WORKSHOP:
TEACHING MATHEMATICS TO SUPPORT THE
MATHEMATICAL WORK OF TEACHING

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WHAT DO WE KNOW ABOUT “MATHEMATICAL KNOWLEDGE” AND TEACHING?

- Teachers need to understand the content they are teaching.
- But it is a different kind of understanding—and a lot more—than the content knowledge typically studied in college courses.
- This is progress.

- BUT THREE BIG CHALLENGES REMAIN.
PREPARING TEACHERS WHO WILL TEACH MATHEMATICS SKILLFULLY: THREE CHALLENGES

- This specialized kind of knowing and using mathematics for teaching can fall “between the cracks” of content courses and methods courses.
- Requirements continue to center on traditional views of content knowledge.
- We need to keep trying to understand with more nuance what the “mathematical demands of teaching” ARE.
MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT)

Subject Matter Knowledge

Common Content Knowledge (CCK)

Horizon Content Knowledge (HCK)

Specialized Content Knowledge (SCK)

Pedagogical Content Knowledge

Knowledge of Content and Students (KCS)

Knowledge of Content and Teaching (KCT)

Knowledge of Content and Curriculum (KCC)
COMMON CONTENT KNOWLEDGE (CCK)

Calculate:

\[
\frac{5}{6} \div \frac{1}{3}
\]
SPECIALIZED CONTENT KNOWLEDGE (SCK)

\[
\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2 \frac{1}{2}
\]

Is this a fluke?
Does it work in general?
If so, why does it work?
Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$?
Knowledge of Content and Students (KCS)

- What are common errors students make when dividing fractions?
- How do students’ experiences with division of whole numbers support their understanding of division of fractions? How does it confuse them?
- What difficulties do students typically have interpreting the answer to a division of fractions problem?

Knowledge of Content and Teaching (KCT)

- Which representation would you use to introduce the meaning of division of fractions? Or to explain the invert and multiply algorithm?
- What sequence of problems would you use to begin work on division of fractions?
- In a whole-class discussion, what solution methods would you want presented, and in what order?
A student comments that “if you divide by smaller and smaller fractions, the answers get bigger.” Is the student right? Is this mathematically significant or interesting?

Are there mathematically significant notions that underlie division of fractions?
KNOWLEDGE OF CONTENT AND CURRICULUM (KCC)

- At what grade level are students typically taught to divide fractions?
- How is division of fractions related to division of whole numbers in the school curriculum?
- What are the models for fractions and for division with which students would be familiar?
WHAT DO WE KNOW ABOUT “MATHEMATICAL KNOWLEDGE” AND TEACHING?

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- This is progress.

BUT THREE BIG CHALLENGES REMAIN.
A PROBLEM IN TEACHING

You ask your fourth graders to explain what a rectangle is. One child offers a definition:

A rectangle is a flat shape. It has four square corners, and it is closed all the way around.
WHAT IS THE WORK FOR THE TEACHER?

- See that something is missing
- Decide what to do or say
- Offer a counterexample

A rectangle is a flat shape. It has four square corners, and it is closed all the way around.

What is a shape that satisfies this definition and yet is not a rectangle?
WHAT IS THE WORK FOR THE TEACHER?

- See that something is missing
- Decide what to do or say
- Offer a counterexample
- “Is this a rectangle?”

A rectangle is a flat shape. It has four square corners, and it is closed all the way around.

(straight sides)  (exactly four square corners)
A rectangle is a flat shape with **straight sides that are connected at exactly four square corners.** It is closed all the way around.
WHAT IS TEACHING?

Teaching is co-constructed

- . . . in broad socio-political, historical, economic, cultural, community, family environments
- . . . through the interpretations and interactions of teachers, students, and “content”

Cohen, Raudenbush, and Ball (2003)
WHAT IS THE WORK OF TEACHING?

Taking responsibility for deliberately maximizing the quality of these interactions . . .

- . . . in ways that maximize the probability that students learn
- . . . worthwhile content and skills

Cohen, Raudenbush, and Ball (2003)
WHY “WORK” OF TEACHING”?

1. To focus our attention on what teachers DO and to distinguish this from other features of classrooms, such as instructional formats, classroom culture and norms, what students are doing, how the curriculum is designed.

But what about small group work, open-ended problems, homework, etc.? Aren’t those what teachers DO?
WHY “WORK” OF TEACHING”?

