tries to capture the relationships among the six approaches.

Six Approaches to Studying the Mathematical Knowledge Needed for Teaching: Illustrating Each Approach in the Domain of Elementary Teachers' Knowledge of Reasoning and Proof

In this part of the paper, we present six approaches for studying the mathematical knowledge needed for teaching: analyzing policy documents, teachers' mathematics curricula, teachers' mathematical knowledge, students' mathematics curricula, students' mathematical knowledge, and school mathematics practice. While we do not argue that the six approaches constitute an exhaustive list of plausible approaches to analyzing teachers' mathematical knowledge for teaching, we suggest that a rather integral understanding of this topic requires the contributions of all six of them.

The approaches roughly correspond to different sources or objects of analysis and different methods or sites of inquiry into the topic of teachers' mathematical knowledge for teaching. Table 1 provides an overview of the approaches, giving also examples of hypothetical studies. We deliberately chose a common theme for our example studies to highlight further the special features of each approach; this common theme focuses on teachers' knowledge of proof by counterexample. The readers can refer to the table as they read through the paper, but they are cautioned that the information provided in the table does not describe sufficiently the six approaches.

The approaches are quite distinct from one another. For example, the primary ways in which one might analyze student knowledge (e.g., the decontextual way as in general psychological theories of development, or the contextual way as in classroom-based studies) are distributed between Approaches 5 and 6. Defining the approaches in ways that eliminate significant overlaps achieves clarity in the domain of action of each approach. This is particularly suitable for the goal of the paper to introduce and illustrate plausible ways to studying the mathematical knowledge for teaching. In actuality, however, a study may utilize a number of approaches.

Before we elaborate the approaches by clarifying their constructs and elucidating their application in the specific area of reasoning and proof, we present three methodological considerations. The first relates to how we might deal with the lack of clarity about the notions of reasoning and proof in school mathematics (see, e.g., Reid, 2002; Steen, 1999). On the one hand, we might use the definitions of reasoning and poof offered in the literature we draw on. On the other hand, we might choose to question those definitions. If we do the first, it is in some sense like unquestionably adopting views of reasoning and proof without sufficient scrutiny. Yet, deciding what are adequate definitions for reasoning and proof is not the purview of this paper. In addition,

Table 1

Six Approaches to Studying the Mathematical Knowledge Needed for Teaching

Approaches	Sources or objects of analysis	Methods or sites of inquiry	Example studies focusing on the knowledge of proof by counterexample needed for teaching
Approach 1: Analyzing policy documents	Policy documents that set standards for teacher education, and others that set standards for school mathematics	Text analysis	Text analysis of policy documents to investigate what policy makers recommend that prospective teachers have an opportunity to learn about proof by counterexample in teacher preparation programs. Developing an understanding of these recommendations can provide insights into what policy makers consider important for prospective teachers to know about this proof method.
Approach 2: Analyzing teachers' mathematics curricula	College mathematics curricula (notably college mathematics textbooks) developed for use in mathematics courses taken by prospective teachers	Curriculum analysis (notably textbook analysis)	Textbook analysis to investigate the opportunities designed for prospective teachers to learn about proof by counterexample in the mathematics courses they take at college. Developing an understanding of these opportunities can provide insights into what textbook authors consider important for prospective teachers to know about this proof method.
Approach 3: Analyzing teachers' mathematical knowledge	(1) Researchers' decisions about which aspects of teachers' mathematical knowledge are worth analyzing; (2) Teachers' current knowledge in specific mathematical areas that relate to the work of teaching	(1) Literature review; (2) Analysis of what teachers' know about mathematics, conducted outside real classroom settings in school (e.g., analysis of experimental situations, situations from teacher preparation programs, interview sessions, responses to test items)	Literature review of available studies on teachers' knowledge of proof by counterexample can illuminate the reasons for which researchers focused on this topic and can provide some insights into what mathematics education researchers consider important for teachers to know about this proof method. Also, empirical analysis of how teachers understand proof by counterexample (e.g., through interview sessions) can help identify areas of common difficulties, which might be addressed in teacher preparation programs, if they are judged as crucial understandings for teachers to have.
Approach 4: Analyzing students' mathematics curricula	School mathematics curricula developed for use in mathematics classes taken by K-12 students	Curriculum analysis (notably analysis of students' textbooks and teachers' guides)	Textbook analysis to investigate the opportunities designed for students, in the mathematics textbooks they use, to learn about proof by counterexample. Developing an understanding of these opportunities can provide insights into what teachers would need to know about this proof method to successfully enact the opportunities in their classrooms.
Approach 5: Analyzing students' mathematical knowledge	K-12 students' current knowledge in specific mathematical areas	Analysis of what students know about mathematics, conducted outside real classroom settings (e.g., analysis of experimental situations, interview sessions, responses to test items)	Empirical analysis of how students understand proof by counterexample (e.g., through interview sessions) can help identify areas of common difficulties. Identifying these areas can provide insights into what teachers would need to know about proof by counterexample to recognize and address the weaknesses of students' understanding of this proof method.
Approach 6: Analyzing school mathematics practice	School mathematics practice	Analysis of records of school mathematics practice (e.g., videotapes of lessons, transcripts of these lessons, student work, teacher journal)	Empirical analysis of the mathematical components of classroom instruction to uncover the demands imposed on teachers' knowledge of proof by counterexample as they implement the curriculum, facilitate discussions, listen to students, manage their difficulties, and actively seek to help them overcome these difficulties.

to question the definitions as they appear in various studies can render the landscape even more chaotic and thereby undercut the framework's value as a tool for organizing the intellectual terrain on reasoning and proof. Therefore, given the primary goal of the paper to illustrate the framework's value, we choose not to question the definitions of reasoning and proof in the literature we use.

The second methodological consideration has to do with the level of schooling on which to focus our analysis. Bounding the inquiry to a particular level of schooling is essential, for the problem is complex and takes, presumably, different forms at different levels. In this paper we choose to focus our inquiry on the *elementary* grades, primarily because elementary teachers' knowledge of reasoning and proof can help achieve considerable leverage on the improvement of students' proficiency in reasoning and proof throughout the grades. Also, setting robust foundations on reasoning and proof from the beginning can have a favorable impact on students' mathematical education more broadly, because reasoning and proof pertains to every mathematical activity.

The third consideration concerns the studies we draw on as we present each approach. Most of these studies have not focused on the topic of teachers' mathematical knowledge for teaching and have not necessarily utilized the approach we use them to illustrate. Given the goal of the paper to illustrate the different approaches, we consider appropriate to discuss how some of these studies could potentially contribute to a particular approach to studying teachers' knowledge of reasoning and proof for teaching. Had our goal been to review these studies, our method would be different.

In discussing how the six approaches are or could be used to study elementary teachers' knowledge of reasoning and proof for teaching, we also examine the logic of the approaches, describe their importance, try to unearth their assumptions, and consider their limitations. The reference to limitations emphasizes our belief that no single approach can be used to study fully the mathematical knowledge need for teaching. Considering both the merits and pitfalls of the approaches highlights their complementarities. Complete presentation of the six approaches in the domain of teachers' knowledge of reasoning and proof lies beyond the scope of the paper.