1. To focus our attention on what teachers DO and to distinguish this from other features of classrooms, such as instructional formats, classroom culture and norms, what students are doing, how the curriculum is designed

2. To honor the effortful and deliberate nature of teaching and not to leave it invisible, implicit, and taken for granted
WHY “MATHEMATICAL” WORK OF TEACHING”?

Teaching involves special kinds of mathematical work:

1. Solving special kinds of mathematical problems
2. Speaking and listening mathematically in ways that are attuned to students
3. Engaging in specialized mathematical reasoning
WHY “MATHEMATICAL” WORK OF TEACHING”?

- To look at how mathematical listening, speaking, interacting, acting, fluency, and doing are part of the work of teaching, not just resources for it.

- This is what I mean by “the special mathematical work of teaching.”
IDENTIFYING THE “MATHEMATICAL WORK OF TEACHING”
What is the mathematical work of teaching in these three minutes?
What is the mathematical work of teaching in these three minutes?

What number does the orange arrow point to? \[ \frac{1}{3} \]

Explain how you know: *Because it's in 3 parts*
What is the mathematical work of teaching in these three minutes?

What number does the orange arrow point to?

Explain how you know: because there are equal parts and you are pointing to the second one so it's $\frac{2}{4}$. 
What is the mathematical work of teaching in these three minutes?

What number does the orange arrow point to?  \( \frac{1}{2} \)

Explain how you know: Because if you look at it and count.
What mathematical work of teaching can you identify?
VIDEO: WHAT DO ANIYAH AND TONI KNOW AND WHAT CAN THEY DO?
WHAT DO MANY “HEAR” IN ANIYAH AND TONI?

ANIYAH
- She has the wrong answer: 1/7

TONI
- She is playing with her hair and trying to get attention
- She is trying to embarrass Aniyah
WHAT DO ANIYAH AND TONI KNOW AND WHAT CAN EACH DO?

ANIYAH

- Uses the definition for a fraction to explain
  - She identifies the “whole”
  - She makes sure the intervals are equal
  - She counts intervals and not tick marks
  - She knows how to write “one-seventh”
- Produces a mathematically well-structured explanation
- Presents her ideas clearly

TONI

- Listens closely to a classmate’s presentation
- Uses the definition for a fraction to ask
  - How Aniyah decided on 7 parts
- Asks a pointed mathematical question
TEACHING INVOLVES SPECIAL KINDS OF MATHEMATICAL WORK

1. Solving special kinds of mathematical problems
2. Speaking and listening mathematically in ways that are attuned to students
3. Engaging in specialized mathematical reasoning
1. SOLVING SPECIAL KINDS OF MATHEMATICAL PROBLEMS

- Selecting or constructing a strategic example, representation, or task
- Analyzing representations, definitions, non-standard correct responses, incorrect responses
- Developing and using appropriate language
2. SPEAKING AND LISTENING MATHEMATICALLY IN WAYS THAT ARE ATTUNED TO STUDENTS

- Using language precisely
- Considering what students know and understand
- Watching and attending while talking
- Noticing, seeking to reduce, and mediating disconnects that interfere with understanding and communication
REPRESENTING MATHEMATICAL IDEAS: SUBTRACTION
WARM UP

- Calculate the answer using the method you were taught growing up
- Identify other methods that students might use to solve
- Identify some common and predictable errors students make when working on multi-digit subtraction tasks

\[
\begin{array}{c}
916 \\
- 478 \\
\hline
438
\end{array}
\]
OVERVIEW OF CLASS

- Representing & explaining mathematical ideas: Subtraction
  - Introduction to a new manipulative
  - Meanings of subtraction
  - Modeling the meanings of subtraction

- Leading a whole class discussion about mathematics:
  - Debrief Discussion #2
  - Considering tasks for discussions
BEAN STICKS

- Loose beans
- Ten sticks
- Hundred rafts
USE BEAN STICKS TO SOLVE THE FOLLOWING PROBLEM:

Joshua ate 16 peas on Monday and 32 peas on Tuesday.