Approach 1: Analyzing Policy Documents

One approach to studying the mathematical knowledge for teaching is to analyze the perspectives registered in policy documents. The perspectives registered in these documents constitute the "policy response to solving [the] problem of what teachers need to know" (Ball et al., 2001, p. 441). The group of people contributing to the development of policy documents is diverse, often a mixture of different professionals (mathematicians, mathematics educators, curriculum developers, professional developers, and teachers) with manifold views about issues of mathematics education. Therefore, the perspectives registered in these documents can be thought of as the "resultant" of the views of people involved in their preparation.

There are two main categories of policy documents: (1) those that set standards for the mathematical preparation of prospective teachers (e.g., CBMS, 2001; NCTM, 1991), and (2) those that set standards for the mathematical education of K-12 students (e.g., NCTM, 1989, 2000). Text analysis of the first kind of policy documents can illuminate the mathematics that policy makers recommend prospective teachers be given the opportunity to learn in teacher preparation programs. Developing an understanding of these recommendations can provide insights into what policy makers consider important mathematical knowledge for teaching. Text analysis of the second kind of policy documents can illuminate the mathematics that policy makers recommend students be given the opportunity to learn in their schooling. Developing an understanding of these recommendations can provide insights into the mathematics that teachers would need to know to successfully enact these opportunities in their classrooms.

To the best of our knowledge, there is no published study focusing on the recommendations about reasoning and proof set forth by different kinds of policy documents. Depending on the kind of policy document the analysis could take different forms.

In the case of policy documents that set standards for the mathematical preparation of teachers, the researcher doing the analysis can try to identify all the relevant parts in the documents that relate to reasoning and proof and codify the recommendations set forth by policy makers. However, these tasks are complicated and call for many methodological decisions on the part of the researcher, primarily because policy documents that set standards for teacher preparation programs often do not provide specific recommendations about what teachers should be given an opportunity to learn about reasoning and proof in these programs. This situation comes in sharp contrast with how the same policy documents treat content areas like number and operations, geometry, and algebra, for which they often provide elaborate sets of recommendations. For example, the only explicit recommendation set forth by the Conference Board of the Mathematical Sciences (CBMS, 2001) about the mathematical experiences that prospective teachers need to obtain with regard to proof and justification is the following:

> Prospective teachers at all levels need experience justifying conjectures with informal, but valid arguments if they are to make mathematical reasoning and proof a part of their teaching. Future high school teachers must develop a sound understanding of what it means to write a formal proof. (p. 14)

Although this excerpt suggests that reasoning and proof should be part of the mathematical preparation of teachers of all grade groupings, the *specific* aspects of reasoning and proof important for elementary teachers' mathematical preparation are not directly addressed. Furthermore, the lack of clarity associated with some key terms used in the quote – like (formal) proof, justification, informal arguments – may allow for different interpretations of its meaning.

In the case of policy documents that set standards for the mathematical education of K-12 students, the researcher doing the analysis can do the same things as in the previous analysis – try to identify all the relevant parts in the documents that relate to reasoning and proof and codify the recommendations set forth by policy makers. These tasks are less complicated in this analysis than in the previous one, primarily because policy documents that set standards for school mathematics have started to provide detailed recommendations, organized by grade level groupings, about what students ought to be given opportunities to learn about reasoning and proof. However, the researcher analyzing policy documents of this kind will need to take the analysis a step further to address the issue of teachers' knowledge of reasoning and proof for teaching. In particular, the researcher will need to investigate how standards for student learning can translate into standards for teacher knowledge, a task that is methodologically complex.

In concluding, we consider potentials and pitfalls of Approach 1. On the one hand, this approach can provide insights into the mathematical knowledge needed for teaching by investigating policy makers' point of view on this topic. This point of view can be particularly useful in identifying common characteristics in the perspectives of different professional groups on the mathematical preparation of teachers and the mathematical education of students. In turn, these characteristics support mathematics teacher educators in making tentative decisions about what teachers need to learn in areas such as reasoning and proof where research is still limited. They can also help achieve some coherence in the development of teacher preparation and school programs in mathematics.

On the other hand, the extent to which the policy view represents the manifold professional groups involved in its development clear and the basis of the policy point view (e.g., research, personal opinions and values) are both unclear. With regard to the latter, Ball et al. (2001) note the policy "approach to specifying knowledge for teaching is rooted primarily in policy deliberations and often does not reflect research evidence" (p. 441). The NCTM (1991) document appears to be one such example, as it is said to be among the early *Standards* documents that are "not well anchored in either research or theory" (Kilpatrick, 2003, p. 1). One important thing to acknowledge, though, is that educational research has not answered all our questions (Hiebert, 2003), one of them being what mathematical knowledge teachers need to have to teach mathematics well (Ball et al., 2001). This, coupled with that it is misleading to believe that research studies have the potential to provide unequivocal answers to each of our questions about educational practice (Silver, 1990), suggest that one may not expect policy recommendations about teachers' knowledge to be totally anchored in research. One *can* expect, however, that these recommendations will acknowledge the relevant scholarship in mathematics education to the extent that this is available. This has started to happen as indicated by the case of the currently released *Standards* document (NCTM, 2000) and its *Research Companion* (Kilpatrick, Martin, & Schifter, 2003). Another issue relates to the impact of the policy view on the development of teacher preparation and school programs in mathematics. Does the study of the recommendations registered in policy documents offer a sense of how reasoning and proof is implemented in teacher preparation or school programs? For example, to what extent do curriculum materials developed purported to be based on curriculum frameworks (e.g., CBMS, 2001; NCTM, 2000) actually embody the percepts of these policy documents? According to Romberg (1997), analysis of curriculum frameworks *cannot* substitute for textbook analysis: "Although curriculum documents describe what is 'intended' policy, these intentions are never really implemented in classrooms. Textbooks are a more detailed source of what mathematics is covered than are curriculum frameworks." (p. 134)

Approach 2: Analyzing Teachers' Mathematics Curricula

A second approach to studying the mathematical knowledge for teaching is to analyze the college mathematics curricula (notably college mathematics textbooks) developed for use in mathematics courses taken by prospective teachers to investigate the opportunities designed for teachers to develop their mathematical knowledge. Understanding these opportunities can support two things. First, it can provide insights into what curriculum developers consider important mathematical knowledge for teaching. The underlying assumption here is that how curriculum developers (notably textbook authors) treat the mathematical content (presentation, emphasis, selection, sequencing, rhetoric, motivation, etc.) reflects their perceptions of mathematical knowledge for teaching. This assumption guided similar analyses in different contexts. For Cooney (1994), in trying to get an idea of the emphases that existed in mathematics teacher education prior to the 1970s, conducted an analysis of three secondary methods textbooks based on the premise that "texts provide some definition for what was considered important knowledge for teachers to have" (p. 615). Second, it can provide a sense of the mathematical experiences prospective teachers are likely to be offered in teacher preparation programs.

Curriculum analyses on reasoning and proof can focus on the issues mentioned above but also on some others. For example, an analysis of this kind can concentrate on the college mathematics textbooks used in mathematics courses for prospective elementary teachers trying to address some of the following questions: What do textbook authors seem to consider important knowledge of reasoning and proof for teaching elementary mathematics? What aspects of reasoning and proof do they emphasize in their books, and what do they expect prospective teachers to understand and be able to do? How do they try to promote teachers' learning of these ideas? How do textbook authors conceptualize reasoning and proof (e.g., as a topic that deserves its own attention, as part of logic or problem solving, etc.)? How does the treatment of reasoning and proof differ, if at all, between textbooks written by mathematicians and textbooks written by mathematics educators? The last question is motivated by the fact that members of the two communities have been found to hold different views about some aspects of school mathematics curricula (Sfard, 1998). Developing an understanding of possible different perspectives of mathematicians and mathematics educators on how reasoning and proof should be treated in teachers' mathematics curricula can inform the field's perception of what teachers need to know in this area.

There is ongoing research on teachers' mathematics curricula focusing on reasoning and proof (McCrory, Stylianides, & Siedel, 2005). Preliminary analysis of 17 textbooks for mathematics courses taken by prospective elementary teachers revealed two notable patterns. First, there is great variability in the place where concepts related to reasoning and proof (e.g., inductive and deductive reasoning, methods of proof, definitions, conjectures, concepts of logic) are explicitly covered in textbooks: in a chapter on mathematical reasoning or proof (e.g., Darken, 2003; Parker & Baldridge, 2003), in a chapter on logic (e.g., Krause, 1991; Musser, Burger, & Peterson, 2003), in a chapter on problem solving (e.g., Beckmann, 2003; Long & DeTemple, 2003), in a chapter that combines logic and problem solving (e.g., Billstein, Libeskind, & Lott, 2001; Masingila, Lester, & Raymond, 2002). There are also textbooks where concepts related to reasoning and proof are not (explicitly) covered (e.g., Jones, Lopez, & Price, 2000). Second, textbooks often provide definitions of inductive and deductive reasoning and contrast the two (e.g., Bassarear, 2001; Bennett & Nelson, 2003; Masingila et al., 2002), but they tend not to define proof or address explicitly issues related to specific methods of proof (e.g., proof by contradiction or mathematical induction).

To conclude, studies utilizing Approach 2 can illuminate the mathematical knowledge perceived important for teaching by textbook authors, and can also provide information about the mathematical experiences designed for teachers in teacher preparation programs. However, studies of this kind also raise a series of issues like the following: How do textbook authors decide which mathematical ideas to cover in their books, and how to promote them? What are their implicit or explicit assumptions about the demands that the work of teaching places on teachers' mathematical knowledge? Are there any empirical warrants for these assumptions? How do the mathematical experiences actually offered to prospective teachers in teacher preparation programs compare with the experiences designed in curriculum materials developed for use in these programs?

Approach 3: Analyzing Teachers' Mathematical Knowledge

Approach 3 has two interrelated objects of analysis. The first comprises mathematics education researchers' decisions about which aspects of teachers' mathematical knowledge are worth analyzing. Literature review of available investigations on teachers' knowledge can illuminate which aspects of teachers' knowledge have become the objects of study, and also which are reasons that researchers investigated these aspects. If the reasons relate to issues of mathematical knowledge for teaching, then it is plausible to say that the objects of study in these investigations identify, from the researchers' point of view, important mathematical understandings for teachers to have. The second object of analysis comprises teachers' knowledge of specific mathematical ideas that are relevant to their work. This object can be examined through different methods of inquiry that take place outside school mathematics practice, such as analysis of experimental situations, situations from teacher preparation programs, interview sessions, and teachers' responses to test items. The investigations of teachers' knowledge can help identify areas of common difficulties, which might be addressed in teacher preparation programs, if they are judged as crucial understandings for teachers to have. Cooney (1999) notes: "If one is trying to understand the status of what teachers know and to build a case for increasing teachers' knowledge of mathematics, this approach [i.e., examining what teachers know and believe] has merit" (p. 164). Along similar lines, Simon (1993) contends: "A research base with respect to prospective teachers' knowledge is essential if we are to develop instructional interventions that will help prospective teachers extend and modify their knowledge" (p. 233).

In trying to apply Approach 3 to determine what might be the knowledge of reasoning and proof needed for teaching elementary school mathematics, we look at the body of research that has examined elementary teachers' knowledge of reasoning and proof. This body of research has focused primarily on teachers' ability to distinguish between empirical and deductive forms of argument, and has revealed the common misconceptions that empirical arguments can prove (Ball, 1988; Goetting, 1995; Ma, 1999; Martin & Harel, 1989; Morris, 2002; Simon & Blume, 1996; Stylianides, Stylianides, & Philippou, 2002) while deductive arguments are not sufficient to prove (Goetting, 1995; Morris, 2002; support of this claim can also be found in Martin & Harel, 1989). Another common misconception is that a single counterexample does not suffice to disprove a general mathematical statement (Goetting, 1995; Simon & Blume, 1996; Stylianides et al., 2002).

Many of the researchers who examined elementary teachers' understanding of empirical and deductive forms of argument justified their decision to study this topic based on the work that teachers do, or are expected to do, in mathematics classrooms. For example, Morris (2002), whose sample consisted of preservice elementary and middle school teachers, noted:

> This population was of particular interest because there is an expectation that elementary and middle school teachers will assist children and adolescents in "recogniz[ing] and apply[ing] deductive and inductive reasoning" in mathematics (National Council of Teachers of Mathematics, 1989, p. 81). (Morris, 2002, p. 86)

Martin and Harel (1989) offered a similar justification for the importance of preservice elementary teachers having solid understanding of empirical and deductive forms of argument:

> The views of proof held by preservice elementary school teachers are important. Because proof receives very limited attention in the elementary school curriculum, the main source of children's experience with verification and proof is the classroom teacher. Classroom teachers' understanding of what constitutes mathematical proof is important, even though they do not directly teach that topic. If teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students. (pp. 41-42)

The two examples suggest that, from the researchers' point of view, elementary school teachers should be able to understand and distinguish between empirical and deductive forms of argument.

Research on elementary teachers' knowledge of reasoning and proof has typically not focused on teachers' understanding of specific methods of proof. Literature search has revealed only a couple of studies that relate to this topic, one about contraposition equivalence rule and the other about proof by mathematical induction (Stylianides et al., 2003, 2004). However, unlike what happens with the studies that examined elementary teachers' ability to distinguish between empirical and deductive forms of argument, the studies that investigated elementary teachers' understandings of specific methods of proof do not appear to be particularly concerned with issues of mathematical knowledge for teaching. The participation of education majors in these studies was not primarily associated with them being prospective *teachers*. Therefore, it is unclear whether researchers believe that elementary school teachers should have robust understanding of specific methods of proof to successfully cultivate proving among their students. The lack of studies in this domain may simply indicate an area that has not yet attracted research attention, or it may suggest that knowledge of specific methods of proof is not central to teaching elementary mathematics.

Ball et al. (2001), in considering the literature on teachers' mathematical knowledge, remark: "researchers have explored the knowledge that they *thought* would matter in teaching" (p. 444; italics added). Simon (1993), in concluding his paper about prospective elementary teachers' knowledge of division, states: "*If* the kinds of understandings discussed in this paper are important, mathematics courses must …" (p. 252; italics added). The remark by Ball et al., coupled with Simon's conditional statement, highlight a challenge implied by Approach 3: How do we, the community of mathematics education researchers, know that we are in fact investigating those aspects of teachers' mathematical knowledge that are important for teaching? On what grounds do we investigate particular aspects of teachers' knowledge of reasoning and proof and not others?

Approaches 1 Through 3

We have thus far discussed three approaches to studying the mathematical knowledge for teaching. A common theme across these approaches is that they all study, to a certain degree, different groups' perspectives on the mathematical knowledge needed for teaching. Approach 1 does so more directly than the other two, for it turns policy makers' recommendations to its main object of study. Approach 2 draws inferences about curriculum developers' perspectives through the study of teachers' mathematics curricula, while Approach 3 examines what mathematics education researchers consider important through analyzing the aspects of teachers' knowledge they choose to study and the reasons that drive these investigations.

All three approaches can make valuable contributions not just to the field's understanding of what mathematical knowledge is needed for teaching but also more broadly. For example, studies utilizing Approach 2 can also offer the field a sense of the intended curriculum for mathematics courses taken by prospective teachers. Despite their merits the three approaches also suffer some limitations. A criticism of all three of them might relate to the utility of their findings and might be based on the fact that their objects of analysis – policy documents, college mathematics curricula, and researchers' choices of what aspects of teachers' knowledge to study – are distant from classroom instruction, the natural setting wherein teachers' mathematical knowledge functions. The development of an integral understanding of what mathematical knowledge is needed for teaching requires that we also consider some factors that come closer to this setting. Two factors that we need to consider are school mathematics curricula and students' mathematical knowledge. Approaches 4 and 5 each focus on one of these factors.

Approach 4: Analyzing Students' Mathematics Curricula

Romberg (1992) noted that "[t]he actual topics taught in classrooms are those that appear in the textbooks that are used, with an emphasis on those concepts and skills that appear on tests" (p. 764). Elsewhere, Romberg and Webb (1993) asserted that "[m]athematics instruction in American schools is driven by textbooks, particularly in the elementary grades" (p. 153). Assuming that these claims are true, no matter whether what they describe is fortuitous, it seems fruitful to examine school mathematics curriculum materials to stipulate what teachers are supposed to teach in order to inform our understanding of what mathematics is important for teachers to know to successfully implement these materials in their classrooms. Approach 4 follows exactly this path.

Research evidence suggests that preservice elementary and secondary school teachers often lack a fundamental understanding of school mathematics (e.g., Ball, 1990; Cooney, Shealy, & Arvold, 1998; Ma, 1999; Simon, 1993). Approach 4 can make an important contribution by articulating what is that students are supposed to learn and use this to indicate the *minimum* knowledge of mathematics that teachers need to know. As Thompson and Thompson (1996) put it, it seems obvious that "successful teaching requires, *at minimum*, that teachers possess the schemes we hope children will build" (p. 21; italics added). At another level, Approach 4 can be used to construct knowledge packages (Ma, 1999) of fundamental concepts of mathematics, thereby capturing some of the complex networks of relationships among mathematical ideas. If designing instruction that promotes making connections is an important educational goal, then it is essential that teachers understand well the networks of ideas they teach (see, e.g., Lampert, 2001).

In the particular domain of reasoning and proof, studies utilizing Approach 4 can describe the place of reasoning and proof in the curricula and how this is promoted, thus helping us think about the knowledge of reasoning and proof called for by teachers who implement these curricula in their classrooms. Despite its importance, this kind of research has not been popular. One reason for the lack of research in this domain is that, until recently, a tool for examining the place and treatment of reasoning and proof in school mathematics curricula has been unavailable. Stylianides and Silver (2004) propose a multifaceted framework to investigate the opportunities school mathematics curricula design for students to engage in reasoning and proving. This framework is deeply rooted in the mathematical discipline and captures important thinking processes associated with manifold aspects of reasoning and proving: identifying patterns, making conjectures, and providing proofs or non-proof arguments to support or refute mathematical claims. The framework also allows the examination of the purposes proofs, patterns, and conjectures serve in the curricula, as well as the guidance designed for teachers in enacting opportunities related to reasoning and proof in their classrooms. Until some results from analyzing the opportunities designed for elementary students, in the curriculum materials they use, to engage in reasoning and proving become available, Approach 4 cannot provide any insights into the knowledge of reasoning and proof needed for teaching elementary mathematics.

To conclude, Approach 4 attempts to stipulate the mathematical knowledge needed for teaching by analyzing the opportunities designed for students to learn mathematics in the curricula they use. This approach can make important contributions to the problem of what mathematical knowledge is needed for teaching, because how teachers enact these opportunities in their classrooms depends on teachers' understanding of the mathematical issues involved. Nevertheless, the following questions are raised: How fully can one specify the knowledge needed for functioning effectively in the dynamic and evolving situation of teaching through studying static and predetermined curriculum materials? What is the relation between *intended* and *implemented* curriculum (see, e.g., Cohen, 1990; Flanders, 1994)? How might this relation be mediated by classroom-based factors that influence high-level mathematical thinking and reasoning (see, e.g., Henningsen & Stein, 1997; Stein et al., 1998)? How do curriculum developers decide which mathematical ideas to cover in their curricula, and how to promote them?

Approach 5: Analyzing Students' Mathematical Knowledge

Many researchers beginning with Shulman (1986, 1987; Even & Tirosh, 1995; Graeber, 1999) identified the "knowledge about students" as an important component of teachers' knowledge. For example, by having a good sense of how students think teachers are in a good position to anticipate common student difficulties. Approach 5, being an approach to studying the *mathematical* knowledge needed for teaching, views the "knowledge about students" more as a source that informs the mathematical understandings teachers need to have to be able to help students develop their mathematical knowledge and less as a component of teachers' knowledge. More specifically, this approach advocates that the examination of students' mathematical knowledge (e.g., ways of thinking, difficulties) through explorations that take place outside of real classroom settings (e.g., analyses of experimental situations, interview sessions, responses to test items) can offer insights into what mathematics teachers need to know well in order to recognize and address the strengths and weaknesses of students' mathematical understandings.

In order to make fruitful contributions to the particular domain of elementary teachers' knowledge of reasoning and proof, Approach 5 requires a research-based body of knowledge about elementary students' common understandings of proof and ways of reasoning. However, only a limited number of studies have examined elementary students' understandings of reasoning and proof through experimental situations, interviews, or responses to test items (the three methods of inquiry under Approach 5). Two studies can be classified into this category (Lester, 1975; Maher & Martino, 1996), although the second one could also be classified under Approach 6 as it draws on settings both within and outside classroom.

Lester (1975) investigated the development of students' ability to write a correct mathematical proof by examining certain developmental aspects of problem-solving abilities in the context of an experimental mathematical system. Lester's sample consisted of 80 students uniformly distributed in the following four grade groupings: 1-3, 4-6, 7-9, and 10-12. The subjects were asked to supply proofs of theorems of the experimental mathematical system. His findings suggest that students in the upper elementary grades (4-6) are able to solve problems in this experimental mathematical system just as successfully as secondary school students, except that they require more time. Students in the lower elementary grades (1-3) appear to be less successful. According to Lester, the results suggest that "even students in the upper elementary grades can be successful at mathematical activities that are closely related to proof. That is, certain aspects of mathematical proof can be understood by children nine years old or younger." (p. 23)

Maher and Martino (1996) report a longitudinal study about the development of one child's understanding of proof. The study draws on data coming primarily from open-ended task-based interviews and classroom observations. Maher and Martino's analysis shows a progression in the subject's ability to justify and make proofs in combinatorics tasks that involve building all different towers of specific heights by using cubes of different colors. In the early grades, the child used trial-and-error to show that she had built all possible towers. In the upper elementary grades, she was able to extend an earlier argument by cases to an argument by mathematical induction.

The two studies provide existence proof that elementary students (at least in the upper grades) *are* capable of logical thinking and serious engagement in reasoning and proof. Elementary teachers would need to have robust understanding of reasoning and proof to cultivate in their students reasoning resources that would enable them to develop the kinds of arguments reported in the studies. At a more basic level, elementary teachers would need to be able to recognize, for example, that the student argument in Maher and Martino's study is a valid form of advanced mathematics.

For Approach 5 to contribute substantially to the effort of determining what knowledge of reasoning and proof is needed by elementary teachers, research that would sketch a broader picture of elementary students' knowledge of reasoning and proof is sorely needed. In the secondary level, a plethora of studies utilizing Approach 5 provide strong evidence that students have difficulties conceptualizing the notion of counterexample and distinguishing between empirical and deductive forms of argument (e.g., Balacheff, 1988; Chazan, 1993; Galbraith, 1981; Senk, 1985). It is plausible to argue that elementary students will generally have similar (if not more serious) difficulties with reasoning and proof. This argument builds on the premise that the younger the students the more difficulties they are likely to have in grasping ideas that span grade levels. If the argument is valid, then one may conclude that elementary teachers need to know well that empirical arguments are not proofs, that deductive arguments provide conclusive evidence that needs not be supplemented by empirical evidence, and that a single counterexample suffices to disprove a general statement. If elementary teachers have robust knowledge of these ideas, they are expected to be in a good position to help their students develop better understanding of the same ideas, thereby preparing them for secondary school mathematics. Unfortunately, the literature reported under Approach 3 shows that elementary teachers typically lack fundamental understanding of these ideas.

To conclude, investigating students' knowledge in experimental situations, interview sessions, or through their responses to specially designed tests provide opportunities to elicit students' mathematical understandings that are not readily available otherwise. Conditions created in *experimental situations* occur rarely, if ever, in real classroom settings (see, e.g., Balacheff, 1988). Rather than having to find an instance of the phenomenon in which they are interested, researchers create the instance and then study it. By so doing, they portray student conceptions in fine detail. *Interview* settings not only help describe students' understandings but also illuminate students' justifications for holding these understandings (see, e.g., Chazan, 1993). The one-on-one interaction between the interviewee and the expert who often conducts the interview allow for deep probing of students' knowledge. *Test items* administered to large numbers of students can offer a global picture of students' knowledge (see, e.g., Senk, 1985). The body of knowledge developed by these different kinds of studies and methods of inquiry can potentially provide valuable insights into the mathematical knowledge teachers need to have in order to recognize manifold aspects of students' knowledge in real classroom settings and address them appropriately. Yet, the following questions are raised: How well can teachers' mathematical knowledge *for teaching* be informed by knowledge about students that is built outside real classroom settings? How do students' understandings elicited outside classroom practice compare with those displayed in real classroom settings?

Approaches 1 Through 5

We have so far discussed five approaches to studying the mathematical knowledge needed for teaching and considered for each of them both potentials and pitfalls. As we have seen, the limitations of one approach are often addressed by other approaches. The question that arises at this point is: Why do we need a sixth approach? What might be missing from the ones we already have?

To address this question it might helpful to revisit a major criticism of the first three approaches we presented in a previous section (cf. "Approaches 1 Through 3"). The criticism related to the utility of the findings of these approaches and was based on the fact that the objects of analysis of these approaches are not directly connected with classroom instruction, the natural setting wherein teachers' mathematical knowledge functions. Approaches 4 and 5 obviously come a step closer to school mathematics practice than the other three, for their objects of study – school mathematics curricula and students' mathematical knowledge – are parts of instruction. But these approaches are still inadequate in capturing the less exposed mathematical entailments of the work teachers and students do together in mathematics classrooms. For example, it has been argued that "[s]imply looking at the math problem or considering the content on which students are working does not lead to a sufficient appreciation of the specific mathematical knowledge or sensibilities that it takes to teach that problem or that content" (Ball & Bass, 2000a, p. 91).

What seems to be missing from our current list of approaches is one that would study the mathematical knowledge needed for teaching by directly examining the mathematical entailments of the work that teachers do in mathematics classrooms. This is the focus of Approach 6.

Approach 6: Analyzing School Mathematics Practice

Approach 6 begins with the study of records of school mathematics practice (e.g., videotapes of classroom lessons, transcripts, student work, teacher journal) to uncover and describe in fine *mathematical* detail the work in which teachers engage in order to understand the mathematical knowledge needed for effective functioning in the course of this work. Some components of this analysis include the investigation of the demands imposed on teacher's mathematical knowledge as they implement the curriculum, facilitate classroom discussions, listen to students, manage their difficulties, and actively seek to help them overcome these difficulties. Closely related is the analysis of students' mathematical knowledge as it unfolds in the context of classroom instruction.

Although a growing number of studies have provided detailed analyses of school mathematics practice with a relative emphasis on elementary students' activity in reasoning and proving (e.g., Ball & Bass, 2003a; Carpenter et al., 2003; Lampert, 1990, 1992; Maher & Martino, 1996; Reid, 2002; Zack, 1997), only few have presented this work as part of a principal interest "to uncover less exposed mathematical entailments of the work for teachers, seeking useful answers to questions about what teachers need to know and be able to do in order to teach" (Ball & Bass, 2000b, p. 198). Ball and Bass (2000b) studied elementary teaching and learning to understand the mathematical component of the work in which teachers and students are engaged with the long-term goal to develop a practiced-based theory of mathematical knowledge for teaching (see Ball & Bass, 2003b). By undertaking a close anatomy of children's thinking and knowledge construction through a disciplinary lens, they aimed to describe what mathematical reasoning might look like with elementary students and understand what it might take for teachers to systematically develop students' capacity for such reasoning.

The work of Ball and Bass (2000b, 2003a), along with the work of others (e.g., Carpenter et al., 2003; Lampert, 1990, 1992; Maher & Martino, 1996; Reid, 2002; Zack, 1997), have advanced the field's understanding of the elementary children's potential with reasoning and proof in classroom environments that promote mathematics as sense-making activity. Elementary school teachers have an important role to play in developing and managing effectively these classroom environments. To be able to carry out successfully this role, teachers need to have thorough knowledge of reasoning and proof, and be able to draw flexibly on this knowledge in the course of their work. Approach 6, when specific to reasoning and proof, targets to describe exactly this body of knowledge.

Next, we present some classroom episodes in which elementary students are engaged in reasoning by contradiction. Then, we use these episodes, together with the one presented at the outset of the paper, to illustrate how Approach 6 can contribute to the field's understanding of the knowledge of reasoning and proof needed for teaching. The episodes are drawn from research studies that have typically not had the same focus as we do here. Also, the selected episodes should be understood not as a sample of normative practice, but rather as a purposeful sample (Patton, 1990) of instances from classroom environments committed to fostering mathematics as reasoning. We argue that the knowledge of reasoning and proof needed for teaching can better be studied in classroom settings of this kind than in more typical classroom settings. Developing an understanding of the critical components of teachers' mathematical knowledge for creating and supporting classroom environments that support the development of students' reasoning resources can serve as a first step to meeting the challenge of designing means to help teachers develop "forms of classroom mathematics practice that foster mathematics as reasoning and that can be carried out successfully on a large scale" (Yackel & Hanna, 2003, p. 234). We argue further that it is meaningful for the examination to focus on *situations for justification* – situations whose issue is the legitimacy of mathematical activity, the accountability "as it occurs during negotiation of mathematical meanings and practices" (Cobb, Wood, Yackel, et al., 1992, p. 576) – as opposed to situations in which students apply procedures and follow memorized rules. The second type of situations can help researchers speculate about the *greatest lower bound* of the set of understandings of reasoning and proof needed by teachers. This bound is not very useful, though, for it helps set basic requirements rather than standards of excellence. Situations for justification allow the setting of standards of excellence, for they provide a context that supports the development of hypotheses about the *least upper bound* of the set of understandings of reasoning and proof needed for teaching. Setting standards of excellence by defining the *least* upper bound is warranted by the pragmatic constraints imposed on the mathematical preparation of teachers (e.g., limited course hours).

Episodes with elementary students engaged in reasoning by contradiction. Several researchers report episodes that show elementary school children utilizing naturally reasoning by contradiction when they construct mathematical arguments (e.g., Carpenter et al., 2003; Lampert, 1990; Reid, 2002; Sáenz-Ludlow & Walgamuth, 1998; Zack, 1997). Here we present just a couple of episodes – one from the lower and another from the upper elementary grades – to illustrate this point.

In the first episode, reported by Carpenter et al. (2003, pp. 48-49), second graders and their teacher work in the context of true/false number sentences.

Mr. C.: How about this number sentence? [*Writes 58 + 0 = 58 on the board*] Is it true or false?
Children: True. True.

Rather than immediately pressing the children to make a generalization, Mr. C. gives them another number sentence illustrating the same principle, but in this case the number sentence is false.

Mr. C.: How about this one? [*Writes 78 + 49 = 78*]
Children: False! No, no false! No way!
Mr. C.: Why is that false?
Jenny: Because it's the same number as in the beginning, and you already put more with it, so it would have to be higher than the number you started with.

Jenny's argument to refute the false statement 78 + 49 = 78 utilizes reasoning by contradiction.

In the second episode, fifth graders in Vicki Zack's (1997) work to find all squares in different n-by-n squares comprised of a total number of n^2 1-by-1 squares. In a small group discussion, all five students in the group (Will, Lew, Gord, Ross, and Ted) agree that there are 385 squares in a 10-by-10 square. However, they disagree on the number of squares in a 60-by-60 square. Ross and Ted argue that one can find the number of squares in this case by multiplying 385 (the answer for the 10-by-10 case) with 6. Hence, they propose that the answer to the problem is 2310. Will and Lew disagree with this answer and use the reasoning described in the following excerpt to refute Ross and Ted's argument: "[Will and Lew say] that 3600 (the result of multiplying 60 by 60) is already bigger than 2310. Lew adds: 'And that's just the [number of] the little squares.'" (p. 295) This argument shows clear that Ross and Ted's answer results in a contradiction.

Discussion. The classroom episodes presented above, together with the one presented at the beginning of the paper, provide existence proof that elementary students are capable of relatively advanced mathematical argumentation. In particular, the episodes suggest that children's mathematical activity has the potential to transcend the empirical level to reach a level where arguments provide conclusive evidence for or against mathematical claims, utilizing logical principles (such as the principle of contradiction) and reasoning tools (such as mathematical definitions). A conclusion that may be drawn from studying these segments of classroom practice is that teachers who aspire to develop and function effectively in classroom environments where students produce arguments like the ones described above need, at the very least, to be able to understand and appreciate the mathematical value of these arguments. If elementary teachers believe that empirical evidence is proof (as it is often the case; cf. Approach 3), it is plausible to assume that they will not expect from their students to justify statements that involve an infinite number of cases, like "odd + odd = even," using non-empirical methods. Nor will they actively seek to help their students develop reasoning skills, like the appropriate use of mathematical definitions, to successfully deal with proving tasks of this kind. Similarly, if teachers think that reasoning by contradiction is invalid, they will neither make sense of student arguments that are based on this reasoning nor will they actively engage in cultivating this way of thinking in their classrooms.

So reasoning by contradiction seems to be an important mode of reasoning for elementary teachers to know and be able to use. But what exactly about reasoning by contradiction do elementary teachers need to know? For example, do they need to be able to devise proofs by contradiction, such as the standard proof for the irrationality of $\sqrt{2}$? Or is it enough for them to be able to recognize and utilize reasoning by contradiction in verbal contexts (e.g., in the setting of whole group discussions)? The specification of the context can have implications for teacher preparation programs, for research has shown that elementary teachers understand considerably better logical principles embedded in verbal than symbolic contexts (Stylianides et al., 2004).

To conclude, analyzing school mathematics practice with a focus on the mathematical entailments of teaching is a fruitful approach to research in studying the mathematical knowledge needed for teaching. This is primarily because the object of study is examined in its natural setting. However, like all other approaches we discussed so far, Approach 6 has some limitations. The mathematical anatomy of the less exposed entailments of teaching through the study of records of practice, no matter how carefully arranged, can never be as complete as it would be had it been supplemented by other means of inquiry, such as interviews and experiments. For example, the study of a classroom episode where a student develops an argument by contradiction does not often allow the investigation of whether other students in the class have been able to make sense of this argument or whether the student who developed it has a good grasp of reasoning by contradiction. Exploring several students' understanding of contradiction in interview sessions particularly designed for this purpose can shed light on this issue. Analyzing data from interview sessions can also help address other related questions such as the following: What aspects of reasoning by contradiction can students grasp spontaneously and what others cause them difficulties? This analysis can illuminate further what it is about reasoning by contradiction that teachers need to know well. Another issue in studying records of practice is that the social complexity arising from the manifold interactions among members of the classroom community can sometimes make it difficult for researchers to isolate and investigate particular aspects of teachers' mathematical knowledge for teaching. Setting up one-on-one teaching experiments (see, e.g., Thompson & Thompson, 1994, 1996) can help address this issue.

A Framework for Studying the Mathematical Knowledge Needed for Teaching

Each of the six approaches to research for studying the mathematical knowledge for teaching, presented in the previous part of the paper, can produce important understandings about this topic. However, if the overall contribution of the six approaches is to be greater than their sum, it is important to develop a framework that will capture, as much as possible, the relationships among the approaches and that will represent a systematic program of research. According to Silver (1990), “the results of systematic programs of research can develop cumulative results that lend themselves to substantive interpretation and important applications for practice” (p. 2).

In this part of the paper, we propose a framework for studying the mathematical knowledge needed for teaching (see Figure 1). We elaborate the framework and use examples from the particular domain of teachers’ knowledge of reasoning and proof for teaching to illustrate some of its key components. In developing the framework we considered theoretical ideas about the relationship between research and practice as well as observations that emerged from our application of the six approaches to the domain of reasoning and proof. Before we present the framework, we mention a caveat about its development. We do not argue that the framework captures fully the study of mathematical knowledge needed for teaching. Our reasons for developing it should be seen in the spirit of the following remark by Simon (1994):

> The creation of a framework is not an attempt to represent “reality”. Indeed we have no direct access to reality. Rather it is an attempt to create a model which is useful and generative in thinking about the phenomena under study. As such, the framework is necessarily simplified (i.e., omits some of the perceived complexity of the situation that it describes) and emphasizes only some of the aspects of knowledge about the situation. (p. 73)

Let us begin with a description of some general features of the framework. The framework represents the study of the mathematical knowledge for teaching as an *iterative cycle* with school mathematics practice situated right at the heart of this cycle. Teachers’ mathematical knowledge for teaching is shown to operate within the setting of school mathematics practice. The framework suggests that practice may serve as a benchmark against which the utility of research contributions made by approaches 1 through 5 may be explored, and also as a pool from which insights and ideas for further research by these approaches can be drawn. Approach 6, which focuses directly on classroom practice, facilitates both of these potential functions of practice.

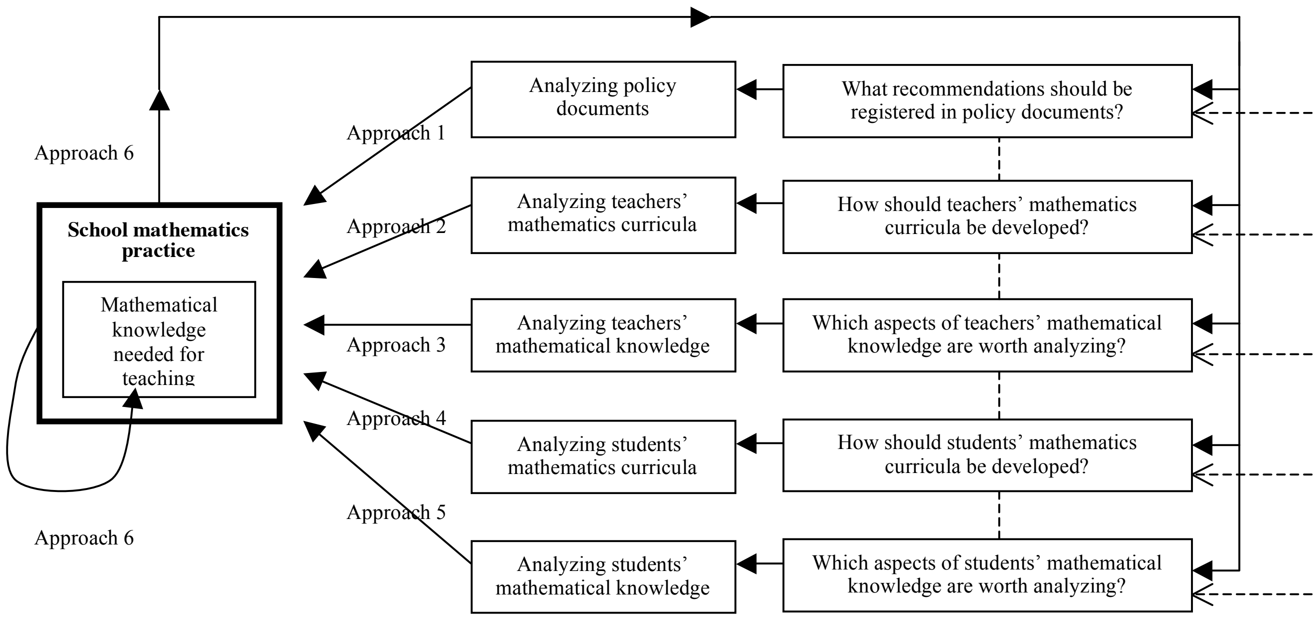

Figure 1. A framework for studying the mathematical knowledge needed for teaching.

The permeation of practice by ideas of research and vice versa, represented by the framework, is compatible with the spirit of Silver's (1990) notion of a bidirectional relationship between research and practice. As he noted,

> [T]he relationship between research and practice is bidirectional. Practitioners will want to consider the multiple ways in which research can positively influence their instructional thinking and actions, but it is equally important for researchers to consider and respond to problems and issues that are raised when practitioners apply research findings, methods, or theoretical perspectives. (p. 9)

The following comment of Hiebert, Gallimore, and Stigler (2002) goes along parallel lines:

> Researchers and teachers could work side-by-side as authentic partners in the new system, each gaining from the others' expertise. Teachers, for example, would use the wealth of their experience to test difficult-to-implement but promising new ideas and then, based on their own and the researchers' observations, new hypotheses could be constructed for future tests. Researchers, in turn, would have greater access to investigational contexts and populations, and gain a rich source of new ideas and hypotheses. They would get ideas from teachers that could be turned into testable hypotheses, much as clinicians make discoveries that are exploited by biomedical scientists to create new generalized knowledge. Rather than being made redundant or obsolete, the work of researchers could become more relevant with a system in place to digest and transform their general findings into professional knowledge for teaching. (p. 13)

The framework is meant to represent the process of studying the mathematical knowledge needed for teaching in a way that resembles the functioning of a *developing system* with the six approaches playing the role of its components. In an *ideal* situation, approaches 1 through 5 would be in coordination and would make compatible contributions to our understanding of mathematical knowledge needed for teaching. For example, Approach 1 would describe policy recommendations that would be completely aligned with both what teachers are taught (Approach 2) and the aspects of teachers' knowledge that mathematics education researchers choose to analyze (Approach 3). In the same ideal situation, Approach 6 would both confirm the utility of these contributions in the terrain of school practice and would have no new insights to offer. Under this hypothetical scenario, the system would be in an optimum state of *equilibrium* and our understanding of teachers' mathematical knowledge needed for teaching complete. However, we are far away from this optimum situation and it is doubtful whether we will ever manage to reach it. Our analysis of the particular domain of teachers' knowledge of reasoning and proof suggests that the system is currently in a state of *disequilibrium*. This disequilibrium can primarily be attributed to insufficiency of the existing body of research at some parts of the system. For example, there I currently limited research in the domain of Approach 4. The gap in our knowledge about what teachers need to know to implement the intended curriculum in their classrooms creates an imbalance in the system because the problem is not informed from the curriculum point of view. This state of disequilibrium in the system can be productive as it can drive efforts toward concordance through the course of research.

The fact that the study of the mathematical knowledge for teaching is represented by the framework as an iterative cycle implicates that the specification of a *starting point* for the process of investigating this topic is irrelevant. One possible way with which the process may begin is by pursuing a position of what might be important for teachers to know as suggested by a recommendation in a policy document, an idea incorporated in curriculum materials created for teachers, an aspect of teachers' knowledge considered essential for the work of teaching by mathematics education researchers, a curricular task that teachers are supposed to teach, or a mathematical topic that students have difficulties with (approaches 1 through 5, respectively). The process may then proceed with close examination of practice in order to explore the utility of this position or to see how it plays out in the actual work of teaching (Approach 6). These different objects – policy recommendations, aspects of teachers' and students' knowledge, and curriculum features – all have the potential to create the momentum that will get the process started. It is important to emphasize that the process does not necessarily require research to get started. It may also begin, for example, with a value-based judgment about what is important for teachers to know. This remark is compatible with the idea that research *cannot* select standards; however, research *can* guide value judgments (see Hiebert, 2003).

Let us illustrate the process described above by drawing on examples from the specific domain of reasoning and proof. Mathematics education researchers consider important that elementary teachers distinguish between empirical and deductive forms of argument. Examination of practice can reveal the extent to and ways in which this understanding is important for teaching elementary mathematics. To offer another example, research on students' understanding of proof suggests that students typically have difficulties realizing that empirical evidence does not establish the validity of a general statement. Examining records of practice has the potential both to sketch a picture of how these kinds of difficulties play out in real classroom settings and to study the demands placed on teachers' knowledge as they manage such student difficulties in the course of their work. Another example is that practice can be used to explore our stipulations of what demands curriculum tasks related to reasoning and proof place on teachers' knowledge as teachers implement these tasks in their classrooms. A different example is that practice can serve as a benchmark against which one can examine the extent to which teaching prospective teachers ideas of logic – as it is presumably the case from looking at some teachers' mathematics curricula – is relevant to their work. A final example is that practice can be used to explore the utility of recommendations registered in policy documents, such as the one set forth long ago by the Cambridge Conference on Teacher Training (1967) regarding the importance of elementary teachers knowing mathematical induction. The recommendation was justified as follows:

> With [mathematical] induction we can prove theorems such as the fundamental theorem of arithmetic, the division algorithm, and the Euclidean algorithm; then we can point out for what kinds of integral domain the proofs hold. But there are other reasons for introducing a prospective K-6 teacher to mathematical induction. A teacher is very likely to be asked two specific questions by children: 'Is there a biggest number?' and 'Is 1/0 equal to infinity?' (p. 64).

The idea of teaching elementary teachers mathematical induction sounds extraneous today; however, this by itself does not provide a reasonable ground for evaluating the recommendation. A close look at school mathematics practice can provide a context for more informed decisions about the extent to which it is useful for elementary teachers to know this specific method of poof.

Thus far we have explained how approaches 1 through 5 can initiate the process of studying the mathematical knowledge for teaching, and also, how their contributions may serve as a point of departure for a practice-based exploration within the domain of Approach 6. The reverse is also possible. One may begin by examining practice (Approach 6) to gain insights into reasonable policy recommendations about what teachers need to know, ways in which teachers' or students' curricula might be developed, or choices of domains of teachers' or students' knowledge worthy of examination. Policy documents, curriculum materials for teachers and students, and aspects of teachers' and students' knowledge that are worth analyzing become, in turn, the objects of study of approaches 1 through 5. In the process just described, practice-based research provides the impetus for rethinking what we recommend for teachers to know, what we promote in curriculum materials, or what aspects of teachers' and students' knowledge we consider important to examine.

We draw again on the particular domain of reasoning and proof to exemplify this process. The examination of records of practice suggests that elementary students can reason by contradiction and that they sometimes use naturally this mode of reasoning to prove or disprove mathematical claims. This observation has the potential to offer new insights into the knowledge of reasoning and proof needed for teaching elementary mathematics, for contradiction has thus far attracted minimal, if any, attention by scholarly work under approaches 1 through 5. We are not aware of any studies that focused on elementary teachers' or students' understanding of contradiction, or of any current policy documents that recommend contraction as an important component of teachers' knowledge for teaching elementary mathematics. Also, we doubt whether this mode of reasoning is treated with any systematic way in the majority of the mathematics curricula for elementary teachers or students. These speculative remarks are not meant to support that reasoning by contradiction deserves more policy and research attention than it has attracted. Rather, they are intended to provide a concrete example of how the study of school practice may bring into (re)consideration recommendations registered in policy documents, tasks included in teachers' and students' curricula, and aspects of teachers' and students' knowledge we regard important to examine.

Let us now turn to some other features of the framework. The *dotted segments* – which connect policy recommendations, development of teachers' and students' curricula, and aspects of teachers' and students' knowledge that are worth analyzing – suggest the existence of relations among these elements. Representing these relations in fine detail lies beyond the scope of this paper, for they are not directly associated with the mathematical knowledge needed for teaching. Generally speaking, it is reasonable to argue for the alignment of policy recommendations, development of curriculum materials, and considerations of what is important knowledge to be examined when concerned with the mathematical knowledge needed for teaching. For example, Sowder, Bezuk, and Sowder (1993) support that teachers' curriculum materials about rational numbers be aligned with recommendations on rational number instruction in schools, because, as they note, "the purpose of the [mathematics content] course is to prepare [pre-service elementary teachers] to become teachers in tomorrow's schools" (p. 256). It seems reasonable to argue further that what teachers are taught in mathematics courses should be compatible not only with the domains of teachers' mathematical knowledge we analyze (when interested in the mathematical knowledge needed for teaching), but also with what teachers are expected to teach and with recommendations registered in curriculum frameworks for school mathematics. The motivation of Morris' (2002) study on teachers' knowledge of reasoning and proof, presented in the discussion of Approach 3, offers an example of the latter. Along similar lines, one may argue that the domains of students' mathematical knowledge researchers analyze should be aligned not only with what students are expected to learn, but also with what is registered in curriculum materials for school mathematics and with recommendations set forth by curriculum frameworks. For example, it is meaningful to anticipate compatibility between the aspects of proof researchers analyze and expect students to know, and the curriculum frameworks through which students learn about proving, for there is evidence that curriculum factors play a role in shaping students' competencies in proof (Healy & Hoyles, 2000).

The *dotted arrows* – which point toward policy documents (and experts' perspectives, more generally), teachers' and students' curricula, and aspects of teachers' and students' knowledge that are worth analyzing – suggest that these elements are influenced by factors other than one another and the study of school mathematics practice. Important such factors include cognitive psychology, the discipline of mathematics, the history and philosophy of mathematics, learning and pedagogical principles, epistemological and sociological convictions. Although in this paper we are not particularly concerned with these influences, we acknowledge their existence and their potential to cause disequilibrium to the system – that is, striving for concordance among the approaches that generates further understanding of the mathematical knowledge needed for teaching.

Let us consider briefly some possible influences of two factors – *cognitive psychology* and *discipline of mathematics* – on the development of mathematics curricula with respect to reasoning and proof. Sowder et al. (1993) note that certain cognitive science principles are expected to guide the structuring of curricula for prospective elementary teachers. A similar argument can be made about the development of school mathematics curricula. In the particular domain of reasoning and proof, it is reasonable to expect that the structure of students' curricula around different modes of reasoning and methods of proof be guided by our understanding of the cognitive development of proof (see Tall, 1999). For example, if cognitive science showed that students as early as elementary or middle school are able to think deductively, then it would make sense for curriculum materials to adjust accordingly based on what it is that students are able to do. Consideration of the discipline of mathematics adds an additional expectation in the treatment of reasoning and proof in school mathematics curricula: making proof a consistent part of these curricula across *all* grade levels. In Hanna's (1995) words, "[a]n informed view of the role of proof in mathematics leads one to the conclusion that proof should be part of any mathematics curriculum that purports to reflect mathematics itself" (p. 42). This expectation finds theoretical support from the idea that school curriculum and students' encounters with disciplinary ideas more generally should be – from the start of students' formal mathematical study – authentic in their relationships to disciplinary structures and scholars' practices. The argument builds on two tenets. First, proof has a pivotal place in the mathematical discipline (see, e.g., Davis & Hersh, 1981; Polya, 1981). Second, school curriculum should be a representation of the structure of the discipline and students should encounter "rudimentary versions" of the subject matter that can be progressively refined through their schooling (Bruner, 1960; Schwab, 1978). Bruner (1960) assumes that "there is a continuity between what a scholar does on the forefront of his discipline and what a child does in approaching it for the first time" (pp. 27-28). Similarly, Schwab (1978) envisions a school curriculum "in which there is, from the start, a representation of the discipline" (p. 269) and in which students have progressively more intensive encounters with the inquiry and ideas of the discipline.

Conclusion

In this paper we have sought to provide insights into how the problem of teachers' mathematical knowledge for teaching can be studied. We have highlighted the need for conceptualizing a systematic program of research for studying this problem, and we have contributed to this goal by proposing a multifaceted framework consisting of six plausible approaches to research for producing understanding about it. The approaches, which roughly correspond to different objects of analysis and methods of inquiry into the mathematical knowledge for teaching, are organized in a network of dynamic relationships that bring together practice, policy, and research. We have exemplified the approaches and have provided evidence of the utility of the framework in organizing research programs related to the mathematical knowledge for teaching by looking closely at the particular domain of elementary teachers' knowledge of reasoning and proof. The framework is offered, though, as a generic tool that can be used in the study of different areas of teachers' mathematical knowledge for teaching.

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