How many more peas did he eat on Tuesday than he did on Monday?
CONTEXT FOR THE VIDEO

- Third grade class
- Early in unit on addition and subtraction computation
- Have been using bean sticks, number line
- Students have already learned conventional algorithm, but few or none know it well yet
- Students have already worked on problem; the clip is during the whole-class discussion of solutions
FOCUS QUESTIONS

- How does Bernadette use bean sticks to solve the problem? How is Bernadette’s method similar/different to the method/s you used?
- How do bean sticks represent the core ideas of place value?
- How do bean sticks compare with bundling sticks?
BUNDLING STICKS VS. BEANSTICKS

BUNDLING STICKS

- Grouping model
- Language of “bundling” and “unbundling” loose sticks and bundles
- Good up through 100s
- Awkward for overhead or board
- Easy to manipulate
- Can be represented with drawings
- Cheap, easy to maintain

BEANSTICKS

- Trading model
- Language of “trading” loose beans for ten-sticks
- Good up to 100s
- Easy for overhead
- Can be represented with drawings
- Small to manipulate
- Cheap, but take work to make and maintain
TWO MEANINGS FOR SUBTRACTION

- Take-away
- Comparison
VIDEO: TWO MEANINGS FOR SUBTRACTION
3. ENGAGING IN SPECIALIZED MATHEMATICAL REASONING

- Comparing mathematical alternatives
- Justifying choices of representations, language, tasks
- Reasoning about starting points and sequences
POSSIBLE PROBLEM:
I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amounts of money might I have?
How does the exact wording of the problem affect the mathematical work for my students? Which best fits my goals for them?
1. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amount of money might I have?

2. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many combinations are possible?

3. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many different amounts of money are possible? Prove that you have found all the amounts that are possible.
USING MATHEMATICAL LANGUAGE PRECISELY BUT ACCESSIBLY

- Coordinating between mathematical details and precision and terms children know or can learn to use
- Coordinating between simple, invented, or everyday language and mathematical details
- Judging what can be left more casual and what cannot
A polygon is a simple closed plane curve composed of finitely many straight line segments.
USING LANGUAGE PRECISELY & ACCESSIBLY #2

a) An even number is a number that can be divided into two equal parts.
b) An even number is any multiple of 2.
c) An even number is any integer multiple of 2.
d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
e) A whole number is even if it is the sum of a whole number with itself.
Are these definitions okay? What are the consequences of any of these?
WHAT MAKES THIS SPECIAL MATHEMATICAL WORK?

1. It is performed in the service of helping others learn mathematics.

2. Its warrants are tied both to pedagogical purpose and mathematical integrity.
HOW CAN TEACHERS LEARN TO DO THE SPECIALIZED MATHEMATICAL WORK OF TEACHING?

- Unpacking mathematics ideas and practices
- Developing horizon knowledge
- Practicing the mathematical work of teaching
DEVELOPING HORIZON KNOWLEDGE

Exploring ideas of density of the rational numbers

Developing awareness of the two kinds of infinity

All the numbers of K-8 mathematics “live” on the number line
PRACTICING THE MATHEMATICAL WORK OF TEACHING

- Working on examples like we have done, including practicing providing reasons
- Developing fluency and speed
- Developing sensibilities about mathematical language
PRACTICE #1: REPRESENTING AND MAPPING ACROSS REPRESENTATIONS

A

\[
\frac{35}{25} = 125
\]

\[+75 \]

\[
\frac{175}{875}
\]

B

\[
\frac{35}{25} = 175
\]

\[+700 \]

\[
\frac{875}{875}
\]

C

\[
\frac{35}{25} = 25
\]

\[100\]

\[+600 \]

\[
\frac{150}{875}
\]
What problem would you pose next to this pupil, and why?
PRACTICE #3: DEVELOPING SENSIBILITIES ABOUT MATHEMATICAL LANGUAGE

- Use of quantifying terms: e.g., exactly, no more than, no less than, at least, at most
- Attention to the wording of mathematical tasks
- Care with definitions, their role, their requirements, and judgments about when, where, and why they are essential
- Awareness of the overlaps and conflicts between mathematical language and everyday language, other school language, and students’ home language
CONCLUSION

1. Teaching involves doing mathematics in some special ways that are connected to the purposes of teaching practice—i.e., helping others learn mathematics.

2. This special kind of work involves special kinds of problems, engaging in special kinds of mathematical reasoning, and speaking and listening in ways that are attuned to students.

3. Teachers need opportunities to develop skills with this special kind of mathematical work.
CREDITS

Graphic on slides 4 and 11: