

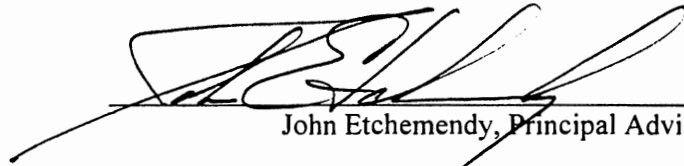
MEDIA AND MEANING:
A SCHEMATIC APPROACH TO REPRESENTATIONAL SEMANTICS
AND ITS APPLICATIONS

A DISSERTATION
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Alan Bush
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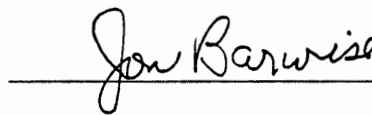
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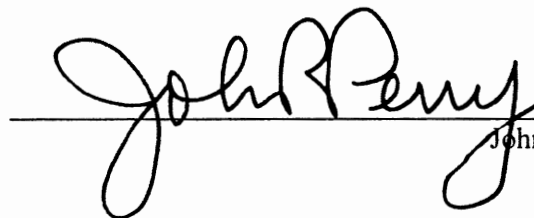
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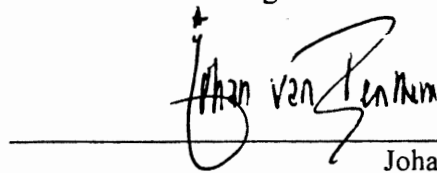
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Abstract

This dissertation presents a general framework for understanding, applying, comparing, and creating techniques for constructing representational semantics. The work begins with John Etchemendy’s positive account of how model-theoretic semantics illuminates the consequence relation for specific languages. Etchemendy uses the term “representational semantics” to identify the approach to logical consequence he advocates. We abstract away from the particulars of Etchemendy’s account, and construct a general conceptual framework we call the “representational schema.” The representational schema gives a general form for techniques used to construct theories of logical consequence implementing representational semantics.

We show how the representational schema is capable of subsuming not just model-theoretic semantics, but also a class of techniques for constructing theories of logical consequence whose central concept can be abstracted from a corollary to Lindenbaum’s Lemma; a class we call order-consistency semantics. We use the schema to describe a general methodology for applying representational techniques to construct theories of logical consequence for particular interpreted languages. That methodology is used to apply (or outline the application of) techniques subsumed by the representational schema to propositional logic, feature logics (sentential languages with feature structures as models), and languages in which feature structures are considered as assertions in their own right. We show how the schema helps us to compare and contrast differing techniques of representational semantics across a number of important dimensions including the mode by which they explain the consequence relation; the range of interpreted languages to which they are applicable; their degree of epistemological commitment; and the ease with which they can be used in particular applications. We show how the representational schema serves as a basis for various vectors of extension, and use the schema to construct several new techniques of

representational semantics. One reduces the epistemological commitments of the specific model-theoretic technique described by Etchemendy, another gives a model-theoretic technique capable of handling partial models, and a third is an order-consistency technique which does not require the assumption of Lindenbaum's Lemma (and thus does not require the assumption of maximal extensions). Applications and benefits of these extensions are described.

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Part I

Introduction

Chapter 1

Representational Semantics and the Representational Schema

1.1 Introduction

“What is logic?” Any language, regardless of its expressive devices, gives rise to a consequence relation, a relation that supports inferences from sentences in the language to other sentences in the language. The study of this relation is the study of the logic of that language” (Etchemendy 1999, 21).

A central aim of logic is the development of theoretical perspectives on the phenomenon of consequence. Carrying out this project includes proposing an answer to the question of what logical consequence is, and giving methods for constructing models of logical consequence for specific languages. The methodology of model-theoretic semantics developed by Alfred Tarski is the generally accepted technique for constructing mathematical models of logical consequence. As argued by John Etchemendy, Tarski interpreted his construction of model-theoretic semantics as giving an analysis of the logical consequence relation, reducing the concept of logical consequence to the simpler concepts of generalization and satisfaction (Etchemendy 1990; Tarski 1956). Etchemendy emphatically praises the power, importance and value of model-theoretic semantics. But at the same time, he demonstrates that Tarski’s reductive interpretation of model-theoretic semantics is seriously flawed, failing

both conceptually and extensionally.

Etchemendy points us in the direction of a different way of thinking about the relationship of model-theoretic semantics to the concept of logical consequence and the task of modelling consequence for specific languages. He shows us how to see model-theoretic semantics as a technique for illuminating the consequence relation for specific languages of assertion; provided that we begin by making certain basic assumptions about what consequence is. On Etchemendy's view, model-theoretic semantics does not reduce logical consequence to more basic concepts. Instead, the model-theoretic approach relies on a prior understanding of logical consequence in general as a way of enabling the construction of models of logical consequence for particular languages. This insight and general approach are at the core of the work in this dissertation.

Etchemendy uses the term “**representational semantics**” to identify the approach to logical consequence he advocates. We are going to take Etchemendy's account of how the technique of model-theoretic semantics implements a representational semantics as the starting point of our investigations. We will abstract away from the particulars of that account, and construct a general conceptual framework we call the “**representational schema**.” The representational schema gives a general form for techniques used to construct theories of logical consequence implementing representational semantics. We use the term “schema” in the sense of a patterned arrangement of constituents within a specified system. In the course of this dissertation, we will use this schema to help us understand, apply, compare, and create a variety of techniques for constructing representational semantics.

There are two natural tests for any proposed general schema. First, does it capture additional instances beyond the one from which it was abstracted? In this case, additional instances would be other techniques for constructing representational theories of consequence beyond the specific model-theoretic approach described by Etchemendy. Secondly, is it productive? That is to say, does the schema increase our understanding of existing instances, and does it enable the development of new ones?

In the pages below, we will show how the representational schema is capable of subsuming not just model-theoretic semantics, but also a class of techniques for constructing theories of logical consequence whose central concept can be abstracted from a corollary to

Lindenbaum’s Lemma; a class we call order-consistency semantics. We will use the schema to describe a general methodology for applying representational techniques to construct theories of logical consequence for arbitrary (partially or fully) interpreted languages. That methodology will be used to apply (or outline the application of) techniques subsumed by the representational schema to a number of interpreted languages, including propositional logic, feature logics (sentential languages with feature structures as models), and languages in which feature structures are considered as assertions in their own right. We will show how the schema helps us to compare and contrast differing techniques of representational semantics across a number of important dimensions: including the mode by which they explain the consequence relation; the range of interpreted languages to which they are applicable; their degree of epistemological commitment; and the ease with which they can be used in particular applications. We will further see how the representational schema serves as a base point for various vectors of extension, and use the schema to construct several new techniques of representational semantics. One will reduce the epistemological commitments of the specific model-theoretic technique described by Etchemendy, another will give a model-theoretic technique capable of handling partial models, and a third will be an order-consistency technique which does not require the assumption of Lindenbaum’s Lemma (and thus does not require the assumption of maximal extensions). Particular applications and potential benefits of these extensions will be identified and described.

1.2 Overview

In the remainder of this chapter, we make an introductory overview of the central concepts and results of the dissertation. We will start out by reviewing Etchemendy’s positive account of model-theoretic semantics as a representational semantics. This discussion will enable us to get a grip on what Etchemendy means by a “representational semantics” and his claims as to what the goals of any specific representational semantics should be. We will also be able to see how Etchemendy interprets the elements of a model-theoretic semantics as achieving those goals. With this understanding in hand, we can then proceed to abstract away from Etchemendy’s account, and formulate the representational schema, a general

form for techniques used to construct theories of consequence which meet the goals of a representational semantics. That schema is the central concept of the dissertation. The remainder of the dissertation is engaged with elaborating and applying that concept.

Our next step in the overview will be to describe the general methodology of applying an instance of the representational schema, using the specific model-theoretic technique described by Etchemendy as an example. The meanings of the elements of the representational schema should be clear by that point.

We will then move on to make the category of order-consistency semantic techniques more tangible. We will present only the most central concepts as a part of the overview, and delay a detailed presentation until Chapter 4. Once the high-level description of order-consistency techniques is given, we will have presented the two main families of representational semantic techniques under consideration: the model-theoretic family and the order-consistency family. The reader should then have a sense of the scope of techniques covered by the representational schema.

The landscape to be surveyed in hand, we will continue with a presentation of various features of that landscape which are of interest, that is, the ways in which the varied specific techniques of representational semantics can be compared with one another. These ways have already been mentioned above, to wit, the mode by which they explain the consequence relation; the range of interpreted languages to which they are applicable; their degree of epistemological commitment; and the ease with which they can be applied in particular cases. We will take the time to expand on what each of these dimensions of comparability mean; and how we intend to carry out the comparisons.

The last material stage in the overview will highlight the various new representational techniques which we have constructed, and their intended applications. We will conclude the overview by giving the structure of the chapters in which all the topics identified above will be discussed.

1.3 Etchemendy's Account of the Relation of Model-Theoretic Semantics to the Concept of Logical Consequence

1.3.1 Etchemendy's presentation of two perspectives on logical consequence

Etchemendy presents two contrasting perspectives on the concept of logical consequence: one which he calls interpretational, and attributes to Tarski; the other which he calls representational, and to which he is committed. Both perspectives can be understood as variants on a common theme; that the relation of logical consequence can be described as follows: A sentence φ is a logical consequence of a set of sentences Σ just in case φ must be true whenever all the sentences in Σ are true. There are two significant concepts left unspecified in that schematic description: the modality which is indicated by “must” and the type of interpretation via which sentences are interpreted as true or false in a particular situation. By specifying different definitions for these concepts we get the interpretational and representational accounts of logical consequence.

The interpretational perspective depends upon a distinction among the terms of a language between logical and non-logical (a distinction whose validity Etchemendy challenges). On the interpretational perspective, interpretations assign objects or properties in the actual world to non-logical terms. The truth value of a sentence is understood as being compositionally derived from these assignments. We can express the interpretational semantics understanding of logical consequence as follows:

Given the way the world actually is, sentence φ is a consequence of a set of sentences Σ just in case
for all the possible interpretations I of the non-logical terms of the language,
if all of the sentences in Σ are true in the actual world when their non-logical terms are interpreted via I ,
then φ is true in the actual world when its non-logical terms are interpreted via I .

The representational perspective does not involve a distinction between logical and non-

logical terms. On the representational perspective, interpretations apply to sentences as a whole. For purposes of understanding the essence of the consequence relation, the role an interpretation plays in determining truth is not analyzed further. There is an assumption that the user of the language has some way by which they evaluate whether sentences are true or false in a given situation, but how that way works is not essential to the representational account of the concept of logical consequence. That unanalyzed mode of evaluation (which we will call a “practice for the categorization of possibility” and discuss in more detail shortly) is what counts as an interpretation on the representational account. Given this background, we can express the representational semantics understanding of logical consequence as follows:

Given a fixed interpretation I of the sentences of the language, sentence φ is a consequence of a set of sentences Σ just in case
for all the possible ways W that the world could be,

if all of the sentences in Σ are true when interpreted with respect to W via I ,
then φ is true when interpreted with respect to W via I .

We see that the key difference between the two approaches is that the interpretational perspective fixes the world, and varies the interpretation of the non-logical terms in the language; whereas the representational perspective fixes the interpretation of the sentences of the language, and varies the possible ways in which the world could be. See Etchemendy 1990 (ch. 2,3,4) for a fuller discussion of the distinction between interpretational and representational semantics.

Etchemendy’s detailed arguments show that the interpretational account is neither conceptually nor extensionally adequate as a concept of logical consequence. For the remainder of the dissertation, we shall consider only the representational view of logical consequence. To understand Etchemendy’s positive account of how the technique of model-theoretic semantics implements a representational semantics, we need to first look at the essential form of that technique.

1.3.2 The essential form of the model-theoretic technique

At the core of the Etchemendy's account is a presentation of the essential form of a model-theoretic construction of consequence. In carrying out such a construction, we are given a set of sentences for which we are to define a consequence relation. The given sentences are formal sentences. They may either be a formalization of the sentences of a natural language, or a formal language constructed for use “as is.” These sentences are assumed to have particular interpretations in advance of the construction. The goal of our construction will be a relation between sets of sentences (considered as premises) and single sentences (considered as conclusions), with the idea that the relation holds just in case the conclusion is a logical consequence of the premises. This relation is defined indirectly. We specify two variable components (in addition to the sentences) and then the model-theoretic framework defines the relation for us. One variable component we specify is a class of set-theoretic structures known alternatively as models, structures, or interpretations. (In this paper we will refer to them as “**models**.”) The other component is a relation between models and sentences called “**truth in a model**” or “**truth in a structure**” and is usually written $m \models \sigma$ for model m and sentence σ . The relation of truth-in-a-model is typically defined using Tarski's semantic techniques (key of which is the concept of satisfaction), though those techniques are not a necessary part of the model-theoretic framework at this level of generality.

Given a set of sentences, class of models, and specified relation of truth-in-a-model, the model-theoretic framework defines a relation and a predicate for us. The framework-defined relation (call it MC , for “**Model-theoretic Consequence**”) is defined on sets of sentences Σ and single sentences φ , such that $MC(\Sigma, \varphi)$ if and only if

$$\text{for every model } m \quad (\text{if } (\text{for every sentence } \sigma \in \Sigma, m \models \sigma) \text{ then } m \models \varphi)$$

Note that we define the relation MC using the defined relation of truth-in-a-model (i.e. $\models$), a relation defined within the model-theoretic framework. That is to say, the relation MC does not essentially involve anything beyond the sentences and the models. The framework-

defined predicate (call it MT , for “**Model-theoretic logical Truth**”) is defined on single sentences φ , such that $MT(\varphi)$ iff for every model m , $m \models \varphi$. Note that $MT(\varphi)$ is equivalent to $MC(\emptyset, \varphi)$.

[Terminological note: sometimes we will say that “ φ is true-in-a-model with respect to m ” for “ $m \models \varphi$ ”. We may also simply say: “ m is a model of φ .” We will use the more verbose style when we wish to emphasize the role of the relation of truth-in-a-model. We will not use the common expression “ φ is true in model m ,” as it could lead to confusion between the concepts of truth and truth-in-a-model.]

A concept which can help unpack the definition of MC is the idea of “**preservation.**” Preservation is a property we apply to arguments (a set of premises and a conclusion) considered relative to an evaluative function (e.g. truth, or truth-in-a-model) and a context of evaluation (e.g. a possibility or model). Some particular argument preserves truth in a possibility if and only if some premise is false or the conclusion is true in that possibility. Similarly, an argument preserves truth-in-a-model in a model m if and only if some premise is not true-in-a-model with respect to m , or the conclusion is true-in-a-model with respect to m . An argument (as a set of premises, conclusion pair) is in the relation MC if and only if that argument preserves truth-in-a-model across all models.

1.3.3 The critical question

At this point in our presentation, the model-theoretic construction is just a mathematical framework. We had a set of interpreted sentences, to which we added a class of models and a specification of the relation of truth-in-a-model. Together, those three components defined for us the relation MC and the predicate MT . We would like to treat MC as if it were equivalent to the representational conception of logical consequence, and MT as if it were equivalent to the representational conception of logical truth. But as it stands, we have not made any claims about how the sentences and models relate to the possibilities they represent. This raises the critical question: What assumptions can we make about the sentences and models and their relations to the possibilities they represent, such that if those assumptions held true, the defined relation MC would be equivalent to the representational conception of logical consequence and the defined predicate MT would be

equivalent to the representational conception of logical truth? Two notes. First, note that Etchemendy considers this as a single question. We will see that we have cause to separate these considerations, and look at the assumptions sufficient to ensure the equivalence of MC and logical consequence independently from the assumptions sufficient to ensure the equivalence of MT and logical truth. Second, notice that since the question is framed in terms of sufficient conditions, there can be many possible responses. Different responses (that is to say different sets of proposed assumptions) will be of varying value. This will be clearer once we consider Etchemendy's characterization of the model-theoretic response, and see the ways in which we can frame its value. As the case of a set of assumptions with least value, consider the assumption: MC is equivalent to logical consequence and MT is equivalent to logical truth. This assumption, while sufficient, is of no value at all!

1.3.4 Etchemendy's characterization of the model-theoretic assumptions

Before we consider Etchemendy's explicitly stated assumptions, we need to make his background assumptions clear. It is a basic assumption of Etchemendy's representational characterization of model-theoretic semantics that sentences bear propositional content, that is, that they can be true or false in different circumstances. So a foundational assumption is that of a space of possibility, canvassing the different ways which things could be. The understanding is that sentences serve to categorize that space of possibility into those possibilities in which the sentence is true, and those possibilities in which the sentence is false. This is what we mean by saying that the sentences have interpretations in advance of applying the model-theoretic construction.

We should note that the assumption of a space of possibility is a conceptual, not an ontological assumption. The assumption does not require that the possibilities be assumed to exist. The only real ontological commitment is to a practice of categorization of possibility; that is, we assume a practice of categorizing possible circumstances as coming or not coming under a certain description. We assume the existence of the practice, but do not need to assume the existence of all the possible circumstances which could be categorized. The practice of categorization constitutes the antecedent interpretation of the sentences.

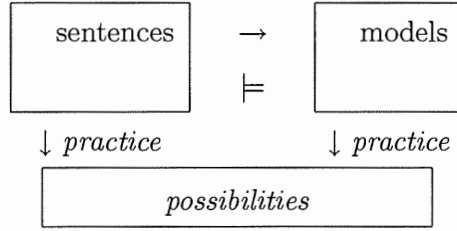
Given this background, the representational perspective we have *chosen* gives us a char-

acterization of logical consequence between a set of sentences (as premises) and a single sentence (as conclusion). By taking the representational perspective, we have chosen to see logical consequence as preservation of truth across all possibilities. That is, for each possible circumstance, if the premises are all true in that possibility (when they are interpreted with respect to the assumed practice of categorization), then the conclusion is true in that possibility (when it is interpreted with respect to the same practice). Further, by taking the representational perspective, we have *chosen* to see the logical truth of a sentence as truth in all possible circumstances (when it is interpreted with respect to the given practice). It is important to stress that these views of logical consequence and logical truth are part of our assumptions in applying the model-theoretic technique, and not results of it. Given that we have chosen to see logical consequence and logical truth in this way, and given that the other assumptions we describe in this section hold, then the relation MC will be equivalent to logical consequence, and the predicate MT equivalent to logical truth.

By the assumptions we have described so far, we have identified what is we mean by logical consequence between premises and conclusion, and what we mean by the logical truth of a sentence. Now we add further assumptions so that the relation MC will be equivalent to our assumed account of logical consequence, and the predicate MT will be equivalent to our assumed account of logical truth.

The key to formulating these assumptions is Etchemendy's proposal that we see models as full-fledged representations, representative in the same way that sentences are. So we further assume the existence of a practice of categorization of possible circumstances by models, by which, given a possible circumstance and a model, the practice says whether or not the model is an accurate description of that possibility. Given a model, the practice divides possible circumstances into two categories: those for which the model is an accurate description, and those for which the model is not so. Again, the only ontological commitment is to the existence of a practice, not to the existence of all possible circumstances in which the practice could be applied.

The following picture shows the different elements under consideration, and their relationships.



Understanding models as representations sets the stage for two additional assumptions. The first is that the models partition the space of relevant possibility. Etchemendy: “if we’ve designed our semantics right, the models impose an exhaustive partition on the possible circumstances that could influence the truth of our sentences” (Etchemendy 1999, 25). This requirement breaks down into three subordinate requirements:

(E1a) “Any individual model represents a logically possible configuration of the world.”

(E1b) “Any two (non-isomorphic) models are logically incompatible; at most one can be accurate.”

(E1c) “Jointly, (the models) are meant to represent all of the possibilities relevant to the truth values of sentences in the language.”

As a result of this compound assumption, the class of models has a very simple (the simplest) form of logical structure. “No model is logically true or logically false, no model follows logically from another, and so forth” (p. 33). Etchemendy describes the logical structure of the class of models as “transparent” since there are no non-trivial consequence relations between representations in the system. This property grounds what Etchemendy takes to be the primary contribution of model-theoretic semantics to illuminating the relation of consequence between sentences (more on this below).

Etchemendy introduces a second assumption based on his view of models as representations.

(E2) “The relation of truth-in-a-model ... satisfies the following constraint: a sentence σ should be true-in-a-model (with respect to) m if and only if σ would be true if the world were as depicted by m , that is, if m were an accurate model” (p. 25).

One way of understanding this assumption is to see it as assuring that the defined relation of truth-in-a-model correctly and completely captures the relation between the categorization of possibility imposed by the sentences and the categorization of possibility imposed by the models.

Given the assumptions described above, Etchemendy claims that “it is a trivial consequence that sentences which are true(-in-a-model) in every model are logically true, and arguments that preserve truth(-in-a-model) in every model are logically valid” (p. 25). In the terms we have used, the assumptions are sufficient to show that *MC* is equivalent to logical consequence, and *MT* is equivalent to logical truth. We will see in the formalization in Chapter 2 that while the proof of this result is not complicated, correct formulation of the assumptions requires and repays careful consideration.

1.3.5 Summarizing Etchemendy’s account

Reviewing the discussion above, we can see what, on Etchemendy’s view, model theoretic semantics does and does not do. On Etchemendy’s account, what model-theoretic semantics does is explain the relation of consequence between sentences in terms of a logically simpler form of representation: the models. This is the primary source of value of the set of assumptions presented above. The assumptions ensure that the models are in fact logically simple, and that they are capable of providing an explanatory ground for the consequence relation between sentences.

Going beyond what Etchemendy says explicitly, but not beyond what is implicit in his formulation, we can say that the explanation of consequence provided by a model-theoretic semantics makes the consequence relation for a given language “intelligible.” What do we mean by the intelligibility of consequence? Let us look into this further.

As we discussed above, the concept of possibility we are using to explicate our notion of logical consequence does not require that possibilities be real in any way accessible to our senses. If we are not able to look at a sentence and see its possibilities; nor compare those possibilities with the possibilities of another sentence; then consequence, though assumed to exist between our sentences, is inaccessible to our senses. We cannot perceive the relation of consequence. On this understanding, model-theoretic semantics can be seen as playing

a very important role: that of making the consequence relation accessible to our thought, or in a word, intelligible. Model-theoretic semantics enables us to visualize consequence in terms we can understand and survey: our sentences and our models, our definition of truth-in-a-model, and the resulting relation of preservation of truth-in-a-model across all models. Given that our assumptions about the relation of a model-theoretic construction to the world hold true, the intelligible relation of preservation of truth-in-a-model across all models is equivalent to the unintelligible (but assumed to exist) relation of consequence (i.e. preservation of truth across all possibilities). This is what we mean by saying that the model-theoretic semantic technique on the representational interpretation “makes consequence intelligible.”

What a model-theoretic semantics does not do is explain what consequence is. An understanding of consequence is part of the assumptions made in order to interpret the relation MC (preservation of truth-in-a-model across all models) as logical consequence. Similarly, an understanding of logical truth is part of the assumptions made in order to interpret the predicate MT (true-in-a-model for all models) as logical truth. On Etchemendy’s representational view, if we are to take our model-theoretic semantics as giving us models of logical consequence and logical truth for particular languages, then we have to have a prior understanding of what we take logical consequence and logical truth in general to be.

1.3.6 Requirements for implementing a representational semantics

Reviewing the discussion presented above, we can see Etchemendy’s (implicit and explicit) understanding of what it takes for a semantic technique to implement a representational semantics. First, the technique must serve to make the representational conception of logical consequence intelligible. But “making consequence intelligible” is not a sufficient condition for the implementation of a representational semantics. The technique must also explain the consequence relation. In the case of model-theoretic semantics, the technique explains the logically complex relation of consequence between sets of sentences and single sentences in terms of a form of representation which is logically simple.

The requirement of an explanation is important because the technique is supposed to support the construction of theories of consequence. An essential property of a theory is

that it somehow simplifies an account from a bare statement of facts observed or expected. Yes, a theoretical account must get the facts right (and that is what intelligibility condition ensures) but to be a theory, the account must also provide some kind of explanation. Part of the reason for this is that the way in which a theory is explanatory supports how that theory is used. We see this in the model-theoretic case. The explanation provided by a model-theoretic account enables us to express the relation of logical consequence for a particular language by a description of a set of models, and a definition of truth-in-a-model. The constructed theory is easier to express than the relation itself, because of the explanatory strategy of the model-theoretic semantic technique.

1.4 Abstracting from Etchemendy's account: The representational schema

Now we are ready to abstract away from Etchemendy's account of how model-theoretic semantics implements a representational semantics. The result of the abstraction will be the representational schema, a general form for techniques used to construct theories of logical consequence implementing a representational semantics. We have seen in the last section that there are two requirements for such techniques. First, the technique must make the representationally understood relation of logical consequence intelligible, and second, the results of the technique must be explanatory of that relation. Let's address these requirements one at a time.

1.4.1 Intelligibility of consequence

First, consider the requirement of intelligibility. We can describe the elements of the model-theoretic technique contributing to the intelligibility of consequence in the following way.

1. The technique is grounded in a foundational characterization of the space of possibility and practices of categorization relating sentences to possibilities; in terms of which the assumed representational concept of logical consequence on arguments is defined. Arguments are modelled as premise/conclusion pairs, where the premise is a set of sentences, and the conclusion a single sentence.

2. The technique supplements the set of sentences with additional mathematical structure: a set of models and a definition of the relation of truth-in-a-model. A practice of categorization is described for the set of models.
3. The technique defines the relation MC (i.e. the relation of preservation of truth-in-a-model across all models) on arguments (premise/conclusion pairs) in terms of the mathematical structure described in (2).
4. There is a set of assumptions ($E1a, E1b, E1c$, and $E2$), expressed in terms of the structure given in (2) and the practices of categorization interpreting the sentences and models. These assumptions, if true, are sufficient to show that the relation MC specified in (3) is equivalent to the assumed relation of consequence between premise/conclusion pairs that was characterized in (1). As a result, the relation MC is able to serve as an intelligible proxy for the relation of consequence assumed in (1).

This way of describing how the model-theoretic technique makes the representational concept of logical consequence intelligible for interpreted sentential languages demonstrates the presence of a more abstract structure. We can see the “intelligibility-producing” aspects of the technique as consisting in four general components, corresponding to the four elements outlined above. These four components comprise the “intelligibility” aspect of the representational schema.

Before giving the four components, we introduce a further abstraction. The account of model-theoretic semantics we have considered so far, has taken sentences as the basic unit of expression whose consequence relation is to be explicated; and considered arguments as having the form of a set of sentences as premises, and a single sentence as conclusion. We would like to frame arguments more generally, for the following reasons. 1) We would like to be able to consider propositional claims which are heterogeneous in form, such as diagrams, charts, maps, etc. as well as sentences. 2) We would like to be able to model various ways of combining expressions beyond simple conjunction. One of the things which differentiates varying forms of making propositional claims are the different ways in which they permit subordinate claims to be combined. 3) Further, we would like to be able to model some of the assumptions which have been built into the standard presentation of argument form. In

order to accommodate these desires, we will introduce the concept of an **assertion type**. An assertion type is simply a type of propositional claim, assumed to categorize the space of relevant possibility into those possibilities of which it is true, and those of which it is false. Given the concept of an assertion type, we can then choose to model arguments as having a single assertion type as premise, and a single assertion type as conclusion.

Now we can give the four general components which comprise the “intelligibility” aspect of the representational schema.

1. A **foundational characterization** of the space of possibility and practices of categorization relating assertion types to possibilities; in terms of which the assumed representational concept of logical consequence on arguments is defined. Arguments are modelled as premise/conclusion pairs.
2. A mathematical structure supplementing and including the original set of assertion types. We will call this structure the **intelligible medium**. Each set of representative elements included in the intelligible medium is assumed to have a practice of categorization of possibility which interprets its members.
3. A relation defined on premise/conclusion pairs in terms of the structure of the intelligible medium which we call the **proxy for consequence relation**.
4. A set of assumptions, expressed in terms of the intelligible medium and the practices of categorization interpreting the representative elements of that structure. These assumptions, if true, are sufficient to show that the proxy for consequence relation is equivalent to the assumed relation of consequence between premise/conclusion pairs that was characterized in the foundational characterization. We call these assumptions the **technique-specific assumptions**. Note that while the technique-specific assumptions must be strong enough to imply the intelligibility of consequence, they may include assumptions unrelated to intelligibility but relevant to the technique’s explanatory strategy.

To qualify as an instance of the representational schema, a technique for producing theories of logical consequence must be describable in terms of these four components: foundational characterization; intelligible medium; proxy for consequence relation, and technique-specific assumptions. Given that it is capable of such description, the technique is capable of making the representational conception of logical consequence intelligible for interpreted languages satisfying the technique-specific assumptions.

1.4.2 Explanatory sufficiency

As we have stated before, but wish to emphasize again, the fact that an account makes the representational conception of logical consequence intelligible is not a sufficient condition for that account to implement a representational semantics. The account must further be explanatory of the consequence relation. Only then will the account be considered a representational theory of consequence.

We capture this requirement as part of the representational schema by adding the following condition. To qualify as an instance of the representational schema, a technique for producing theories of logical consequence must produce accounts which are explanatory of the consequence relation, in virtue of having been produced by the technique.

This “definition” is open in that we do not precisely characterize what we mean by “explanatory.” We can provide some guidelines. First, to be considered explanatory, a theory produced by a candidate technique should be simpler than the relation of logical consequence being modelled. Secondly, to be considered explanatory, the way in which a theory produced by a candidate technique is simpler should be a guide to applying the technique in particular cases. But beyond this, we do not give a specification.

This openness is an advantage of the schema rather than a weakness. As we will see, it is the source of much of the power of the schema. First, this openness shows us a path to seek out additional forms for explanatory theories of consequence. The approach of order-consistency semantics which we will consider shortly is just such a form. Secondly, many interpreted languages will be capable of having their consequence relation modelled by multiple techniques with different explanatory strategies. This will enable additional insight into how the relation of consequence works for those languages. We will be able to see the

relation of consequence for those languages from each of the explanatory perspectives.

1.4.3 Summary: The representational schema stated

We recapitulate the preceding section and express the representational schema. We can think of the representational schema as a set of conditions to be applied to techniques intended to produce theories of logical consequence.

Given a technique intended to produce theories of logical consequence,

- (1) if that technique is describable in terms of the four components: foundational characterization; intelligible medium; proxy for consequence relation, and technique-specific assumptions sufficient for the intelligibility of consequence (as described above); and
- (2) if the theories generated by that technique are explanatory of the consequence relations they model, in virtue of having been produced by the technique; where explanatoriness is characterized by the produced theories being simpler than the consequence relations they model, and by the explanatory strategy being a guide to applications of the technique in constructing such theories;

then the technique is an instance of the representational schema.

Given that it is an instance of the representational schema, a technique is capable of making the representational conception of logical consequence intelligible for interpreted languages satisfying the technique's technique-specific assumptions; and those theories will be explanatory of the consequence relations they model.

As a shorthand, we will use the phrase “**representational technique**” for a technique which is an instance of the representational schema. In this dissertation, we will discuss two primary categories of representational technique. There will be **model-theoretic representational techniques** (of which the technique described by John Etchemendy is an example) and **order-consistency representational techniques** (which we will discuss shortly). We will call the representational technique which Etchemendy describes, the **Set E model-theoretic representational technique**. (This name comes from the name we

will give the technique-specific assumptions used by the technique, that is, “Set E .”) Sometimes we will simply call this technique the **Set E technique**. We will use similar naming conventions for other techniques introduced.

1.5 Applying the representational schema

1.5.1 A general mode of application

The first attribute of the representational schema we will consider is the way in which the schema offers a standard approach to constructing a representational semantics for an interpreted language.

In what do such applications consist?

We begin to answer this question by considering what will be the given in any application of the representational schema. The answer: the given in any application of the representational schema will be some language assumed to have an antecedent interpretation. The result of every application of a representational technique is a **theory of consequence** for the language as interpreted.

In some cases, the given interpretation of the language under consideration will be assumed complete. In other cases, and in particular, those involving conventional logical languages, the given interpretation will be partial. These latter languages are really language-forms, in that in their general form they are only partially interpreted. For example, in the case of the language of propositional logic, in advance of any particular use, the sentential connectives are interpreted, but the sentence symbols are not. In the case of a partially interpreted language, the given constrains the set of acceptable complete interpretations.

To carry out an application of some representational technique we have to perform the following tasks. Note, however, that the tasks need not be ordered in time, in the same order that they are presented below. We could think of these three tasks as being carried out simultaneously.

1. Identify the set of assertion types used to model the propositional claims of the language.
2. Form an intelligible medium of the type used by the technique, by supplementing the set of assertion types with additional structure (and possibly additional representative elements, like the models in model-theoretic semantics).
3. Characterize the set of acceptable complete interpretations for the representative elements in the intelligible medium. This includes, but may not be limited to the assertion types.
 - (a) The results of this activity can take the form of specifying a set of **semantic conventions** for the representative elements in the medium (e.g. sentences, models). These conventions define what count as acceptable interpretations of those representative elements. For instance, in defining the semantic conventions for the sentences of propositional logic, we would deem unacceptable any practice of categorization counting both a sentence and its negation as true in the same possibility.

An application is considered to have been carried out correctly if the characterizations of the acceptable complete interpretations of the representative elements of the constructed intelligible medium are sufficient to imply that given acceptable complete practices interpreting the representative elements of that medium, that medium satisfies the technique-specific assumptions with respect to those practices.

In that case, for any acceptable complete interpretations of its representative elements, the proxy for consequence relation for the technique, when evaluated with respect to the constructed intelligible medium, will be equivalent to the representational conception of logical consequence for the assertion types of the language as interpreted.

Terminological note: When we say “**interpreted set of assertion types**” we are referring to a set of assertion types together with a practice of categorization that interprets them.

1.5.2 A specific example

To make the abstract description above more concrete, let us sketch the application of the representational schema to the language of propositional logic. We will carry out this example in detail in Chapter 3.

For our application we use the Set E model-theoretic representational technique (the one informally defined by Etchemendy and described above).

Step 1: Identify a set of assertion types for the language. In this case, we could use the powerset of the set of propositional sentences as the set of assertion types.

Step 2: The Set E technique involves supplementing the set of assertion types with a set of models, and a relation of truth-in-a-model between sentences and models. We use the set of total truth assignments as models, and the standard definition of truth-in-a-model for the propositional language. Together the powerset of sentences, function characterizing truth-in-a-model, and set of total truth assignments form the intelligible medium for the application.

Step 3: We have two kinds of representative elements in the medium: sets of sentences and models. We define semantic conventions defining acceptable interpretations for both kinds of element. (We do this in the actual application in Chapter 3).

In the actual application, we prove that if a practice of categorization for sets of sentences and a practice of categorization for models satisfy the semantic conventions described for those kinds of practices in Step 3, then the intelligible medium we have constructed satisfies the Set E technique-specific assumptions with respect to those practices. Given that proof, we know that for every pair of acceptable interpretations for sets of sentences and models, the proxy relation of preservation of truth-in-a-model across all models is equivalent to the representational conception of logical consequence (preservation of truth across all possibilities). Further, because of the technique used, we know that the theory so constructed explains the consequence relation between sets of sentences in terms of the logically simple form of representation given by the total truth assignments. That completes (in outline) the application of the representational schema to the language of propositional logic.

1.5.3 Specific applications we will consider

In addition to various simple examples to illustrate the basic properties of the techniques under discussion, we will discuss three specific applications: propositional logic, feature logics (sentential languages which use feature structures as models) and languages in which feature structures are the assertion types. We will carry out complete applications of both model-theoretic and order-consistency techniques to the language of propositional logic, and outline the application of the representational schema to feature logics and languages in which feature structures are the assertion types.

1.6 Order-consistency representational techniques

In this section, we will introduce the category of order-consistency representational techniques. The purpose of this introduction is to give the category some substance, and in so doing, give a sense of the scope of the representational techniques we are considering in this dissertation. Note that we will postpone most of the development of order-consistency representational techniques until Chapter 4.

1.6.1 The role of order-consistency representational techniques

Order-consistency representational techniques, like model-theoretic representational techniques, make the representational conception of logical consequence intelligible, and provide an explanation of that relation. Importantly, however, the explanatory strategy employed by order-consistency representational techniques is different in kind than that provided by model-theoretic representational techniques. Whereas a model-theoretic representational technique explains the consequence relation among assertion types in terms of a logically simpler form of representation (the models) and a relation of truth-in-a-model between the assertion types and the models, an order-consistency representational technique explains the consequence relation among assertion types in terms of the simpler concepts of ordering and consistency.

Recall that an interpreted set of assertion types is that set together with a practice of categorization of possibility that interprets its members. Given our foundational assumptions,

every interpreted set of assertion types has an assumed representational relation of logical consequence which we have characterized as preservation of truth across all possibilities. We will see that for many interpreted sets of assertion types, this relation of consequence is amenable to being made intelligible by both model-theoretic and order-consistency techniques.

As we will see, order-consistency representational techniques provide important additional theoretical and practical perspectives on the concept of logical consequence. These techniques extend our theoretical understanding by giving us additional sources of explanation for why one assertion type is a consequence of another. Further, the relations between the different techniques make it clearer what any one specific technique is doing, and help us see the nature of these techniques in general. Order-consistency representational techniques may also offer the practical advantage of enabling better ways to specify the relation of consequence for certain interpreted languages.

1.6.2 Conceptual roots of order-consistency representational techniques

The fundamental intuition underlying order-consistency representational techniques can be found in a corollary to Lindenbaum’s Lemma. Before we present the Lemma and the relevant corollary, let us introduce some background. Lindenbaum’s Lemma and its corollaries are frequently expressed in relation to some syntactical system of propositional logic. In this case, we will be drawing on a presentation of the Lemma in Chellas’ book on propositional modal logic (Chellas 1980). Relative to that context, several concepts can be characterized. First, consider that the relevant “unit of assertion” is a set of sentences. One set of sentences Q **extends** a set of sentences P , just in case P is a subset of Q . Since the context is syntactic, consistency is defined in terms of the derivability of contradictions. A set of sentences is **consistent** just in case the contradictory sentence cannot be derived from them (p. 47). A set of sentences is **maximal** if it is consistent, and has only inconsistent proper extensions (p. 53). Another way to express maximality is that a set of sentences is maximal if it is consistent, and no sentence not already in the set can be added to it without the result being inconsistent.

Lindenbaum’s lemma itself can be expressed as follows: Every consistent set of sentences

has a maximal extension (p. 55). The relevant corollary can be expressed: A sentence q is deducible from a set of sentences P just in case q belongs to every maximal extension of P (p. 57). Equivalently, one could say: A sentence q is deducible from a set of sentences P just in case every maximal extension of P extends $\{q\}$. Looking at this corollary, we can see a key concept, that a proxy for consequence (in this case deducibility) can be expressed in terms of consistency and ordering. Translated into a semantic setting, this is the core idea behind order-consistency representational techniques.

1.6.3 Assimilating the concept of order-consistency semantics to the representational schema

In order to assimilate the concept that a proxy for consequence can be stated in terms of order and consistency to the representational schema we have to take several steps, in line with the four components articulated by the schema.

1. Foundational characterization. We can accept the same foundational characterization as that used by the model-theoretic representational techniques.
2. Intelligible medium. We here use a different intelligible medium than that used by model-theoretic representational techniques. Instead of supplementing the set of assertion types with a set of models and a definition of a relation of truth-in-a-model, we supplement the set of assertion types with a partial order (which will be assumed to imply extension of commitment), and a specification of which assertion types are taken to be consistent.
3. Proxy for consequence relation. Our first order-consistency representational technique will use the concept expressed in the corollary to Lindenbaum's Lemma as its proxy for consequence relation. We will define a relation LLC such that for assertion types p, q , $LLC(p, q)$ just in case every maximal extension of p is a maximal extension of q .
4. Technique-specific assumptions. Here we will specify four general conditions which are sufficient to imply that the proxy relation LLC is equivalent to the representational conception of logical consequence. We will discuss these assumptions in great detail in

Chapter 4. For now, let us just say that in the first order-consistency representational technique we define, there will be four assumptions (a set we will call “Set CL ”).

- (a) One ensures that the partial order given implies monotonic extension of commitment.
- (b) One ensures that the specification of consistency is correct.
- (c) One ensures that there are “enough” assertion types to express certain conditions.
- (d) The last is Lindenbaum’s Lemma.

We will call this first order-consistency representational technique, the “**Set CL order-consistency representational technique**,” after the name of its assumption set.

We shall develop (or outline) several example applications of order-consistency representational techniques. One example we will work out fully is the language of propositional logic. That language is just one example of a language whose consequence relation is capable of being modelled by multiple techniques with different explanatory strategies.

We have now presented (in outline) the two broad classes of representational techniques we will be discussing in this dissertation. The general picture we have now:

- Representational techniques
 - 1. Model-theoretic representational techniques
 - (a) Set E model-theoretic representational technique
 - (b) ... more to come
 - 2. Order-consistency representational techniques
 - (a) Set CL order-consistency representational technique
 - (b) ... more to come

We now proceed to consider how the various techniques in these classes can be compared with one another.

1.7 Comparing representational techniques

One of the primary values of having a common schema to which all of the representational techniques can be assimilated is that we then have a common ground on which to compare them with one another. In this section, we will discuss four of the different dimensions along which we can (and in this dissertation do) compare representational techniques. The dimensions we will talk about include: (1) the mode by which the techniques explain the consequence relation; (2) the range of interpreted languages to which the techniques are applicable; (3) the degree of epistemological commitment which the techniques require; and (4) the ease with which the techniques can be applied in particular cases.

1.7.1 Explanatory strategy

A fundamental way in which representational techniques can differ from one another is in the explanatory strategy they use to explain the relation of logical consequence. We have seen two broad classes of strategy: that used by the model-theoretic representational techniques, and that used by the order-consistency representational techniques. A theory of consequence constructed using a model-theoretic representational technique explains the (usually) logically complex relation of logical consequence between assertion types in terms of a logically simpler form of representation (the models) and a defined relation of truth-in-a-model between assertion types and models. A theory of consequence constructed using an order consistency representational technique explains the relation of logical consequence between assertion types in terms of the simpler concepts of order and consistency.

We will see that within a category, there can also be some subtler differences in explanatory strategy. For example, while the Set E model-theoretic representational technique explains the logical consequence relation between assertion types in terms of a form of representation which has the simplest logical structure; we can describe other techniques in which the form of representation used for the models is not logically simplest but is logically simple, and typically simpler than the form used for the assertion types.

1.7.2 Range of applicability

This is one of the most important dimensions on which we can compare representational techniques, and one to which we will devote quite a bit of attention. One of the benefits of the representational schema is that it gives us a standard way to establish relations of inclusion between the range of interpreted sets of assertion types to which one representational technique is applicable, and the range to which some other technique is applicable.

Here we will present the standard approach used to establish these relations of inclusion. In Part IV, we will establish results linking all of the techniques presented in a chain of inclusion. That is, every interpreted set of assertion types to which the first technique can be applied, is an interpreted set of assertion types to which the second technique can be applied, etc. We express the standard approach here, and will apply this approach repeatedly in Part IV. The most important concept is that there is a standard method for comparing ranges of application across techniques.

The central idea can be expressed as follows. We are given a (source) interpreted set of assertion types for which an application of some (source) representational technique can construct an explanatory intelligible proxy for the representational relation of logical consequence. We want to know if some other (target) representational technique can do the same.

The application of the (source) technique will have made use of an interpreted intelligible medium of the type used by that technique. That (source) intelligible medium will satisfy the technique-specific assumptions of the source technique with respect to the practices of categorization by which its representative elements are interpreted. The key will be to construct from the (source) interpreted intelligible medium, a (target) interpreted intelligible medium, such that

1. the representational relation of logical consequence for the (source) interpreted set of assertion types is embedded in the representational relation of logical consequence for the (target) interpreted set of assertion types; and
2. the target interpreted intelligible medium is of the type used by the (target) representational technique, and the practices of categorization by which its representative

elements are interpreted satisfy the technique-specific assumptions of the target technique.

By (1), we mean that there is a function h mapping elements of the source set of assertion types to elements of the target set of assertion types, such that for any p, q in the source set of assertion types, q is a representational consequence of p when p and q are interpreted with respect to the source practice of categorization just in case $h(q)$ is a representational consequence of $h(p)$ when $h(p)$ and $h(q)$ are interpreted with respect to the target practice of categorization.

Then if (1) and (2) hold, the application of the target representational technique makes the representational relation of logical consequence for the source interpreted set of assertion types explanatorily intelligible. To see this, realize that given (1) and (2), it is the case that for all p, q in the source set of assertion types,

q is a representational consequence of p when p and q are interpreted with respect to the source practice of categorization just in case
 $h(q)$ is a representational consequence of $h(p)$ when $h(p)$ and $h(q)$ are interpreted with respect to the target practice of categorization just in case
the intelligible proxy for consequence relation for the target representational technique considered with respect to the target intelligible medium holds for the pair $\langle h(p), h(q) \rangle$.

The discussion above gives us a definition. An arbitrary interpreted set of assertion types is in the **range of applicability** of a representational technique X exactly when there is a intelligible medium A of the kind used by X and practices of categorization P interpreting the representative elements of A , such that A satisfies the Set X technique-specific assumptions with respect to the practices in P , and the representational relation of logical consequence for the given interpreted set of assertion types is embedded (via some function h) in the representational relation of logical consequence for the assertion types of medium A as interpreted by the practice in P interpreting those types.

We will use the notation $ra(\text{Set } X)$ to indicate the range of applicability of the Set X technique.

1.7.3 Degree of epistemological commitment

Another way in which techniques can differ is the degree of epistemological commitment which their technique-specific assumptions require. Some techniques require stronger assumptions than others. For example, the Set *E* model-theoretic representational technique requires that the set of models contains, for each sentence, a subset whose members collectively represent all of the possibilities in which that sentence could be false and another subset whose members collectively represent all of the possibilities in which that sentence could be true. We will show in an extension (the Set *PP* model-theoretic representational technique) that we can make the representational relation of logical consequence explanatorily intelligible with a weaker assumption. It will be sufficient that there be, for each sentence, a subset of the set of models whose members collectively represent all of the possibilities in which that sentence could be true. Set *PP* is epistemologically weaker than Set *E* since it does not require the user to assume that their models represent the entire space of possibility. Instead, they need only assume that their models represent those possibilities represented by some one of their assertion types.

1.7.4 Ease of applicability

Another way in which we will see that techniques differ is in the ease with which they can be applied in particular cases. Part of these differences in ease of applicability stem from differences in the strength of the technique-specific assumptions used by the technique. A technique with weaker assumptions may be easier to apply. Part of these differences in ease of applicability stem from differences in explanatory strategy. For example, in a particular application, it may be easier to specify ordering and consistency than to specify a set of models and a defined relation of truth-in-a-model.

1.8 Extending the set of representational techniques

An important and useful attribute of the representational schema is that it establishes base points and vectors along which new representational techniques can be developed. There are three levels at which extension is possible:

1. Foundational characterization. It is possible that a new family of techniques could be generated by altering the foundational characterization upon which the families of model-theoretic and order-consistency techniques are based.
2. Explanatory strategies. It is possible that new explanatory techniques beyond those used by model-theoretic and order-consistency semantics could be expressed.
3. Techniques within categories of explanatory strategies. This is the level at which we extend the set of representational techniques.

(a) Within the category of model-theoretic representational techniques we will define two new techniques:

- i. the Set *PP* model-theoretic representational technique... which as mentioned above has lesser epistemological commitments than the Set *E* technique; but is capable of modelling every representational relation of logical consequence that the Set *E* technique is capable of; and
- ii. the Set *BE* model-theoretic representational technique... which allows the use of partial models, and will turn out to be easier to apply in the case of feature logics, than the Set *E* or Set *PP* techniques.

(b) Within the category of order-consistency representational techniques we will define one new technique:

- i. the Set *CG* order-consistency representational technique... which uses the same technique-specific assumptions as the Set *CL* technique, less the assumption of Lindenbaum's Lemma. The Set *CG* technique uses a somewhat more involved proxy for consequence relation than the Set *CL* technique. This new technique removes the requirement that the language under consideration have a maximal extension for each consistent assertion type.

One idea which should be clear from the above. This project is not seeking to define one particular technique which is capable of handling all interpreted languages. Instead, we are presenting a way of organizing, understanding, comparing, and creating techniques.

The picture which we end up with looks like this:

- Representational techniques

1. Model-theoretic representational techniques

- (a) Set E model-theoretic representational technique
- (b) Set PP model-theoretic representational technique
- (c) Set BE model-theoretic representational technique
- (d) ... more to come?

2. Order-consistency representational techniques

- (a) Set CL order-consistency representational technique
- (b) Set CG order-consistency representational technique
- (c) ... more to come?

Through our comparisons of the ranges of applicability of the various techniques, we will be able to establish the following ordering:

$$ra(\text{Set } E) \subseteq ra(\text{Set } PP) \subseteq ra(\text{Set } CL) \subseteq ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$$

This ordering indicates our current knowledge. When we say that the range of application of the Set X technique is a subset of the range of applicability of the Set Y technique, what we are saying is that we know that every interpreted set of assertion types for which an applicability of the Set X technique can make the representational relation of consequence explanatorily intelligible is an interpreted set of assertion types to which an application of the Set Y technique can do the same.

1.9 Outline of the dissertation

That completes our overview of the content of the dissertation. Before proceeding to the main text, let us present an outline of the chapters.

- Part 1. Introduction

- Chapter 1. Representational Semantics and the Representational Schema (this chapter).

- Part 2. Assimilating Model-theoretic Semantics to the Representational Schema.
 - Chapter 2. Model-theoretic Representational Techniques.
 In this chapter, we formally present and discuss the Set E , Set PP , and Set BE model-theoretic representational techniques.
 - Chapter 3. Applications of Model-theoretic Representational Techniques.
 In this chapter, we give several example applications of the model-theoretic representational techniques. We apply the Set E technique to a simple language; and then to the language of propositional logic. We also outline the application of the Set BE technique to languages of feature logics.
- Part 3. Assimilating Order-consistency Semantics to the Representational Schema.
 - Chapter 4. Order-consistency Representational Techniques.
 In this chapter, we formally present and discuss the Set CL , and Set CG order-consistency representational techniques. As a part of this presentation, we carry out a more complete presentation of the background and motivation for order-consistency semantics. We also show that the range of applicability of the Set CL technique is included in the range of applicability of the Set CG technique.
 - Chapter 5. Applications of Order-consistency Representational Techniques.
 In this chapter, we present several example applications of the order-consistency representational techniques, and outline the application of the Set CG technique to languages where feature structures are used as assertion types.
- Part 4. Comparing the Ranges of Applicability of the Representational Techniques Presented.
 This part presents detailed proofs of the relations between the ranges of applicability of the different representational techniques we have discussed.
 - Chapter 6 demonstrates that the range of the Set PP technique is a subset of the range of the Set CG technique.

- Chapter 7 demonstrates that the range of the Set *PP* technique is a subset of the range of the Set *CL* technique.
 - Chapter 8 demonstrates that the range of the Set *CG* technique is a subset of the range of the Set *BE* technique.
 - Chapter 9 demonstrates for every application of the Set *CG* technique there is an application of Set *CG* technique which makes the same representational relation of consequence intelligible, but with a minimal number of assertion types.
- Part 5. Reflections and next steps.
 - Chapter 10. Reflections and next steps.

In this chapter, we reflect on the concepts developed and results obtained, and look ahead to the next steps in the project.

Part II

Assimilating Model-theoretic Semantics to the Representational Schema

Chapter 2

Model-theoretic Representational Techniques

2.1 Introduction

We have chosen, as our primary dimension of conceptual organization, to categorize the techniques of representational semantics by the way in which they seek to explain the consequence relation; what we could call in a phrase their “**explanatory strategy**.” The reason for this is simple. The explanatory strategy of a representational technique is the end to which all the other variable elements of the technique (foundational characterization, intelligible medium, proxy for consequence relation, and technique-specific assumptions) are the means. These various latter elements are all developed with the intention of making explanations of the kind described by the explanatory strategy possible.

In this chapter, we will investigate and discuss the category of representational techniques which explain the logical consequence relation between assertion types in terms of a logically simpler form of representation (the models) and a defined relation of truth-in-a-model. This is the category we have termed: “**model-theoretic representational techniques**.” We will call their shared explanatory strategy the “**model-theoretic explanatory strategy**.” In the following chapter, we will describe and discuss various applications of these techniques.

Our plan for the chapter is as follows. We will begin by identifying and characterizing

the three model-theoretic representational techniques we have considered as a part of our investigations. This initial characterization will set our expectations from the remainder of the discussion. We will proceed to carry out a formalization of the techniques. The presentation of the formalization will begin with general concepts required for all three techniques, and then describe those aspects in which the techniques differ. This discussion will make clear the ways in which the techniques fulfill the model-theoretic explanatory strategy. The next step will be to prove that all three techniques are capable of making consequence intelligible. We will then discuss certain relations between the ranges of applicability of the techniques, and finish by considering the relations of the techniques to the separate task of making logical truth intelligible.

2.2 Three model-theoretic representational techniques

Our investigations in this chapter will concern three different representational techniques sharing the model-theoretic explanatory strategy. In this section, we will present the techniques generally, and discuss their properties. The formal presentation of these techniques will come in Section 2.3, and proofs of these properties follow that formalization.

All three techniques share the same foundational characterization, intelligible medium, and proxy for consequence relation. They differ only in the sets of technique-specific assumptions that they use. We have chosen to name the techniques according to the sets of technique-specific assumptions they use. The three techniques are: the Set E technique, the Set PP technique, and the Set BE technique. Let us consider these techniques and their properties one at a time.

2.2.1 The Set E technique

The Set E technique is a formalization of Etchemendy's account of how a model-theoretic semantics implements a representational semantics. We have already described this technique informally above (in Section 1.3). One property of the Set E technique is that theories developed using the Set E technique make logical consequence intelligible. As we discussed above, theories developed using the Set E technique explain the relation of logical conse-

quence between assertion types in terms of a logically transparent form of representation, the models. Recall that logical transparency is the simplest logical form. The logical transparency of the set of models used by an application of the Set *E* technique is due to the fact that the sets of possibilities they represent are required (by the Set *E* assumptions) to be disjoint and nonempty.

A special property of the Set *E* technique is that theories developed with it make logical truth intelligible as well. That is to say, given that an intelligible medium of the type used by the Set *E* technique satisfies the Set *E* assumptions with respect to the practices of categorization interpreting its representative elements (the assertion types and the models), there is a relation which can be defined in terms of the intelligible medium which is equivalent to logical truth. We will explore this aspect of the Set *E* technique further in Section 2.6 below.

2.2.2 The Set *PP* technique

The Set *PP* technique is a derivative of the Set *E* technique. It uses a set of technique-specific assumptions which are slightly weaker than the Set *E* assumptions. Like the Set *E* technique, theories developed with the Set *PP* technique make logical consequence intelligible, and they explain the relation of logical consequence between assertion types in terms of a logically transparent set of models.

The logical transparency of the set of models used by an application of the Set *PP* technique is due to the fact that the sets of possibilities they represent are required (by the Set *PP* assumptions) to be disjoint and nonempty.

There are two primary effects of the weaker set of assumptions used by Set *PP*. One is a gain, the other a loss. The gain is that it is epistemologically easier to commit to the Set *PP* assumptions than to the Set *E* assumptions. We will see that to assume the Set *E* assumptions, one must assume that the set of models being used represent every possibility. The Set *PP* assumptions have a lesser requirement. They only require that the set of models being used represent every possibility we can represent with our assertion types. We mentioned a gain and a loss. What we lose in moving from the the Set *E* technique to the Set *PP* technique is that theories developed using Set *PP* do not make logical truth

intelligible in the way that theories developed using Set E did.

2.2.3 The Set BE technique

The Set BE technique is like the Set E and Set PP techniques in that theories developed using Set BE make logical consequence intelligible, and that theories developed using Set BE explain the logical consequence relation between assertion types in terms of a logically simpler form of representation, the models. The fundamental difference between the Set BE technique and the other two model-theoretic techniques we have described is that in theories developed using the Set BE technique, the set of models is not guaranteed to be logically transparent. In the case of Set BE , the set of models is logically simple but not logically simplest.

We'll discuss the way in which the models used by the Set BE technique are logically simple below. The key thing to keep in mind at this point is that the Set BE technique does not include the requirement that the sets of possibilities represented by the models be disjoint and nonempty. This makes the Set BE technique easier to apply in the case of languages for which partial models are the most natural way to express the semantics.

We should also note that theories developed using Set BE do not make logical truth intelligible in the way that theories developed using Set E did (but neither do they assume that the models represent all possibilities).

2.2.4 Known relations between techniques in terms of range of applicability

From our investigations, we have discovered that

1. the range of application of the Set E technique is a subset of the range of application of the Set PP technique; and
2. the range of application of the Set PP technique is a subset of the range of application of the Set BE technique.

These results will be proven below.

2.3 Formalizing the model-theoretic representational techniques

Now we will present a means for formalizing the model-theoretic techniques we have been discussing. This formalization will include all of the elements of a technique as understood from the representational schema; that is, foundational characterization, intelligible medium, proxy for consequence relation, and technique-specific assumptions. All the techniques we are considering share the same foundational characterization, intelligible medium, and proxy for consequence relation. We will introduce formalizations for those first. Then we shall present formalizations of the technique-specific assumptions for each technique.

2.3.1 The intelligible medium

All the model-theoretic representational techniques we are defining (Set E , Set PP , Set BE) use the same intelligible medium. The structure of this medium follows directly from the model-theoretic explanatory strategy. A model-theoretic representational technique explains the relation of logical consequence for a set of assertion types in terms of a logically simpler form of representation (the models) and a function characterizing the relation of truth-in-a-model between assertion types and models. We will combine these three elements: set of assertion types, set of models, and defined function characterizing the relation of truth-in-a-model into a single structure and call it a “**truth medium**.”

Formally, a truth medium consists of a triple $\langle \Sigma, M, t \rangle$ where Σ is a set of assertion types, M is a set of models, and t is a function from Σ to $\mathcal{P}(M)$. We use the function t to specify the set of models for each assertion type, instead of the relation $\models$ defined on M and Σ . The expression $t(\sigma)$ gives the models of σ . The two forms are interchangeable: $\forall \sigma \in \Sigma, \forall m \in M, m \models \sigma$ iff $m \in t(\sigma)$.

Given a truth medium A , we refer to the constituents of A as follows: Σ_A is the set of assertion types of A , M_A is the set of models of A , and t_A is the function characterizing truth-in-a-model for A . Using this notation: $t_A(\sigma)$ is the set of models of σ for medium A .

For every truth medium A , we assume that Σ_A and M_A are non-empty. A medium without assertion types is worthless for expressing information; a medium without models

cannot implement a model-theoretic semantics.

By definition, we can see that truth media are instances of the classification structure used in Barwise and Seligman, and therefore, are instances of Chu spaces (Barwise and Seligman 1997, 28ff). Furthermore, these structures are of the kind studied in Abstract Model Theory, which considers various general postulates that are supposed to hold for all ‘logics,’ e.g. “isomorphism implies truth equivalence” and closure under negation and conjunction. The level of generality of the postulates in abstract model theory is similar to that of the technique-specific assumptions we discuss here (Barwise and Feferman 1985). Another area with a similar degree of abstractness is Abstract Data Types (Cardelli and Wegner 1985).

We give the truth media structure the name “**medium**” for two reasons. First, as we will see, it is not simply a classification, but a classification used in a particular way, with associated practices and assumptions. Secondly, we use the term “medium” to reinforce the connection between truth media and the “**consistency media**” which we will introduce later.

2.3.2 The proxy for consequence relation MC

All the model-theoretic representational techniques we are defining (Set E , Set PP , Set BE) use the same proxy for consequence relation MC , which was defined informally as preservation of truth-in-a-model across all models, and more formally as:

For set of sentences Σ , and sentence φ ,

$MC(\Sigma, \varphi)$ if and only if,

for every model m (if (for every sentence $\sigma \in \Sigma$, $m \models \sigma$) then $m \models \varphi$)

We make several adjustments. First, we are now considering arguments as consisting of a single assertion type as premise and a single assertion type as conclusion. Second, we wish to make the definition media-relative, that is, every truth medium A will have its own defined relation MC_A . Third, our truth media use the function t from assertion types to sets of models, instead of the relation $\models$ on models and assertion types.

Making these adjustments, we can give the formal definition of MC_A (MC for truth medium A) as follows:

For any pair of assertion types $p, q \in \Sigma_A$,
 $MC_A(p, q)$ if and only if $t_A(p) \subseteq t_A(q)$.

That is to say, every model of p is a model of q .

2.3.3 Foundational characterization

All the model-theoretic representational techniques we are defining (Set E , Set PP , Set BE) share the same foundational characterization of practices of categorization of possibility, and a representational conception of logical consequence.

Practices and possibilities

As we discussed in section 1.3.4, the assumptions we are using to interpret the model-theoretic structure contain an implicit usage of the notion of practices, by which assertion types (subsuming sentences) and models categorize a space of relevant possibility. These practices appear in two guises: the practice of assertion types (sentences) being true, and the practice of models being accurate. In our formalization, we treat these two kinds of practices (assertion types being true, and models being accurate) in parallel since both assertion types and models are full-fledged representational schemes on the representational view.

In our formalization, we make the assumption that the possibilities under consideration are discrete, and that exactly one of the possibilities is actual. So we can model the space of possibility by a set of possibilities $\mathcal{U}$, and the practices by means of functions.

The practice of categorizing a possibility as being one in which an assertion type is true, we can model by a function: $\mathcal{T}_\Sigma : \Sigma \rightarrow \mathcal{P}(\mathcal{U})$, such that, for any $\sigma \in \Sigma$, $\mathcal{T}_\Sigma(\sigma)$ is the set of possibilities in which σ is true. We say that $\mathcal{T}_\Sigma(\sigma)$ is the set of possibilities **represented** by σ .

The practice of categorizing a possibility as being one of which a model is accurate, we can model by a function: $\mathcal{T}_M : M \rightarrow \mathcal{P}(\mathcal{U})$, such that, for any $m \in M$, $\mathcal{T}_M(m)$ is the

set of possibilities in which m is accurate. We say that $\mathcal{T}_M(m)$ is the set of possibilities **represented by m** .

In most cases, we will be considering $\mathcal{T}_\Sigma$ and $\mathcal{T}_M$ relative to some truth medium A , in which case we write $\mathcal{T}_{\Sigma_A}$, and $\mathcal{T}_{M_A}$.

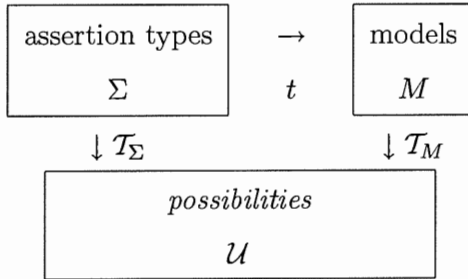
We wish to emphasize the difference in our epistemological relationship with respect to the space of possibility and the practices by which we assume to categorize it, on one hand, and the elements of our model-theoretic intelligible medium, i.e. the sentences, class of models, and specified function characterizing the relation of truth-in-a-model, on the other. We are not committed to being able to know the possibilities *as* possibilities; nor to being able to know a practice in its totality. We are only committed to being able to know a practice through its individual applications. On the other hand, we can intelligibly comprehend the totality of our sentences, models, and function characterizing the relation of truth-in-a-model.

We represent the epistemological inaccessibility of the space of possibility and the assumed practices by using the calligraphic typeface, that is, $\mathcal{U}$, $\mathcal{T}_{\Sigma_A}$, $\mathcal{T}_{M_A}$.

Also: since we will be using the term quite a bit, we will often shorten the phrase “practice of categorization of possibility” to simply **“practice of categorization.”**

While practices of categorization of possibility provide interpretations for assertion types and models, we have chosen not to call them simply “interpretations” for two reasons: first, because the usage could be confusing given the other uses for the term “interpretation” already in the literature; and secondly, because we wish to draw attention to the fact that the various methods under investigation are dependent upon a specific kind of interpretation, namely, the kind of practice described above.

We can update the informal diagram from Section 1.3.4 with our new definitions.



The representational concept of logical consequence

In our informal discussion, we chose to model the representational concept of logical consequence as preservation of truth across all possibilities given a fixed interpretation of the language. We can capture this concept relative to our formalization as follows: for any assertion types p and q , q is a logical consequence of p if and only if every possibility in which p is true is a possibility in which q is true.

Formally, we say, for a set of assertion types Σ related to a space of relevant possibility $\mathcal{U}$ via a practice of categorization $\mathcal{T}_\Sigma$,

for any elements p, q of Σ , q is a consequence of p iff $\mathcal{T}_\Sigma(p) \subseteq \mathcal{T}_\Sigma(q)$.

We will define a relation $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$ (*Assumed Logical Consequence*) on $\Sigma \times \Sigma$ as follows:

$\forall p, q \in \Sigma, ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p, q)$ iff $\mathcal{T}_\Sigma(p) \subseteq \mathcal{T}_\Sigma(q)$.

The subscripting on ALC is important, because every assumed concept of logical consequence is relative, not just to some set of assertion types Σ , but also to the practice of categorization of possibility $\mathcal{T}_\Sigma$ by which the elements of Σ are interpreted. We use the phrase “**interpreted set of assertion types**” to refer to the combination of a set of assertion types and a practice of categorization of possibility interpreting that set. So we could say that our assumed concept of logical consequence is relativized to interpreted sets of assertion types.

When we wish to refer to an interpreted set of assertion types on its own, we may use the notation $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ to indicate the set of assertion types Σ as interpreted by practice of categorization $\mathcal{T}_\Sigma$.

The representational account of consistency

Looking ahead, we will also need an account of **consistency**. We will assume the following concept of consistency: an assertion type is consistent just in case it represents some possibility. We can express this condition formally as follows:

$\forall p \in \Sigma, p$ is consistent iff $\mathcal{T}_\Sigma(p) \neq \emptyset$.

As in the case of the concept of logical consequence, we should note that this is not an analysis of consistency, but an foundational assumption for the representational account.

2.3.4 Prelude to the technique-specific assumption sets

Now we are about to describe three sets of technique-specific assumptions (the Set E assumptions, the Set PP assumptions, and the Set BE assumptions). As a shorthand, when we are talking about assumption sets (vs. techniques) we may simply say “Set E ” for assumption Set E . We will claim that, each set, when taken together with the foundational characterization, intelligible medium, and proxy for consequence relation described above, constitutes a representational technique.

Demonstrating this claim will require that we show (for each technique) two subsidiary claims.

1. First, we will have to show that the technique’s technique-specific assumption set is sufficient to imply the intelligibility of consequence. That is to say, we will need to prove that: given any truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting its assertion types and models, if the medium A satisfies the assumptions in the set with respect to the practices, then the proxy for consequence relation MC_A is equivalent to the representational conception of logical consequence ($ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$). Relative to some truth medium A and the practice of categorization interpreting its assertion types $\mathcal{T}_{\Sigma_A}$, we can express this goal condition formally as follows:
(ICC) intelligibility of consequence condition: $MC_A = ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$

2. Secondly, we will have to show that for any theory constructed using the technique, the set of models is logically simple. Theories constructed using the technique will consist in a truth medium, such that the truth medium is assumed to satisfy the technique-specific assumptions with respect to the practices of categorization interpreting the assertion types and models. So we will need to show that the technique-specific assumptions are sufficient to imply that the models are logically simple.

As we consider each technique-specific assumption set below, we will show that that set is

sufficient to imply that the models are logically simple. We will prove that each assumption set is sufficient for the intelligibility of consequence in a later section. Now we consider the three technique-specific assumption sets Set *E*, Set *PP*, and Set *BE* corresponding to the three model-theoretic representational techniques we are defining.

2.3.5 Assumption Set *E*

The Set *E* technique is a formal carrying-over of the informal account of model-theoretic semantics presented by Etchemendy. (“Set *E*” is a mnemonic for the “Etchemendy” set). So in this section we will define formal equivalents for each of the assumptions which Etchemendy outlined. We express the assumptions relative to an arbitrary truth medium *A*, and practices of categorization of possibility $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. As a naming convention, we name those assumptions involving just $\mathcal{T}_M$ with a “*A*” prefix, and we name those involving both $\mathcal{T}_M$ and $\mathcal{T}_{\Sigma}$ with a “*B*” prefix.

Assumption 1a: “Any individual model represents a logically possible configuration of the world”

We can restate this as the equivalent: “Every model could be accurate,” that is, relative to each model, there is some possibility in which it would be accurate. Recall that for any model *m*, $\mathcal{T}_{M_A}(m)$ is the set of possibilities in which *m* is accurate. So we can write this assumption as follows:

$$A5 : \forall m \in M_A, \mathcal{T}_{M_A}(m) \neq \emptyset$$

Assumption 1b: “Any two (non-isomorphic) models are logically incompatible; at most one can be accurate”

For now, we leave out treatment of isomorphic models. We can restate this as the equivalent: “There is no possibility in which two distinct models are accurate.” So we can write this assumption as follows:

$$A4 : \forall m, n \in M_A, m \neq n \rightarrow (\mathcal{T}_{M_A}(m) \cap \mathcal{T}_{M_A}(n) = \emptyset)$$

Alternatively we can write the contrapositive

$$A4 : \forall m, n \in M_A, (\mathcal{T}_{M_A}(m) \cap \mathcal{T}_{M_A}(n) \neq \emptyset) \rightarrow m = n$$

Assumption 1c: “Jointly, (the models) are meant to represent all of the possibilities relevant to the truth values of (assertion types) in the (medium)”

Taken together with *A4* and *A5*, Assumption 1c is to ensure that “the models impose an exhaustive partition on the possible circumstances that could influence the truth of our (assertion types)” (Etchemendy 1990, 25).

This assumption as written permits of a number of interpretations. The simplest is that the set of possibilities represented by any model (considered collectively) equals the entire space of relevant possibility. Formally, we could express this as follows:

$$A6 : \bigcup_{m \in M_A} (\mathcal{T}_{M_A}(m)) = \mathcal{U}$$

A notational consideration: From this point forward in our presentation, we will use function image notation to express sets like $\bigcup_{m \in M_A} (\mathcal{T}_{M_A}(m))$ above. It will simplify the readability of many of the proofs to come. Given a function f and a subset c_0 of the domain of f , the image of c_0 under f is $\{f(a) \mid a \in c_0\}$ which we write as $f[c_0]$. Using function image notation, we can write assumption *A6* as

$$A6 : \bigcup \mathcal{T}_{M_A}[M_A] = \mathcal{U}$$

Assumption *A6* is definitely **not** Etchemendy’s desired interpretation. For as we will see below, this condition, when taken together with the other assumptions Etchemendy proposes (*A4*, *A5* from above and *B5* introduced below), is not sufficient to guarantee the intelligibility of consequence condition (*ICC*). (See Proposition 16).

So let’s look for another interpretation; one that does work. That is, we are looking for an interpretation of assumption 1c, such that that interpretation, taken together with *A4*, *A5*, and *B5* is sufficient to guarantee the intelligibility of consequence (*ICC*) condition. Consider taking the requirement that “models represent all of the possibilities relevant to the truth values of assertion types” to apply to each assertion type on a one-by-one basis. Let’s explore what this means in detail.

We have been given a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. For each assertion type $\sigma \in \Sigma_A$, we will consider two sets: 1) the set of possibilities represented by σ (that is, the set of possibilities in which σ would be true, formally $\mathcal{T}_{\Sigma_A}(\sigma)$); and 2) the set of possibilities not represented by σ (that is, the set of possibilities in which σ would be false, formally $\mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$). Our proposed assumption will require that, for every assertion type σ in the medium A , there be a set of models collectively representing exactly those possibilities in $\mathcal{T}_{\Sigma_A}(\sigma)$, and a set of models collectively representing exactly those possibilities in $\mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$. The entire assumption could be written formally as:

$$B8 : \forall \sigma \in \Sigma_A \left(\begin{array}{l} (\exists S \subseteq M_A \text{ such that } \bigcup \mathcal{T}_{M_A}[S] = \mathcal{T}_{\Sigma_A}(\sigma)) \text{ and} \\ (\exists R \subseteq M_A \text{ such that } \bigcup \mathcal{T}_{M_A}[R] = \mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)) \end{array} \right)$$

When considered relatively to some truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that A satisfies $A4$ and $A5$ with respect to $\mathcal{T}_{M_A}$, assumption $B8$ is sufficient to assure that, for every assertion type σ in the truth medium A , there exist two sets of models: one (S) which imposes a partition (via $\mathcal{T}_{M_A}$) on the possibilities in which σ can be true, and another (R) which imposes a partition (via $\mathcal{T}_{M_A}$) on the possibilities in which σ can be false. Formally, $\mathcal{T}_{M_A}[S]$ is a partition of $\mathcal{T}_{\Sigma_A}(\sigma)$, and $\mathcal{T}_{M_A}[R]$ is a partition of $\mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$. As we will see in the proofs below, the conjunction of assumptions $B8$ with $A4$, $A5$, and $B5$ is sufficient to imply the intelligibility of consequence condition. (Spelling this result out fully: given a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumptions $A4$, $A5$, $B5$, and $B8$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then A satisfies ICC with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.) We will accept $B8$ as our formal interpretation of Assumption 1c.

Assumption 2: “An (assertion type) σ should be true-in-a-model (with respect to) m iff σ would be true if the world were as depicted by m , that is, if m were an accurate model”

In the formalization we have introduced:

- that σ is true-in-a-model with respect to m , is expressed by $m \in t_A(\sigma)$

- that the world is as depicted by m , would be expressed by claiming that the possibility which is actual is an element of $\mathcal{T}_{M_A}(m)$
- that σ is true, would be expressed by claiming that the possibility which is actual is an element of $\mathcal{T}_{\Sigma_A}(\sigma)$

We can then formalize the phrase: “ σ would be true, if the world were as depicted by m ” as follows: $\forall u \in \mathcal{U}, u \in \mathcal{T}_{M_A}(m) \rightarrow u \in \mathcal{T}_{\Sigma_A}(\sigma)$, which is equivalent to $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma)$. This allows us to formalize the complete assumption as follows:

$$B5 : \forall \sigma \in \Sigma_A \forall m \in M_A (m \in t_A(\sigma) \text{ iff } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma))$$

Sometimes we will wish to split this assumption into two independent assumptions:

$$B2 : \forall \sigma \in \Sigma_A \forall m \in M_A (\text{ if } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma), \text{ then } m \in t_A(\sigma))$$

$$B3 : \forall \sigma \in \Sigma_A \forall m \in M_A (\text{ if } m \in t_A(\sigma), \text{ then } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma))$$

$B2$ is the claim that if the possibilities represented by a model m are within the possibilities represented by an assertion type σ , then σ is true-in-a-model with respect to m .

$B3$ is the claim that if an assertion type σ is true-in-a-model with respect to a model m , then the possibilities represented by m are within the possibilities represented by σ .

Assumption Set E

We can now assemble the elements of assumption Set E in one place.

Assumption Set E consists of:

- $A5 : \forall m \in M_A, \mathcal{T}_{M_A}(m) \neq \emptyset$
- $A4 : \forall m, n \in M_A, m \neq n \rightarrow (\mathcal{T}_{M_A}(m) \cap \mathcal{T}_{M_A}(n) = \emptyset)$
- $B8 : \forall \sigma \in \Sigma_A \left(\begin{array}{l} (\exists S \subseteq M_A \text{ such that } \bigcup \mathcal{T}_{M_A}[S] = \mathcal{T}_{\Sigma_A}(\sigma)) \text{ and} \\ (\exists R \subseteq M_A \text{ such that } \bigcup \mathcal{T}_{M_A}[R] = \mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)) \end{array} \right)$
- $B5 : \forall \sigma \in \Sigma_A \forall m \in M_A (m \in t_A(\sigma) \text{ iff } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma))$

We note that assumption $A5$ says that the every model represents at least one possibility; and that assumption $A4$ says that the possibilities represented by any pair of models are disjoint. This is sufficient to show that, for any truth medium A satisfying assumption Set

E , the set of models M_A will be logically transparent. The only relation of consequence between models is the reflexive case.

As a note, it may at first seem strange to talk about consequence between models. But on the representational view, models are full-fledged representations with their own practice of categorization of possibility, so there is an assumed relation of logical consequence between models just like the one between assertion types. We can express that relation as follows: Given some truth medium A , for any two models $m, n \in M_A$, n is a consequence of m just in case $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{M_A}(n)$.

Looking more closely at assumption $B8$, we can see that it implies that there exists a $T \subseteq M_A$ such that $\bigcup \mathcal{T}_{M_A}[T] = \mathcal{U}$. This is a strong assumption. It implies that every possibility is covered by some model.

2.3.6 Assumption Set PP

As stated above, assumption Set PP is a derivative of assumption Set E . To form the Set PP assumptions, we are going to take the Set E assumptions, and relax $B8$, replacing it with $B6$.

Assumption Set PP first form:

- $A5 : \forall m \in M_A, \mathcal{T}_{M_A}(m) \neq \emptyset$
- $A4 : \forall m, n \in M_A, m \neq n \rightarrow (\mathcal{T}_{M_A}(m) \cap \mathcal{T}_{M_A}(n) = \emptyset)$
- $B6 : \forall \sigma \in \Sigma_A (\exists S \subseteq M_A \text{ such that } \bigcup \mathcal{T}_{M_A}[S] = \mathcal{T}_{\Sigma_A}(\sigma))$
- $B5 : \forall \sigma \in \Sigma_A \forall m \in M_A (m \in t_A(\sigma) \text{ iff } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma))$

$B6$ is the new assumption here, replacing $B8$. It says that, for every assertion type σ , there is a set of models such that the set of possibilities represented collectively by those models is equal to the set of possibilities represented by σ . It is just the half of $B8$ talking about the possibilities represented by σ (versus the half of $B8$ talking about the possibilities NOT represented by σ). The above set is equivalent to the following set, which is the one we will use.

Assumption Set PP second form:

- $A5 : \forall m \in M_A, T_{M_A}(m) \neq \emptyset$
- $A4 : \forall m, n \in M_A, m \neq n \rightarrow (T_{M_A}(m) \cap T_{M_A}(n) = \emptyset)$
- $B1 : \forall \sigma \in \Sigma_A (\bigcup T_{M_A} [t_A(\sigma)] = T_{\Sigma_A}(\sigma))$

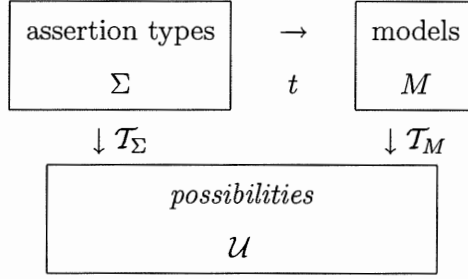
Proposition 1 *The two forms of Set PP assumptions are equivalent. See proof at the end of the chapter.*

$B1$ is the new assumption here. It says that, for every assertion type σ , the set of possibilities represented by the models of σ is equal to the set of possibilities represented by σ itself.

We can immediately notice two things about assumption Set PP . First, there is no longer an assumption that every possibility is represented by some model. The requirements imposed on the models by the assumptions in Set PP involve only those possibilities for which some assertion type is true. This is epistemologically less demanding than Set E , since it does not require the user of the technique to construct models for possibilities which no assertion type in the language under consideration can represent. Secondly, we can immediately see that since Set PP includes assumptions $A5$ and $A4$, it is the case that for any truth medium A satisfying assumption Set PP , the set of models M_A will be logically transparent.

Understanding assumption $B1$

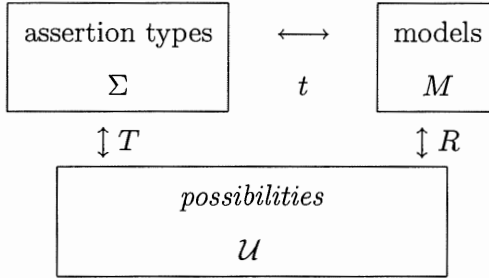
We can look at assumption $B1$ from another perspective, one which may make its role clearer. We can think about the model-theoretic semantic picture as having three components: sentences, models, and possibilities. Given some truth medium with practices of categorization $\mathcal{T}_\Sigma$ and $\mathcal{T}_M$, we can see the following structure: the assertion types in Σ are linked to the models in M by the function t , the models in M to possibilities in $\mathcal{U}$ by the function $\mathcal{T}_M$, and the assertion types in Σ to the possibilities in $\mathcal{U}$ by the function $\mathcal{T}_\Sigma$. We repeat the picture from Section 2.3.3.



We can rewrite the various maps as relations. So consider

1. a relation T on $\mathcal{U} \times \Sigma$ such that $u T p$ iff $u \in \mathcal{T}_\Sigma(p)$;
2. a relation R on $\mathcal{U} \times M$ such that $u R m$ iff $u \in \mathcal{T}_M(m)$;
3. a relation t on $M \times \Sigma$ such that $m t p$ iff $m \in t(p)$.

This allows us to redraw the picture as follows:



Then we can express a constraint:

T = the relational composition of t and R in that order: i.e.

$u T p$ iff there exists an m such that $u R m$ and $m t p$

This constraint is a principle of commutation for relations. And it is equivalent to assumption $B1$.¹

2.3.7 Assumption Set BE

Assumption Set BE adds a new wrinkle. The Set BE technique will require that the set of models in the truth medium be partially ordered. For medium A , we indicate the ordering on M_A as follows: $\preceq_{M_A}$.

The Set BE assumptions can be expressed as follows:

¹This correspondence was pointed out to me by Johan van Benthem.

- $A7 : \forall m, n \in M_A, \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{M_A}(n) \text{ iff } n \preceq_{M_A} m$
- $B1 : \forall \sigma \in \Sigma_A (\bigcup \mathcal{T}_{M_A} [t_A(\sigma)] = \mathcal{T}_{\Sigma_A}(\sigma))$
- $B2 : \forall \sigma \in \Sigma_A \forall m \in M_A (\text{ if } \mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma), \text{ then } m \in t_A(\sigma))$

In this case, the assumption $A7$ ensures that the set of models is logically simple. One model n is a consequence of another model m just in case $n \preceq_{M_A} m$. So there are non-trivial consequence relations, but the relations that do exist fit a specific simple and specified pattern. (For most interesting sets of assertion types, the consequence relation is more complex than a partial order). An example of a set of models which would satisfy the assumption $A7$, would be a set of partial models, for which the ordering $\preceq_M$ was the subsumption relation (so $n \preceq_{M_A} m$ would indicate that m was more articulated than n).

As we will show later, the second two assumptions: $B1$ and $B2$, are sufficient, by themselves, to ensure the intelligibility of consequence. So this assumption set is different than the first two. For assumption Sets E and PP , some of the assumptions involved in ensuring intelligibility of consequence (i.e. $A4$ and $A5$) also grounded the explanatory strategy. Here, one assumption ($A7$) grounds the explanatory strategy, and others ($B1$ and $B2$) ensure the intelligibility of consequence. Note that assumptions $B1$ and $B2$ alone would not ground the explanatory strategy. Taken alone, they do not ensure that the set of models is logically simple.

The Set BE technique will be easier to use than the other model-theoretic techniques when the natural set of models has the structure of a partial order, in which the ordering completely characterizes consequence among the models (i.e. when the natural set of models satisfies $A7$). We will see this in the example of applying the representational schema to feature logics (in Chapter 3), and also in the proof showing that the range of application of the Set CG technique is included within the range of the Set BE technique (Chapter 8).

2.4 Proofs that theories developed with the model-theoretic representational techniques presented do in fact make consequence intelligible

In order to show that that theories constructed with any of the model-theoretic representational techniques Set E , Set PP , and Set BE make consequence intelligible, it is sufficient to show the following results:

- For any truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,
 1. if A satisfies assumptions $B1$ and $B2$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then the proxy for consequence relation (MC_A) is equivalent to the representational conception of the logical consequence relation $\left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle} \right)$. This is the central proof.
 2. if A satisfies assumption Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then A satisfies assumptions $B1$ and $B2$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
 3. if A satisfies assumption Set E with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then A satisfies assumption Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

We show the first claim immediately. Following that we will introduce some additional named assumptions, identify implications between named assumptions, and then show the other two.

2.4.1 Additional named assumption sets

For ease of reference, we name two additional assumption sets. But note that these do not correspond to representational techniques.

Set B is assumptions $B1$ and $B2$.

Set LT is assumptions $B5$ and $A6$.

2.4.2 Intelligibility of consequence theorem for truth media (Set B version).

Theorem 2 *For any truth medium A satisfying assumption set B ($B1$ and $B2$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, it is the case that the proxy for consequence relation (MC_A) is equivalent to the representational conception of the logical consequence relation ($ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$) (that is, $\forall p, q \in \Sigma_A, t_A(p) \subseteq t_A(q)$ iff $\mathcal{T}_{\Sigma_A}(p) \subseteq \mathcal{T}_{\Sigma_A}(q)$).*

Proof:

Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization such that A satisfies assumptions $B1$ and $B2$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Let p, q be arbitrary elements of Σ_A .

($\rightarrow$)

1. Assume that $t_A(p) \subseteq t_A(q)$.
2. Let $u \in \mathcal{T}_{\Sigma_A}(p)$.
3. By $B1$, $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$.
4. So there is some $m \in t_A(p)$ such that $u \in \mathcal{T}_{M_A}(m)$.
5. Since $t_A(p) \subseteq t_A(q)$, $m \in t_A(q)$, and therefore $u \in \bigcup \mathcal{T}_{M_A}[t_A(q)]$.
6. By $B1$, $\bigcup \mathcal{T}_{M_A}[t_A(q)] = \mathcal{T}_{\Sigma_A}(q)$.
7. So $u \in \mathcal{T}_{\Sigma_A}(q)$. ■ ($\rightarrow$)

($\leftarrow$)

1. Assume that $\mathcal{T}_{\Sigma_A}(p) \subseteq \mathcal{T}_{\Sigma_A}(q)$.
2. Let $m \in t_A(p)$.
3. Since $m \in t_A(p)$, $\mathcal{T}_{M_A}(m) \subseteq \bigcup \mathcal{T}_{M_A}[t_A(p)]$.
4. By $B1$, $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$; so $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(p)$.
5. Since $\mathcal{T}_{\Sigma_A}(p) \subseteq \mathcal{T}_{\Sigma_A}(q)$, we know $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(q)$.
6. By $B2$, we know $m \in t_A(q)$. ■ ($\leftarrow$) ■

2.4.3 Additional named assumptions

We introduce some additional named assumptions.

Assumption A2

$$A2 : \forall x \in M_A (\exists y \in \mathcal{T}_{M_A}(x) \text{ such that } \forall z \in M_A (y \in \mathcal{T}_{M_A}(z) \rightarrow z = x))$$

Every model in M_A has at least one possibility that it uniquely represents.

Assumption B4

$$B4 : \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] = \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A]$$

The possibilities represented by any model of any sentence equals the possibilities represented by any sentence.

Assumption B7

$$B7 : \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] \subseteq \bigcup \mathcal{T}_{M_A} [M_A]$$

The possibilities represented by any sentence are a subset of the possibilities represented by any model.

2.4.4 Relations of implication between named assumptions

The following relations of implication among the named assumptions can be easily identified. Proofs can be found at the end of the chapter.

Proposition 3 *B8 implies B6. Immediate from definitions.*

Proposition 4 *B8 implies A6.*

Proposition 5 *A4, A5 imply A2.*

Proposition 6 *B1 implies B4.*

Proposition 7 *B4 implies B7. Immediate from the definitions, since $\bigcup t_A [\Sigma_A] \subseteq M_A$, so $\bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] \subseteq \bigcup \mathcal{T}_{M_A} [M_A]$.*

Proposition 8 *B1 implies B7. By transitivity.*

Proposition 9 *B1 implies B3. Immediate from the fact that if $m \in t_A(\sigma)$, then $T_{M_A}(m) \subseteq \bigcup T_{M_A}[t_A(\sigma)]$.*

Proposition 10 *(B2 and B3 and B6) imply B1.*

Proposition 11 *(B2 and B3) iff B5. Immediate from the definitions.*

Proposition 12 *(B1 and A2) imply B2. (Proposition)*

2.4.5 Relations of implication between assumption sets

In this section, we demonstrate various relations which exist between the named proposition sets.

Proposition 13 *Based on the above propositions, we can see the following relations of implication between the assumption sets presented.*

For any truth medium A , and practices of categorization T_{Σ_A} and T_{M_A} interpreting A ,

- Set E ($B5, A4, A5, B8$) implies Set PP ($B1, A4, A5$).

- (a) $B8$ implies $B6$ by Proposition 3.
- (b) $B5$ implies $B2$ and $B3$ by Proposition 11.
- (c) $B2, B3, B6$ imply $B1$ by Proposition 10.

- Set PP ($B1, A4, A5$) implies Set B ($B1, B2$)

- (a) $A4, A5$ imply $A2$ by Proposition 5.
- (b) $B1, A2$ imply $B2$ by Proposition 12.

- Set BE ($B1, B2, A7$) implies Set B ($B1, B2$).

- Set E ($B5, A4, A5, B8$) implies set LT ($B5, A6$).

- (a) $B8$ implies $A6$ by Proposition 4.

Sets **E**, **PP** are each individually sufficient to imply the intelligibility of consequence

Corollary 14 *For any truth medium A satisfying assumption Set E (or assumption Set PP) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, it is the case that MC_A is equivalent to $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.*

This is immediate from Theorem 2 and Proposition 13 establishing the relations among assumption sets.

This completes the proofs that theories constructed with any of the model-theoretic representational techniques Set E , Set PP , and Set BE make consequence intelligible.

2.5 Relative ranges of applicability among the model-theoretic representational techniques

We can show the following relationships between the ranges of applicability of the model-theoretic representational techniques we have introduced:

Set E technique $\subseteq$ Set PP technique $\subseteq$ Set BE technique

This ordering indicates our current knowledge. When we say that the range of applicability of the Set X technique is a subset of the range of applicability of the Set Y technique, what we are saying is that we know that every interpreted set of assertion types for which an application of the Set X technique can make the representational relation of consequence explanatorily intelligible is an interpreted set of assertion types to which an application of the Set Y technique can do the same.

Since the Set E assumptions imply the Set PP assumptions, we can show that every interpreted set of assertion types in the range of applicability (remember this a technical term defined in Section 1.7.2) of the Set E technique is a member of the range of applicability of the Set PP technique as well.

Proposition 15 *The range of applicability of the Set E technique is included in the range of applicability of the Set PP technique.*

1. Let $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ be an interpreted set of assertion types in the range of applicability for the Set *E* technique.
2. Then there is a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that A satisfies the Set *E* assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$ is embedded in $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.
3. In that case, by Proposition 13, A satisfies the Set *PP* assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Then it is the case that $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ is in the range of applicability of the Set *PP* technique.
4. So the range of applicability of the Set *E* technique is a subset of the range of applicability of the Set *PP* technique.

The other result, that the range of application for the Set *PP* technique is a subset of the range of application for the Set *BE* technique, will be shown by two later proofs. In Chapter 6 we will show that the range of applicability of the Set *PP* technique is a subset of the range of applicability of the Set *CG* technique, and in Chapter 8 we will show that the range of applicability of the Set *CG* technique is a subset of the range of applicability of the Set *BE* technique.

2.6 Model-theoretic representational techniques and logical truth

2.6.1 Introduction

In this section, we show that it is possible to model the representational conception of logical truth analogously to the way we modelled the representational conception of logical consequence. We have already chosen to characterize the representational conception of logical truth as truth in all possibilities. In the context of an interpreted truth medium (that is, a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting A) there is a proxy predicate for logical truth, such that if certain sets of assumptions are satisfied, that

proxy predicate is true of an assertion type just in case that assertion type is a representational logical truth. That is, under those conditions, the proxy predicate will make the representational conception of logical truth intelligible.

2.6.2 The proxy for logical truth predicate MT

We will have already informally identified just such a proxy predicate (the predicate MT). Our informal characterization of the predicate MT was that MT held of a sentence just in case the sentence was true-in-a-model for all models. Given our definitions so far, we could define MT_A (MT for truth medium A) formally as:

For any assertion type $p \in \Sigma_A$, $MT_A(p)$ if and only if $t_A(p) = M_A$.

2.6.3 The representational concept of logical truth

In our informal discussion, we chose to model the representational conception of logical truth as truth in all possibilities. We can capture this understanding in a formal definition as follows. For set of assertion types Σ related to a space of relevant possibility $\mathcal{U}$ via a practice of categorization $\mathcal{T}_\Sigma$,

for any element p of Σ , p is a logical truth iff $\mathcal{T}_\Sigma(p) = \mathcal{U}$.

We will define a predicate $ALT_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$ (Assumed Logical Truth) on Σ as follows:

$\forall p \in \Sigma$, $ALT_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p)$ iff $\mathcal{T}_\Sigma(p) = \mathcal{U}$.

The subscripting on ALT is important, because every assumed concept of logical truth is relative, not just to some set of assertion types Σ , but also to the practice of categorization of possibility $\mathcal{T}_\Sigma$ by which the elements of Σ are interpreted. That is, our assumed concept of logical truth is relativized to interpreted sets of assertion types. One could argue that the concept of logical truth is also relative to the space of relevant possibility $\mathcal{U}$, but that involves issues we choose not to consider at this time. For a discussion of some of the issues involved here, see Barwise 1999. For purposes of our discussion, we consider all instances of the assumed concept of logical truth to refer to a common space of relevant possibility.

2.6.4 Goal condition

We can express the goal condition for the **intelligibility of logical truth** (*ITC*) relative to a truth medium A and practice of categorization $\mathcal{T}_{\Sigma_A}$ as follows:

$$MT_A = ALT_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}.$$

Expanding defined predicates, *ITC* is: $\forall p \in \Sigma_A, t_A(p) = M_A$ iff $\mathcal{T}_{\Sigma_A}(p) = \mathcal{U}$.

We will use this goal condition like we used the intelligibility of logical consequence condition (*ICC*), that is, a set of assumptions are sufficient for the intelligibility of logical truth, just in case, for every truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies that assumption set with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then *ITC* is true.

2.6.5 Results about the intelligibility of logical truth

In the proofs at the end of the chapter, we show that given a truth medium, either of the assumption sets *LT* or *E*, are sufficient to make logical truth intelligible via the proxy relation *MT* (Propositions 18 and 19); but that neither *PP* nor *BE* are (Propositions 20 and 21). We have already shown that given a truth medium, any of the assumption sets *E*, *PP*, or *BE*, in conjunction with the relation *MC* are sufficient to ensure the intelligibility of logical consequence (Propositions 2 and 14); whereas below we show that assumption set *LT* is not (Proposition 17). This highlights an important point, that logical truth and logical consequence can be made intelligible independently of one another.

2.6.6 We can usefully study logical consequence independently of logical truth

Knowledge of logical consequence is epistemologically significant even in the absence of knowledge of logical truth. Our investigations above show how we can study the construction of intelligible proxies for logical consequence separately from the construction of intelligible proxies for logical truth. For the remainder of this dissertation (with the exception of an appendix), we will focus on the relation of logical consequence.

This concludes the linear text of the chapter. The remainder of the material in this chapter are the proofs of propositions mentioned earlier in the text.

2.7 Proofs of supporting propositions

2.7.1 Proofs relating named assumptions

In this section, we give proofs of supporting propositions mentioned in the text of the chapter.

Proof of Proposition 1: The two versions of Set PP are equivalent.

Version 1 = $\{A5, A4, B6, B5\}$; Version 2 = $\{A5, A4, B1\}$

(1 $\rightarrow$ 2)

Let A be a truth medium satisfying $\{A5, A4, B6, B5\}$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

1. $B5$ implies $B2$ and $B3$ by Proposition 11.
2. $B2, B3, B6$ imply $B1$ by Proposition 10. ■ (1 $\rightarrow$ 2)

(2 $\rightarrow$ 1)

Let A be a truth medium satisfying $\{A5, A4, B1\}$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

1. $B1$ implies $B3$ by Proposition 9.
2. $A4, A5$ implies $A2$ by Proposition 5.
3. $B1, A2$ implies $B2$ by Proposition 12.
4. $B2, B3$ implies $B5$ by Proposition 11.
5. Let $\sigma \in \Sigma_A$. By $B1$, $\bigcup \mathcal{T}_{M_A} [t_A(\sigma)] = \mathcal{T}_{\Sigma_A}(\sigma)$.
6. So $S = t_A(\sigma)$ satisfies $B6$. ■ (2 $\rightarrow$ 1)

Proof of Proposition 4: $B8$ implies $A6$.

Let A be an arbitrary truth medium satisfying $B8$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Show that A satisfies $A6$ with respect to $\mathcal{T}_{M_A}$.

Proof:

1. [Show $\bigcup \mathcal{T}_{M_A} [M_A] = \mathcal{U}$]

2. ($\subseteq$)

(a) Immediate since $\mathcal{T}_{M_A} : M_A \rightarrow \mathcal{P}(\mathcal{U})$. ■ ($\subseteq$)

3. ($\supseteq$)

(a) Let $u \in \mathcal{U}$.

(b) Let $\sigma \in \Sigma_A$. We know such a σ exists since A is a truth medium.

(c) Since $\mathcal{T}_{\Sigma_A}(\sigma) \subseteq \mathcal{U}$, either $u \in \mathcal{T}_{\Sigma_A}(\sigma)$ or $u \in \mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$.

(d) Case 1. $u \in \mathcal{T}_{\Sigma_A}(\sigma)$

i. By B8, $\exists S \subseteq M_A$ such that $\bigcup \mathcal{T}_{M_A}[S] = \mathcal{T}_{\Sigma_A}(\sigma)$.

ii. So there exists an $s \in M_A$, such that $u \in \mathcal{T}_{M_A}(s)$.

iii. So $u \in \bigcup \mathcal{T}_{M_A}[M_A]$ ■ (Case1)

(e) Case 2. $u \in \mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$

i. By B8, $\exists R \subseteq M_A$ such that $\bigcup \mathcal{T}_{M_A}[R] = \mathcal{U} - \mathcal{T}_{\Sigma_A}(\sigma)$.

ii. So there exists an $r \in M_A$, such that $u \in \mathcal{T}_{M_A}(r)$.

iii. So $u \in \bigcup \mathcal{T}_{M_A}[M_A]$ ■ (Case2) ■ ($\supseteq$) ■ Proposition.

Proof of Proposition 5: A4, A5 implies A2.

Let A be an arbitrary truth medium satisfying A4 and A5 with respect to practice of categorization $\mathcal{T}_{M_A}$.

Show that A satisfies A2 with respect to practice of categorization $\mathcal{T}_{M_A}$.

Proof:

Let $x \in M_A$. By A5, $\mathcal{T}_{M_A}(x) \neq \emptyset$. Pick an arbitrary element y of $\mathcal{T}_{M_A}(x)$. Let z be an element of M_A such that $y \in \mathcal{T}_{M_A}(z)$. Then $z = x$ by A4. ■ Proposition.

Proof of Proposition 6: B1 implies B4.

1. Let A be an arbitrary truth medium satisfying B1 with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

2. $\left[\begin{array}{l} \text{Show that } A \text{ satisfies } B4 \text{ with respect to practices of categorization } \mathcal{T}_{\Sigma_A} \text{ and } \mathcal{T}_{M_A}, \text{ that is,} \\ \text{show } \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] = \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] \end{array} \right]$

3. $(\subseteq)$

- (a) Let $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]]$.
- (b) Then there is an $m \in \bigcup t_A [\Sigma_A]$ such that $u \in \mathcal{T}_{M_A}(m)$.
- (c) Then there is a $\sigma \in \Sigma_A$ such that $m \in t_A(\sigma)$.
- (d) So $u \in \bigcup \mathcal{T}_{M_A} [t_A(\sigma)]$.
- (e) By $B1$, $u \in \mathcal{T}_{\Sigma_A}(\sigma)$.
- (f) So $u \in \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A]$.

4. $\supseteq$

- (a) Let $u \in \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A]$.
- (b) So there is some $\sigma \in \Sigma_A$ such that $u \in \mathcal{T}_{\Sigma_A}(\sigma)$.
- (c) By $B1$, $u \in \bigcup \mathcal{T}_{M_A} [t_A(\sigma)]$.
- (d) So there is an $m \in t_A(\sigma)$ such that $u \in \mathcal{T}_{M_A}(m)$.
- (e) Since $\sigma \in \Sigma_A$, $m \in \bigcup t_A [\Sigma_A]$.
- (f) So $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]]$. ■ Proposition.

Proof of Proposition 10: $B2, B3, B6$ imply $B1$

Let A be an arbitrary truth medium satisfying $B2, B3$, and $B6$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Show that A satisfies $B1$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Proof:

Let σ be an arbitrary element of Σ_A .

$(\subseteq)$

1. Let $u \in \bigcup \mathcal{T}_{M_A} [t_A(\sigma)]$.

2. So there is some $m \in t_A(\sigma)$ such that $u \in \mathcal{T}_{M_A}(m)$.

3. By B3, $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma)$. ■ ($\subseteq$)

($\supseteq$)

1. Let $u \in \mathcal{T}_{\Sigma_A}(\sigma)$.

2. So by B6, there exists $\exists S \subseteq M_A$ such that $\bigcup \mathcal{T}_{M_A}[S] = \mathcal{T}_{\Sigma_A}(\sigma)$.

3. So there is some $m \in S$ such that $u \in \mathcal{T}_{M_A}(m)$.

4. Since $m \in S$ we know $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma)$.

5. By B2, $m \in t_A(\sigma)$. ■ ($\supseteq$) ■ Proposition.

Proof of Proposition 12: B1, A2 imply B2.

Let A be an arbitrary truth medium satisfying B1, A2 with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Show that A satisfies B2 with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Proof:

Let σ be an element of Σ_A , and let m be an element of M_A such that $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma)$.

[Show $m \in t_A(\sigma)$]

1. Since A satisfies A2 with respect to $\mathcal{T}_{M_A}$, we know that $\exists y \in \mathcal{T}_{M_A}(m)$ such that $\forall z \in M_A (y \in \mathcal{T}_{M_A}(z) \rightarrow z = m)$. Pick one such y .

2. Since $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(\sigma)$, $y \in \mathcal{T}_{\Sigma_A}(\sigma)$.

3. Since A satisfies B1 with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know $\bigcup \mathcal{T}_{M_A}[t_A(\sigma)] = \mathcal{T}_{\Sigma_A}(\sigma)$.

4. So there is an $n \in t_A(\sigma)$ such that $y \in \mathcal{T}_{M_A}(n)$.

5. So by above, $m = n$.

6. So $m \in t_A(\sigma)$. ■ Proposition.

2.7.2 Proofs that certain sets of assumptions are insufficient for the intelligibility of logical consequence

Proposition 16 *The conjunction of assumptions $B4, B5, A4, A5, A6$ is insufficient to imply the intelligibility of consequence.*

Here we construct a case in which a truth medium A satisfies the assumptions $B4, B5, A4, A5$ and $A6$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and yet the intelligibility of consequence condition fails.

Consider the following:

$$\Sigma_A = \{p, q, r\}$$

$$M_A = \{m, n, o\}$$

$$\mathcal{U} = \{1, 2, 3, 4, 5\}$$

Define $\mathcal{T}_{\Sigma_A}$ as follows:

x	$\mathcal{T}_{\Sigma_A}(x)$
p	$\{1, 2, 4, 5\}$
q	$\{2, 3\}$
r	$\{4\}$

$$\text{So } \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] = \{1, 2, 3, 4, 5\}$$

Define $\mathcal{T}_{M_A}$ as follows:

x	$\mathcal{T}_{M_A}(x)$
m	$\{1\}$
n	$\{2, 3\}$
o	$\{4, 5\}$

The above definitions satisfy assumptions $A4$ (the models are disjoint) and $A5$ (the models are consistent). $A6$ is satisfied as well.

We use the statement of $B5$ to construct t_A .

x	$t_A(x)$
p	$\{m, o\}$
q	$\{n\}$
r	$\emptyset$

$$\text{So } t_A [\Sigma_A] = \{m, n, o\} \text{ and therefore } \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] = \{1, 2, 3, 4, 5\} = \bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A].$$

Given the above definitions, assumptions $B4$ and $B5$ are satisfied.

Now we show that the intelligibility of consequence condition is NOT satisfied.

From the above, we can compute the relation $t_A(x) \subseteq t_A(y)$ as follows:

		y		
		p	q	r
x	p	t	f	f
	q	f	t	f
	r	t	t	t

Whereas the relation $\mathcal{T}_{\Sigma_A}(x) \subseteq \mathcal{T}_{\Sigma_A}(y)$ is as follows:

		y		
		p	q	r
x	p	t	f	f
	q	f	t	f
	r	t	f	t

Consider the ordered pair $\langle r, q \rangle$. $t_A(r) \subseteq t_A(q)$, but $\mathcal{T}_{\Sigma_A}(r) \not\subseteq \mathcal{T}_{\Sigma_A}(q)$. Since the above example satisfied $B4, B5, A4, A5$ and $A6$, we conclude that the conjunction of these assumptions is NOT sufficient to guarantee the intelligibility of consequence condition.

Corollary 17 *Set LT is not sufficient to imply the intelligibility of consequence condition.*

2.7.3 Proofs indicating which sets of assumptions are sufficient or insufficient for intelligibility of logical truth

Assumption sets which are sufficient for the intelligibility of logical truth

This theorem demonstrates that LT is sufficient for the intelligibility of logical truth (ITC).

Proposition 18 *For any truth medium A satisfying assumption set LT ($B5$ and $A6$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, it is the case that that ITC is true, that is, that $MT_A = ALT_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.*

1. Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization such that A satisfies assumptions $B5$ and $A6$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Let p be an arbitrary element of Σ_A .

2. $(\rightarrow)$

(a) Assume $t_A(p) = M_A$. [Show $\mathcal{T}_{\Sigma_A}(p) = \mathcal{U}$.]

(b) $(\subseteq)$

i. Immediate by definition of $\mathcal{T}_{\Sigma_A}$. ■ $(\subseteq)$

(c) $(\supseteq)$

i. Let $u \in \mathcal{U}$.

ii. By A6, $u \in \bigcup \mathcal{T}_{M_A} [M_A]$.

iii. So there exists an $m \in M_A$, such that $u \in \mathcal{T}_{M_A}(m)$.

iv. By $(\rightarrow)$ assumption, $m \in t_A(p)$.

v. By B5, $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(p)$.

vi. So $u \in \mathcal{T}_{\Sigma_A}(p)$. ■ $(\supseteq)$ ■ $(\rightarrow)$

3. $(\leftarrow)$

(a) Assume $\mathcal{T}_{\Sigma_A}(p) = \mathcal{U}$. [Show $t_A(p) = M_A$.]

(b) $(\subseteq)$

i. Immediate by definition of t_A . ■ $(\subseteq)$

(c) $(\supseteq)$

i. Let $m \in M_A$.

ii. $\mathcal{T}_{M_A}(m) \subseteq \mathcal{U}$, by definition of $\mathcal{T}_{M_A}$.

iii. So $\mathcal{T}_{M_A}(m) \subseteq \mathcal{T}_{\Sigma_A}(p)$, by $(\leftarrow)$ assumption.

iv. By B5, $m \in t_A(p)$. ■ $(\supseteq)$ ■ $(\leftarrow)$ ■ Theorem.

Corollary 19 *For any truth medium A satisfying assumption Set E with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, it is the case that ITC is true, that is, that $MT_A = ALT_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.*

This is immediate from Theorem 18 and Proposition 13.

Assumption sets which are not sufficient for the intelligibility of logical truth

Assumption set PP ($B1, A4, A5$) is not sufficient to imply the intelligibility of logical truth condition.

Proposition 20 *It is not the case that: for any truth medium A satisfying assumption set PP ($B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, that ITC is true, i.e. that $MT_A = ALT_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.*

We construct an example of a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set PP ($A1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and yet the intelligibility of logical truth condition (ITC) is false.

$$\Sigma_A = \{p\}$$

$$M_A = \{m, n, o\}$$

$$\mathcal{U} = \{1, 2, 3, 4, 5\}$$

Define $\mathcal{T}_{\Sigma_A}$ as follows:

x	$\mathcal{T}_{\Sigma_A}(x)$
p	$\{1, 2, 3, 4\}$

Define $\mathcal{T}_{M_A}$ as follows:

x	$\mathcal{T}_{M_A}(x)$
m	$\{1\}$
n	$\{2, 3\}$
o	$\{4\}$

The above definitions satisfy assumptions $A4$ (the models are disjoint) and $A5$ (the models are consistent).

We use the statement of $B1$ to construct t_A .

Construct t_A as follows:

x	$t_A(x)$
p	$\{m, n, o\}$

Given the above definitions, assumption $B1$ is satisfied, as $\bigcup \mathcal{T}_{M_A} [t_A(p)] = \{1, 2, 3, 4\} = \mathcal{T}_{\Sigma_A}(p)$.

In our example, $t_A(p) = M_A$, yet $\mathcal{T}_{\Sigma_A}(p) \neq \mathcal{U}$. So ITC is false. ■ Proposition.

Proposition 21 *It is not the case that: for any truth medium A satisfying assumption set $BE (B1, B2, A7)$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, that ITC is true, i.e. that $MT_A = ALT_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.*

We construct an example of a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set $BE (B1, B2, A7)$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and yet the intelligibility of logical truth condition (ITC) is false.

For this example, take the example from Proposition 20. Define the ordering $\preceq_{M_A}$ on M_A as the antichain ordering. Then the medium A given in the example satisfies $B1$, $B2$, and $A7$. Yet it does not satisfy ITC . ■ Proposition.

Chapter 3

Applications of Model-theoretic Representational Techniques

3.1 Introduction

In this chapter we will explore the application of model-theoretic representational techniques to three different interpreted languages: (1) a simple language meant to illustrate the basic concepts involved; (2) the language of propositional logic; and (3) the languages of feature logics (sentential languages with feature structures as models). As you will recall from Chapter 2, all the model-theoretic representational techniques we have defined share the same foundational characterization, form of intelligible medium, and intelligible proxy for consequence relation. They differ only in the sets of technique-specific assumptions used. We can therefore begin by presenting a general methodology for applying any of the defined model-theoretic representational techniques. Our next step will be to give a high-level overview of the example applications to be presented. For each application, we will explain (1) what the language(s) in question are; (2) which techniques we plan to use in building theories of consequence for those languages; and (3) why those applications are of interest. We will then proceed to apply the techniques. We will carry out complete applications in the case of the initial example and the language of propositional logic; and sketch the application in the case of the languages of feature logics.

3.2 A general methodology for applying model-theoretic representational techniques

In Section 1.5.1, we introduced a general methodology for applying instances of the representational schema. We can now specialize that methodology in the context of model-theoretic techniques. At the current time, we have defined three such techniques: the Set *E* technique, the Set *PP* technique, and the Set *BE* technique. The only variable element between these three techniques is the set of technique-specific assumptions which they employ. So we can describe a common methodology for applying any of these techniques.

Given a selected technique,

Step 1: Identify a set of assertion types for the language.

Step 2: Form a truth medium by supplementing the set of assertion types with a set of models, and a relation of truth-in-a-model between sentences and models.

Step 3: We have two kinds of representative elements in a truth medium: assertion types and models. Define semantic conventions describing acceptable interpretations for these elements.

Step 4: Prove that if a practice of categorization for assertion types and a practice of categorization for models satisfy the semantic conventions described for those kinds of practices in Step 3, then the truth medium we have constructed satisfies the technique-specific assumptions with respect to those practices.

Given that proof, we know that for every pair of acceptable interpretations for assertion types and models, the proxy relation of preservation of truth-in-a-model across all models is equivalent to the representational conception of logical consequence (preservation of truth across all possibilities). Further, because of the technique used, we know that the theory so constructed explains the consequence relation between assertion types in terms of the logically simple form of representation given by the models.

3.3 Overview of the applications

We will carry out or outline three applications of model-theoretic representational techniques. The first example will be a simple one, intended to bring out key concepts. The

second will be the language of propositional logic. The third will be the languages of feature logics.

3.3.1 The “roll of the die” example

The first application we will consider involves a very simple language used to describe the numerical value of a roll of a six-sided die. This language will only have three assertion types *odd*, *greaterThanFour*, and *five*; and a single acceptable interpretation. The simplicity of the language will make it easy for us to carry out the application; and also will demonstrate just how little it takes to get a model-theoretic semantics off the ground. We will use the Set E technique for this application.

One attribute of this application that is of special interest is that it demonstrates that the model-theoretic techniques we are considering have no requirement that the set of assertion types be “complete” in the sense of representing the space of relevant possibility in some regular way. For example, in the case of this language, we can say “odd” but we cannot say “not odd;” we can say “greater than four” but we cannot say “odd or greater than four.” For the model-theoretic techniques we have considered, the sets of technique-specific assumptions place no requirements on assertion types involving the possibilities represented by some other type or types. Call such requirements “**mutual constraints on assertion types.**” As an example of one such potential requirement consider: we could require, for every assertion type p , that there exist some assertion type q such that q represents all relevant possibilities not represented by p ; that is, the requirement that every assertion type have an absolute negation in the language.

As should be obvious (and as we will explore in some detail in Chapter 4), such mutual constraints on assertion types can play an important role in determining the usefulness of a language for various practical applications. As a matter-of-fact, most of our standard logical languages do obey such mutual constraints on assertion types and, as a result, represent the space of relevant possibility in a regular way. Such regularities will serve as part of the basis for the order-consistency representational techniques we shall discuss in Chapter 4.

3.3.2 The language of propositional logic

The second application we will consider is the language of propositional logic. The standard analysis of propositional logic is of course model-theoretic. By taking such a standard example, this application will help make the less-familiar concepts of our representational schema more clear.

As we mentioned above, the language of propositional logic, as defined, is only partially interpreted. The language characterizes a class of acceptable interpretations, but does not specify a single interpretation. We will use the concept of semantic conventions to describe the class of acceptable interpreting practices of categorization for the language of propositional logic. An important aspect of this application is that it will allow us to work through and apply a complete analysis of a set of semantic conventions.

An important attribute of the language of propositional logic (as used to represent and reason about arguments) is that it does obey mutual constraints on assertion types, and as such, does represent the space of relevant possibility in a regular way. We will explore some of these regularities in Chapter 4.

There are some technical attributes of this application that will be of special interest. For one thing, we consider the question of what counts as the appropriate set of assertion types for our application. Do we take the set of assertion types to be the set of propositional sentences, or the powerset of the set of propositional sentences? We will argue that we need to consider the latter case if we are to model the concept of logical consequence as we use it to represent and reason about arguments in propositional logic. Our application will actually be factored into two levels. First we will apply the Set E technique to a version of the propositional language which has single sentences for its assertion types; then we will use that construction as a basis for applying the same technique to a version of the propositional language which has sets of propositional sentences for its assertion types.

This two-level application will introduce two important ideas: the concept of **partially ordered truth media**, that is, a truth medium where the set of assertion types has a partial order defined upon it; and the concept of **operators on intelligible media**, that is, operators which take one or more interpreted intelligible media as arguments, and constructs from them an interpreted intelligible medium. Such operators will be important

to many of the proofs to come.

3.3.3 The languages of feature logics

Feature structures, or “**attribute-value matrices**,” are a certain kind of structure enabling an organized presentation of information. An example feature structure is shown here. This example is from Rounds 1997.

$$\left[\begin{array}{c} \text{vp} \\ \text{AGR [1]} \left[\begin{array}{c} \text{NUM sing} \\ \text{PERS 3rd} \end{array} \right] \\ \text{SUBJ[1]} \end{array} \right]$$

Informally, feature structures work by allowing information to be expressed as the values of features. There are various types of features (e.g. “*AGR*,” “*NUM*,” in the above). Features take values (e.g. “sing”). The value of a feature can itself be a set of features with values (e.g. “ $\left[\begin{array}{c} \text{NUM sing} \\ \text{PERS 3rd} \end{array} \right]$ ”). Parts of the structure can be shared (as indicated by the “[1]” above, the *AGR* feature and the *SUBJ* feature share the same value, namely “ $\left[\begin{array}{c} \text{NUM sing} \\ \text{PERS 3rd} \end{array} \right]$ ”). Circularity in the structure is also possible. We will describe feature structures in more detail in the outline of the application below.

Structures of this kind have a rich tradition in linguistic theories (Chomsky 1957; Chomsky 1965; Kaplan and Bresnan 1982; Gazdar, Klein, Pullum, and Sag 1985; Pollard and Sag 1987) and have also been used for a number of computational frameworks (Kay 1979; Shieber 1986). More recently, feature structures are being used as the foundation for XML, the next general standard for information interchange on the Internet.

Feature logics define sentential languages which are interpreted with respect to feature structures. In a feature logic, (1) the assertion types are sets of sentences in a defined sentential language, (2) feature structures play the role of models, and (3) there is a definition of a relation of truth-in-a-feature-structure between sentences and feature structures. The relation of logical consequence is characterized by the preservation of truth-in-a-feature-structure across all feature structures (Rounds 1997, 506). The actual relation is slightly

more involved, due to the modal nature of the feature logic language; we will deal with this once we have had a chance to introduce feature structures more formally.

Here we have something very important to notice. From the representational perspective we have taken, this story giving an account of logical consequence for a feature logic is not complete. Given that we take the powerset of the set of sentences in the feature logic as the set of assertion types, and given that we take feature structures to be the models, and given that we take the relation of truth-in-a-feature-structure to play the role of the relation of truth-in-a-model, the relation of preservation of truth-in-a-feature-structure across all feature structures is the model-theoretic candidate for a proxy relation for the representational conception of logical consequence for the language of the feature logic. But what is missing... is a characterization of the semantic conventions of the language of the feature logic, and of the feature structures themselves, and a proof that given some acceptable (i.e. semantic conventions satisfying) practice of categorization interpreting the sentences of the language, and some acceptable practice of categorization interpreting the feature structures, the intelligible medium (powerset of set of sentences, feature structures, and definition of truth-in-a-feature structure) satisfies one of the technique-specific assumption sets with respect to those practices. Only then would we have a complete justification of the claim that the proposed proxy relation of preservation of truth-in-a-feature-structure across all feature structures is in fact equivalent to the representational conception of logical consequence for acceptable interpretations of the sentences and models of the logic. The most important attribute of this application is the recognition of the need for this further investigation.

There is a second important attribute of this application. Though we have not yet worked out the semantic conventions for a feature logic, from our intuitive understanding of how feature structures are used, it is the case that the semantic conventions of a feature logic will allow the interpretations of feature structures (i.e. the sets of possibilities they represent) to overlap one another. If the set of all feature structures is being used as the set of models in the application, then neither the Set *E* nor the Set *PP* techniques will be appropriate; since both techniques have the requirement that the sets of possibilities represented by models be disjoint. Either some other set of models is required (not feature structures or a disjoint subset of feature structures), or a new technique. This opens up a

potential application for the Set BE technique.

The overview completed we now proceed to the specific applications.

3.4 The “roll of the die” application

3.4.1 Introduction to the example

In this section we apply a model-theoretic representational technique to construct and make explanatorily intelligible the relation of logical consequence for a simple language. Our example will concern a very primitive language used to describe the numerical value of a roll of a single six-sided die. This language will have three assertion types: *odd*, *greaterThanFour*, and *five*. The practice of categorization by which these assertion types are interpreted coincides with the ordinary English meanings of their names, applied as predicates to the number of dots on the side of the die facing upwards after a roll. For example, if a die is rolled such that the side with five dots is facing up, all three assertion types would be assessed true by the practice. Whereas, if a die is rolled such that the side with six dots is facing up, the assertion types *odd* and *five* would be assessed false, and the assertion type *greaterThanFour* would be assessed true. Our task is to apply a model-theoretic representational technique to make the representational relation of logical consequence between these assertion types intelligible.

3.4.2 Details of the application

We could carry out the application of a model-theoretic representational technique as follows. (Note that this example does not go through the full mechanism of a proof. Our purpose here is to highlight the main ideas of an application of a model-theoretic technique as simply as possible. We will conduct a full proof of correctness in the propositional logic example).

We choose the Set E technique.

We will define a truth medium $Roll = \langle \Sigma_{Roll}, M_{Roll}, t_{Roll} \rangle$.

We are given the set of assertion types $\Sigma_{Roll} = \{odd, greaterThanFour, five\}$.

Take the set of models M_{Roll} to be $\{1, 2, 3, 4, 5, 6\}$.

Take the set $\mathcal{U}$ to be the space of possible rolls of the die. Note that on this view $\mathcal{U}$ is not simply a set with six elements. Rather, partial descriptions of some of the elements of $\mathcal{U}$ might include “a white die with black spots was rolled on felt and the side with three dots came up...,” or “a red die with white dots was rolled in Las Vegas on Tuesday and the side with four dots came up...,” etc. The given practice of categorization which interprets the elements of Σ_{Roll} can then be seen as the function $\mathcal{T}_{\Sigma_{Roll}} : \Sigma_{Roll} \rightarrow \mathcal{P}(\mathcal{U})$. The representational relation of logical consequence we assume, and wish to make intelligible, is given by $ALC\langle \Sigma_{Roll}, \mathcal{T}_{\Sigma_{Roll}} \rangle$.

Define $\mathcal{T}_{M_{Roll}}$ as follows. Given any possible roll $r \in \mathcal{U}$ of the die, and element m of M_{Roll} , $r \in \mathcal{T}_{M_{Roll}}(m)$ just in case the number of dots on the side of the die facing up in r is the numeric value, on the usual interpretation of the digits, of the digit naming m .

We can then define the function t_{Roll} as follows:

$$t_{Roll}(odd) = \{1, 3, 5\}$$

$$t_{Roll}(greaterThanFour) = \{5, 6\}$$

$$t_{Roll}(five) = \{5\}$$

In accepting these definitions as implying the Set E assumptions, we would be making the following claims:

1. Assumption A5: Every model is possible, that is, each of the integral numbers of dots 1, 2, 3, 4, 5, 6 may appear on the side of the die facing up after some roll.
2. Assumption A4: No two distinct models accurately describe the same possibility, that is, after every roll, the side of the die facing up shows at most one of the numbers of dots 1, 2, 3, 4, 5, 6.
3. Assumption B8: For each assertion type p , we can give a set of models which collectively and precisely represent the possibilities in which that assertion type is true; and a set of models which collectively and precisely represent the possibilities in which that assertion type is false. For example, for *odd*... we could claim that the possibilities collectively represented by the models 1, 3, and 5 precisely exhaust the possibilities in which *odd* is true; and that the possibilities collectively represented by the models 2, 4, and 6 precisely exhaust the possibilities in which *odd* is false. Making this assumption

means that we are excluding certain cases from consideration, such as the case where the side of the die facing up shows two and one-half dots, for example. No model as we have specified them would accurately represent such a case.

4. Assumption *B5*: “An (assertion type) σ should be true-in-a-model (with respect to) m iff σ would be true if the world were as depicted by m , that is, if m were an accurate model.” We can claim that the function t_{Roll} reflects this relation between the English interpretations of *odd*, *greaterThanFour*, and *five*, and the usual interpretations of the digits 1,2,3,4,5,6.

Given the truth medium *Roll* as defined, the relation MC_{Roll} would be as shown in the following table:

$MC_{Roll}(p, q)$		q		
		<i>odd</i>	<i>greaterThanFour</i>	<i>five</i>
p	<i>odd</i>	true	false	false
	<i>greaterThanFour</i>	false	true	false
	<i>five</i>	true	true	true

If the truth medium *Roll* did in fact satisfy the Set *E* assumptions (*A4*, *A5*, *B8*, *B5*) with respect to practices of categorization $\mathcal{T}_{\Sigma_{Roll}}$ and $\mathcal{T}_{M_{Roll}}$, then by the Corollary to the Intelligibility of Truth Media Theorem (Corollary 14), MC_{Roll} would be equivalent to $ALC_{\langle \Sigma_{Roll}, \mathcal{T}_{\Sigma_{Roll}} \rangle}$. So under those conditions, the relation MC_{Roll} makes the assumed relation of logical consequence $ALC_{\langle \Sigma_{Roll}, \mathcal{T}_{\Sigma_{Roll}} \rangle}$ explanatorily intelligible. The relation of consequence between the assertion types is explained in terms of the logically simpler models, and the definition of truth-in-a-model.

3.5 The propositional logic application

3.5.1 Introduction

The standard semantics of the language of propositional logic is defined in a model-theoretic style. In this section, we will describe that semantics using the Set *E* technique.

Our first step is to define the set of assertion types for the language of propositional logic. Let's begin by considering two alternative candidates.

1. One approach would be to take the assertion types of this language to be the set of propositional sentences, i.e. the set of sentences formed from the sentence symbols by construction operators building conjunctions, disjunctions, implications, and negations (Enderton 1972, 17ff). One could take this position, as the propositional sentences do in fact make propositional claims, and each propositional sentence is assumed to divide the space of possibility into those possibilities of which it is true and those of which it is false. For certain purposes this approach would be appropriate.
2. But if we wish to model the propositional language as it is in fact used to make claims and evaluate arguments, we should take a wider view. In our standard usage, we take the premise of an argument to be a *set* of sentences, and the conclusion to be a single sentence. This means that if we wished to model our standard usage of the language of propositional calculus, we should take the set of assertion types to be given by the set of sets of propositional sentences (i.e. the powerset of the set of propositional sentences). Note that in formally implementing this approach, we will use the approach described in (1) (which considers single propositional sentences as assertion types in their own right) as a foundation.

So in modelling the language of propositional logic, there are at least two candidates for the set of assertion types Σ . We could take Σ to be the set of propositional sentences WFF , or we could take Σ to be the powerset of WFF (with the empty set interpreted as a logical truth, and nonempty sets interpreted as the conjunction of the individual claims made by their members). Rather than choose between these candidates, we will first apply the Set E technique to the case where the set of assertion types is taken to be WFF , and later to the case where the set of assertion types is taken to be $\mathcal{P}(WFF)$. When we have a need to distinguish them, we will refer to the version of the language of propositional logic in which the assertion types are single sentences as the WFF -language of propositional logic, and the version of the language of propositional logic in which the assertion types are sets of sentences as the $\mathcal{P}(WFF)$ -language of propositional logic.

3.5.2 Application of the Set E technique to the WFF -language of propositional logic

In this section, we apply the Set E technique to construct a representational account of logical consequence for the WFF -language of propositional logic. In the next section (at this level) we will use the results of this section, and the Set E technique, to construct a representational account of logical consequence for the $\mathcal{P}(WFF)$ -language of propositional logic.

For this application, we assume that the given set of assertion types is the set of propositional sentences WFF . This application will take the following steps.

1. We will describe the truth medium ST for the application;
2. We will give semantic conventions describing acceptable interpretations of
 - (a) the set of propositional sentences (the set of the assertion types for the WFF -language of propositional logic); and
 - (b) the set of total truth assignments (the set of models for the application);
3. We will prove that if practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for interpretations of the set of propositional sentences and interpretations of the set of total truth assignments, respectively; then the truth medium ST satisfies the Set E assumptions with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$.

Then, for any practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ which together satisfy the semantic conventions for interpretations of the set of propositional sentences and interpretations of the set of total truth assignments, respectively; the relation MC_{ST} makes the representational relation of logical consequence $ALC_{\langle WFF, \mathcal{T}_{WFF} \rangle}$ explanatorily intelligible. This will complete the application.

The truth medium ST

We will now describe how we form the truth medium in this application. We will call the truth medium we are constructing ST . As with any truth medium, ST will be composed

of three elements: a set of assertion types, a set of models, and a function identifying the models of each assertion type.

The set of assertion types in ST is the set of propositional sentences WFF . In the standard semantics for propositional logic, the set of models is given by the set of functions from the set SS of propositional sentence symbols to the set of truth values $\{T, F\}$. Each such function is called a “total truth assignment” or “tta” for short. We will call the set of all such total truth assignments by the name TTA . We define the set of models in ST as equal to TTA .

The last step in constructing ST is to give the function for truth-in-a-model. We will call this function t_{ST} . In the standard semantics for propositional logic, this is given by the relation $\models$ of truth-in-a-truth-assignment defined on $TTA \times WFF$, such that $m \models p$ just in case p is true-in-a-truth-assignment with respect to m . We can adopt that definition directly, defining t_{ST} as follows. For all $p \in WFF$, $t_{ST}(p) = \{m \in TTA \mid m \models p\}$

Semantic conventions for interpretations of the sets WFF and TTA

We will now define semantic conventions for practices interpreting the set of sentences WFF and the set of models TTA . The important idea here is that acceptable interpretations of WFF and TTA come in pairs; and that acceptable pairs will share an acceptable interpretation of the sentence symbols. So first we shall characterize the set of acceptable interpretations for the set of sentence symbols SS . Then we can give an account of the semantic conventions on the interpretations of WFF and TTA in the following form:

A practice of categorization $\mathcal{T}_{WFF} : WFF \rightarrow \mathcal{P}(\mathcal{U})$ in conjunction with a practice of categorization $\mathcal{T}_{TTA} : TTA \rightarrow \mathcal{P}(\mathcal{U})$ are acceptable interpretations of WFF and TTA respectively, provided that there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ bear certain further relations to f .

So we will first characterize the set of acceptable interpretations for the set of sentence symbols SS , then consider what further conditions we need to impose on $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$. Following that, we will pull the semantic conventions together into a single statement.

Semantic conventions defining acceptable interpretations of SS So let us now define what counts as an acceptable interpretation of the sentence symbols. What we will be doing here is defining a condition, such that, if a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ satisfies that condition, then f counts as an acceptable interpretation of the sentence symbols. The condition we will use for our analysis is that an interpretation must ensure that the sentence symbols are logically independent to be acceptable. That is, no knowledge of the truth values of a set S of sentence symbols should imply knowledge of the truth values of any set of sentence symbols not contained in S . It is not necessary to use such an assumption to ground a semantics for the language of propositional logic (Barwise and Etchemendy, 1999), but we will use it here.

We can capture this convention of logical independence as follows.

We begin with the concept of a truth assignment. A function g is a **truth assignment** if g is a partial function from the set of propositional sentence symbols SS to $\{T, F\}$. We will call the set of all truth assignments TA (to distinguish it from the set of total truth assignments TTA). The set of truth assignments TA has a standard ordering as a set of partial functions. For any $f, g \in TA$, $f \preceq_{TA} g$ just in case $dom(f) \subseteq dom(g)$ and for all $x \in dom(f)$ $f(x) = g(x)$.

We can capture the standard interpretation $\mathcal{I}_{TA}$ of a truth assignment g relative to an interpretation of the set of sentence symbols f as follows. (We use the notation $\mathcal{I}_{TA(f)}$ to name this interpretation).

For all $g \in TA$, for all $f : SS \rightarrow \mathcal{P}(\mathcal{U})$

$$\mathcal{I}_{TA(f)}(g) = \bigcap_{A_n \in dom(g)} \text{if } g(A_n) = T \text{ then } f(A_n) \text{ else } \mathcal{U} - f(A_n).^1$$

What this definition says is that, if truth assignments are being interpreted relative to an interpretation f of sentence symbols, then... for every truth assignment g , the possibilities

¹(Note that one of the subexpressions in this specification has the form “if *condition* then *expression*₁ else *expression*₂”. This type of expression is called a “conditional expression.” The value of a conditional expression is defined as follows. If the value of the subexpression *condition* is true, then the value of the conditional expression as a whole is the value of *expression*₁ else the value of the conditional expression as a whole is the value of *expression*₂. Giving the specification “ $x = \text{if } g(A_n) = T \text{ then } f(A_n) \text{ else } \mathcal{U} - f(A_n)$ ” is equivalent to the specification “if $g(A_n) = T$ then $x = f(A_n)$ else $x = \mathcal{U} - f(A_n)$ ”)

in which g is true are just those possibilities in which every sentence symbol that g assigns ‘ T ’ is true, and every sentence symbol that g assigns ‘ F ’ is false.

The following is an immediate consequence of this definition.

Proposition 22 *For any practice $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ interpreting the sentence symbols, and any $g, h \in TA$, if $h \preceq_{TA} g$ then $\mathcal{T}_{TA(f)}(g) \subseteq \mathcal{T}_{TA(f)}(h)$.*

Now we can express the condition of logical independence of sentence symbols as follows.

Definition 23 *Semantic convention for a practice interpreting SS*

(*LI*) *A practice $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ interpreting the sentence symbols is acceptable, providing that, for every pair of truth assignments $g, h \in TA$, $\mathcal{T}_{TA(f)}(g) \subseteq \mathcal{T}_{TA(f)}(h)$ then $h \preceq_{TA} g$.*

Let’s consider this convention *LI*. This convention says that if a truth assignment is a logical consequence of another, then every assignment of truth value in the conclusion is included within the premise.

As a case of a f which would violate convention *LI*, consider the following. Say that $f(A_1) \cap f(A_2) \subseteq f(A_3)$. Note that such a f would mean that A_3 was a logical consequence of the conjunction of A_1 and A_2 . This is not to be allowed, for then the A_3 is not independent of A_1 and A_2 . We can show that this f fails convention *LI* as follows. Consider truth assignments $g, h \in TA$ such that g assigns A_1 the value T and A_2 the value T and makes no other assignments; and that h assigns A_3 the value T , and makes no other assignments. Then given this f , by the definition of $\mathcal{T}_{TA}$, $\mathcal{T}_{TA(f)}(g) \subseteq \mathcal{T}_{TA(f)}(h)$. This violates *LI*, since $h \not\preceq_{TA} g$.

Proposition 24 *Combining the result of Proposition 22 with Convention *LI*, we know that for any f satisfying the semantic conventions for a practice interpreting SS , $\forall g, h \in TA$, $\mathcal{T}_{TA(f)}(g) \subseteq \mathcal{T}_{TA(f)}(h)$ iff $h \preceq_{TA} g$.*

Additional semantic conventions constraining acceptable interpretations of WFF .

Now that we have defined what it is for an interpretation of sentence symbols to be acceptable, we can define the additional semantic conventions governing acceptable interpretations of *WFF*. Remember that the semantic conventions for practices interpreting *WFF* and

TTA apply to practices in pairs, and that the interpretation for WFF and the interpretation for TTA in an acceptable pair will both refer to a common acceptable interpretation for the set of sentence symbols. So as we characterize the semantic conventions for an interpretation of WFF , let us do so in the context of a practice $f : SS \rightarrow \mathcal{P}(\mathcal{U})$, which is understood to be the shared acceptable interpretation of the sentence symbols.

We will define an acceptable interpretation of the set of sentences WFF as one that (1) agrees with f for its interpretation of simple sentences (those which are simply a sentence symbol); and (2) interprets complex sentences in accordance with our intended interpretations of the sentential connectives.

In line with this intention we give the following definition:

Definition 25 *Additional semantic conventions for a practice interpreting WFF*

To be an acceptable interpretation of WFF , a practice of categorization $\mathcal{T}_{WFF} : WFF \rightarrow \mathcal{P}(\mathcal{U})$ must, with respect to the acceptable practice f it shares with the interpretation of TTA , obey the following conventions:

1. Convention *SIM* (for “simple sentences”)

(*SIM*) for all $A_n \in SS$, $f(A_n) = \mathcal{T}_{WFF}(A_n)$; and
2. Convention *COM* (for “complex sentences”)

(*COM*) $\forall p, q \in WFF$, $\mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) = \mathcal{U} - \mathcal{T}_{WFF}(p)$ and $\mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q)) = \mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(q)$ ²

The convention *COM* assures that the interpretations of the sentences of WFF obey the intended interpretations of the sentential connectives. In our discussion here and below, we will use only the truth functional connectives $\neg$ (for negation) and $\wedge$ (for conjunction) as together these form a complete set of connective symbols.

The convention *COM* assures that every assertion type and its negation partition the space of relevant possibility.

²Recall that $\mathcal{E}_{\neg}$ constructs the negation of its argument, and $\mathcal{E}_{\wedge}$ constructs the conjunction of its arguments.

Additional semantic conventions constraining acceptable interpretations of TTA

Now we give the additional semantic conventions defining acceptable interpretations of the set TTA of total truth assignments, the models in our truth medium ST . Again, we must remember the semantic conventions for practices interpreting WFF and TTA apply to practices in pairs, and the interpretation for WFF and the interpretation for TTA in an acceptable pair will both refer to a common acceptable interpretation for the set of sentence symbols. So as we characterize the semantic conventions for an interpretation of TTA , we do so in the context of a practice $f : SS \rightarrow \mathcal{P}(\mathcal{U})$, which is understood to be the shared acceptable interpretation of the sentence symbols.

Definition 26 *Additional semantic conventions for a practice interpreting TTA*

To be an acceptable interpretation of TTA , a practice of categorization $\mathcal{T}_{TTA} : TTA \rightarrow \mathcal{P}(\mathcal{U})$ must, with respect to the acceptable practice f it shares with the interpretation of WFF , obey the following conventions:

(AMC) $\mathcal{T}_{TTA} = \mathcal{T}_{TA(f)} \upharpoonright TTA$. (that is, $\mathcal{T}_{TTA}$ is equal to the function $\mathcal{T}_{TA}$ interpreted with respect to f and restricted to TTA).³

Recall that $\mathcal{T}_{TA}$ gives a truth assignment (partial or total) its standard interpretation, based upon an interpretation of the sentence symbols.

This constraint assures that an interpretation of TTA follows the standard interpretation of a truth assignment, given the acceptable shared interpretation of the sentence symbols.

The semantic conventions for interpretations of WFF and TTA We can now pull together the full specification of semantic conventions:

Definition 27 *A practice of categorization $\mathcal{T}_{WFF} : WFF \rightarrow \mathcal{P}(\mathcal{U})$ in conjunction with a practice of categorization $\mathcal{T}_{TTA} : TTA \rightarrow \mathcal{P}(\mathcal{U})$ are acceptable interpretations of WFF and TTA respectively, provided that*

1. there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and

³ We use the notation $\upharpoonright$ to indicate function restriction, so $m \upharpoonright S$ indicates the function m restricted to the domain S .

(a) (*WFF* interpretation conditions)

- i. (*SIM*) $\forall A_n \in SS, f(A_n) = \mathcal{T}_{WFF}(A_n)$; and
- ii. (*COM*) $\forall p, q \in WFF, \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) = \mathcal{U} - \mathcal{T}_{WFF}(p)$ and $\mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q)) = \mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(q)$

(b) (*TTA* interpretation conditions)

- i. (*AMC*) $\mathcal{T}_{TTA} = \mathcal{T}_{TA(f)} \upharpoonright TTA$

Consequences of the semantic conventions The propositions below are immediate consequences of the semantic conventions:

Proposition 28 *For any $\mathcal{T}_{WFF}, \mathcal{T}_{TTA}$ satisfying the semantic conventions for interpretations of *WFF* and *TTA*, no sentence and its negation are true in the same possibility, that is,*

$$\forall p \in WFF \ (\mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) = \emptyset)$$

Proposition 29 *For any $\mathcal{T}_{WFF}, \mathcal{T}_{TTA}$ satisfying the semantic conventions for interpretations of *WFF* and *TTA*, for any sentence, it is the case that together, a sentence and its negation exhaust the space of relevant possibility (Law of Excluded Middle), that is,*

$$\forall p \in WFF \ (\mathcal{T}_{WFF}(p) \cup \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) = \mathcal{U})$$

Proposition 30 *For any $\mathcal{T}_{WFF}, \mathcal{T}_{TTA}$ satisfying the semantic conventions for interpretations of *WFF* and *TTA*, the set of possibilities represented collectively by the set of all sentences in *WFF* is $\mathcal{U}$, that is, $\bigcup \mathcal{T}_{WFF}[WFF] = \mathcal{U}$.*

Proposition 31 *For any pair $\mathcal{T}_{WFF}, \mathcal{T}_{TTA}$ satisfying the semantic conventions for interpretations of *WFF* and *TTA*, the set of possibilities represented collectively by the set of all elements of *TTA* is $\mathcal{U}$, that is, $\bigcup \mathcal{T}_{TTA}[TTA] = \mathcal{U}$.*

1. $\bigcup \mathcal{T}_{TTA}[TTA] \subseteq \mathcal{U}$ since $\mathcal{T}_{TTA}$ is a function from *TA* to $\mathcal{P}(\mathcal{U})$.
2. For $\mathcal{U} \subseteq \bigcup \mathcal{T}_{TTA}[TTA]$, consider that it is possible, for any $u \in \mathcal{U}$, to construct an element m of *TTA* such that $u \in \mathcal{T}_{TTA}(m)$. Since $\mathcal{T}_{TTA}$ satisfies *AMC* there is an f such that f is an acceptable interpretation of the sentence symbols, and $\mathcal{T}_{TTA} =$

$\mathcal{T}_{TA(f)} \upharpoonright TTA$. Simply define m as follows:

$\forall A_i \in SS$ if $u \in f(A_i)$ then $m(A_i) = T$ else $m(A_i) = F$.

Proof of correctness of the application

We claim the following:

Proposition 32 *Given practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ that together satisfy the semantic conventions for practices interpreting WFF and TTA , the truth medium $ST = \langle WFF, TTA, t_{ST} \rangle$ satisfies assumption Set E with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$.*

Proving this claim will show the correctness of the application, permitting us to claim that given any practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ that together satisfy the semantic conventions for practices interpreting WFF and TTA , the proxy for consequence relation MC_{ST} will be equivalent to the representational relation of logical consequence $ALC_{\langle WFF, \mathcal{T}_{WFF} \rangle}$.

Goal of the proof The assumptions in Set E considered in relation to truth medium ST , with the sentences interpreted by $\mathcal{T}_{WFF}$ and the models interpreted by $\mathcal{T}_{TTA}$, can be expressed as follows.

- Assumption $A5$: Every element of TTA (every total truth assignment) is possible (on interpretation by $\mathcal{T}_{TTA}$).
- Assumption $A4$: No two distinct elements of TTA are accurate of the same possibility (when interpreted via $\mathcal{T}_{TTA}$).
- Assumption $B8$: For every sentence $p \in WFF$,
 1. there exists a set $S \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{WFF}(p)$; and
 2. there exists a set $R \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{WFF}(p)$.

The set $S = t_{ST}(p)$, that is, the set of models of p satisfies the first condition; the set $R = t_{ST}(\mathcal{E}_{-}(p))$, that is, the set of models of the negation of p , satisfies the second.

- Assumption *B5*: A sentence p is true-in-a-model (with respect to) total truth assignment m iff p would be true if the world were as depicted by m , that is, if m were an accurate model. Symbolically we can write, $\forall p \in WFF, \forall m \in TTA, m \models p$ iff $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

We will need to show that all of the above are true, given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA .

Proof (top level) First, we will prove a generally useful proposition. Then we will show that if practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , then truth medium ST satisfies all of the Set E assumptions with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$. We will sketch the proofs here, and give the full proofs at the end of the chapter.

Proposition 33 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that for every sentence, and every total truth assignment, if the possibilities represented by the sentence and the total truth assignment overlap at all, then the possibilities represented by the total truth assignment are completely contained within the possibilities represented by the sentence.*

Formally, we state:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA ,

for all $p \in WFF, \forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) \neq \emptyset$ implies $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

- This proof shows that the granularity of total truth assignments is finer than the granularity of sentences. The proof is an induction on the structure of WFF .

Proposition 34 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that every element of TTA is possible when interpreted with respect to $\mathcal{T}_{WFF}$, that is,*

(A5 applied) $\forall m \in TTA, (\mathcal{T}_{TTA}(m) \neq \emptyset)$

- Proof by reductio. A total truth assignment which was inconsistent would have every other truth assignment as a consequence, including ones which were not less than or equal to it in the ordering of truth assignments. This would violate convention *LI* on acceptable interpretations of the sentence symbols.

Proposition 35 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that no two distinct elements of TTA represent the same possibility (when interpreted with respect to $\mathcal{T}_{TTA}$), that is,*

(A4 applied) $\forall m, n \in TTA, (m \neq n \text{ implies } \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{TTA}(n) = \emptyset)$.

- If two total truth assignments are distinct, then they differ on the value of some sentence symbol, and then by the definition of $\mathcal{T}_{TA}$, their interpretations are distinct.

Proposition 36 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that*

(B5 applied) For every $p \in WFF, \forall m \in TTA, (m \models p \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p))$

- This proof is by induction on WFF . It shows that the definition of $\models$ behaves correctly given our semantic conventions. Proposition 33 plays a special role.

Proposition 37 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that*

For every $p \in WFF$, the set $S = \{m \in TTA \mid m \models p\}$ is such that $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{WFF}(p)$. (This implies B8a).

- Shows that the set of models of a sentence collectively represent exactly the possibilities represented by the sentence. To show that the possibilities represented by a model of p are included within the possibilities represented by p , use Proposition 36.

To show that the possibilities represented by a sentence p are included in the possibilities represented by a model of p , let u be a possibility represented by

p . Construct a total truth assignment that represents u . Since the possibilities represented by the total truth assignment and the sentence overlap (at u), by Proposition 33, the possibilities represented by the truth assignment are a subset of the possibilities represented by p . By Proposition 36, the truth assignment is a model of p . So u is a possibility represented by a model of p .

Proposition 38 *Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that For every $p \in WFF$, the set $R = \{m \in TTA \mid m \not\models p\}$ is such that $\bigcup \mathcal{T}_{TA}[R] = \mathcal{U} - \mathcal{T}_{WFF}(p)$. (This implies B8b).*

- Show that the set of non-models of a sentence collectively represent exactly the possibilities not represented by the sentence. Like the proof of Proposition 37 with an extra twist thrown in to handle negation.

Summary of the proof Given the above propositions, we have shown that, given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , then the truth medium $ST = \langle WFF, TTA, t_{ST} \rangle$ satisfies assumption Set E (A4, A5, B8, B5) with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$. In that case, by the Corollary to the Intelligibility of Truth Media Theorem (Corollary 14), MC_{ST} would be equivalent to $ALC_{\langle WFF, \mathcal{T}_{WFF} \rangle}$. So for any such $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$, the relation MC_{ST} makes the assumed relation of logical consequence $ALC_{\langle WFF, \mathcal{T}_{WFF} \rangle}$ intelligible. This completes our application of the Set E model-theoretic representational technique to the WFF -language of propositional logic.

3.5.3 Application of Set E technique to the $\mathcal{P}(WFF)$ -language of propositional logic

Now we are ready to consider the application of the Set E technique in the case when the set of assertion types is $\mathcal{P}(WFF)$. We will use the results of the previous section.

For this application, we will assume that the given set of assertion types is the powerset of the set of propositional sentences, that is, $\mathcal{P}(WFF)$. This application will take the following steps.

1. We will describe the truth medium PT for the application;
2. We will give semantic conventions describing acceptable interpretations of
 - (a) the powerset of the set of propositional sentences (the set of assertion types for the $\mathcal{P}(WFF)$ -language of propositional logic); and
 - (b) the set of total truth assignments (the set of models for the application);
3. We will prove that if practices $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for interpretations of the powerset of the set of propositional sentences and interpretations of the set of total truth assignments, respectively; then the truth medium PT satisfies the Set E assumptions with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$.

Then, for any practices $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ which together satisfy the semantic conventions for interpretations of the powerset of the set of propositional sentences and interpretations of the set of total truth assignments, respectively; the relation MC_{PT} makes the representational relation of logical consequence $ALC_{\langle \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)} \rangle}$ explanatorily intelligible. This will complete the application.

The truth medium PT

We will now describe how we form the truth medium in this application. We will call the truth medium we are constructing PT . As with any truth medium, PT will be composed of three elements: a set of assertion types, a set of models, and a function identifying the models of each assertion type.

The set of assertion types in PT is the powerset of the set of propositional sentences $\mathcal{P}(WFF)$. The set of models in PT will be the same as the models we used in modelling the WFF -language of propositional logic, namely, the set of total truth assignments TTA .

The last step in constructing PT is to give the function for truth-in-a-model. We will call this function t_{PT} . In the standard semantics for propositional logic, a total truth assignment is a model of a nonempty set of sentences just in case it is a model (on the single sentence level) of every sentence in the set. Every total truth assignment is a model of the empty set of sentences. Following this, we will define t_{PT} as follows:

$\forall P \in \mathcal{P}(WFF), t_{PT}(P) = \text{if } P = \emptyset \text{ then } TTA \text{ else } \bigcap t_{ST}[P].$

Here we use the function t_{ST} , defined for the truth medium ST , to describe the relation of “being a model at the single sentence level.”

Semantic conventions for interpretations of the sets WFF and TTA

We will now define semantic conventions for practices interpreting the powerset of the set of sentences WFF and the set of models TTA . The important idea here is that acceptable interpretations of $\mathcal{P}(WFF)$ and TTA come in pairs; and that acceptable pairs will share an acceptable interpretation of the sentence symbols.

We will give our account of the semantic conventions on the interpretations of $\mathcal{P}(WFF)$ and TTA in the following form:

A practice of categorization $\mathcal{I}_{\mathcal{P}(WFF)} : \mathcal{P}(WFF) \rightarrow \mathcal{P}(\mathcal{U})$ in conjunction with a practice of categorization $\mathcal{I}_{TTA} : TTA \rightarrow \mathcal{P}(\mathcal{U})$ are acceptable interpretations of $\mathcal{P}(WFF)$ and TTA respectively, provided that there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and $\mathcal{I}_{\mathcal{P}(WFF)}$ and $\mathcal{I}_{TTA}$ bear certain further relations to f .

The ‘further relations to f ’ we need characterize in the case of a practice interpreting TTA will be the same as those characterized in the case of the WFF -language of propositional logic. So we need now only consider those additional constraints in the case of a practice interpreting $\mathcal{P}(WFF)$.

Additional semantic conventions constraining acceptable interpretations of $\mathcal{P}(WFF)$.

Now we define the additional semantic conventions governing acceptable interpretations of $\mathcal{P}(WFF)$. Remember that the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA apply to practices in pairs, and that the interpretation for $\mathcal{P}(WFF)$ and the interpretation for TTA in an acceptable pair will both refer to a common acceptable interpretation for the set of sentence symbols. So as we characterize the semantic conventions for an interpretation of $\mathcal{P}(WFF)$, let us do so in the context of a practice $f : SS \rightarrow \mathcal{P}(\mathcal{U})$, which is understood to be the shared acceptable interpretation of the sentence symbols.

In the standard semantics for propositional logic, a nonempty set of sentences represents

those possibilities in which every sentence in the set (considered as an assertion in its own right) is true, and an empty set of sentences is true in every possibility. We will say that $\mathcal{T}_{\mathcal{P}(WFF)}$ is an acceptable interpretation of $\mathcal{P}(WFF)$ just in case there is an acceptable interpretation $\mathcal{T}_{WFF}$ of the sentences in WFF (considered as assertions in the own right) such that $\mathcal{T}_{\mathcal{P}(WFF)}$ interprets sets of sentences in accordance with the way that $\mathcal{T}_{WFF}$ interprets individual sentences.

In line with this intention we give the following definition:

Definition 39 *Additional semantic conventions for a practice interpreting $\mathcal{P}(WFF)$*

To be an acceptable interpretation of $\mathcal{P}(WFF)$, a practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)} : \mathcal{P}(WFF) \rightarrow \mathcal{P}(\mathcal{U})$ must, with respect to the acceptable practice f it shares with the interpretation of TTA , obey the following:

1. There exists an interpretation $\mathcal{T}_{WFF}$ of WFF that satisfies *SIM* and *COM* with respect to f ,
such that $\forall P \in \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)}(P) = (\text{if } P = \emptyset \text{ then } \mathcal{U} \text{ else } \bigcap \mathcal{T}_{WFF}[P])$

The semantic conventions for interpretations of $\mathcal{P}(WFF)$ and TTA We can now pull together the full specification of semantic conventions:

Definition 40 *A practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)} : \mathcal{P}(WFF) \rightarrow \mathcal{P}(\mathcal{U})$ in conjunction with a practice of categorization $\mathcal{T}_{TTA} : TTA \rightarrow \mathcal{P}(\mathcal{U})$ are acceptable interpretations of $\mathcal{P}(WFF)$ and TTA respectively, provided that*

1. there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and
 - (a) ($\mathcal{P}(WFF)$ interpretation conditions)
 - i. There exists an interpretation $\mathcal{T}_{WFF}$ of WFF that satisfies *SIM* and *COM* with respect to f ,
such that $\forall P \in \mathcal{P}(WFF),$
 $\mathcal{T}_{\mathcal{P}(WFF)}(P) = (\text{if } P = \emptyset \text{ then } \mathcal{U} \text{ else } \bigcap \mathcal{T}_{WFF}[P]);$ and

- (b) (TTA interpretation conditions)
 - i. $\mathcal{T}_{TTA}$ satisfies AMC with respect to f .

Proof of correctness of the application

We claim that given practices of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ that together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA , that the truth medium $PT = \langle \mathcal{P}(WFF), TTA, t_{PT} \rangle$ satisfies assumption Set E with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$.

Proving this claim will show the correctness of the application, permitting us to claim that given any practices of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ that together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA , the proxy for consequence relation MC_{PT} will be equivalent to the representational relation of logical consequence $ALC_{\langle \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)} \rangle}$.

Goal of the proof The assumptions in Set E considered in relation to truth medium PT , with the sentences interpreted by $\mathcal{T}_{\mathcal{P}(WFF)}$ and the models interpreted by $\mathcal{T}_{TTA}$, can be expressed as follows.

- Assumption $A5$: Every element of TTA (every total truth assignment) is possible (on interpretation by $\mathcal{T}_{TTA}$).
- Assumption $A4$: No two distinct elements of TTA are accurate of the same possibility (when interpreted via $\mathcal{T}_{TTA}$).
- Assumption $B8$: For every $P \in \mathcal{P}(WFF)$,
 1. there exists a set $S \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{\mathcal{P}(WFF)}(P)$; and
 2. there exists a set $R \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{\mathcal{P}(WFF)}(P)$.
- Assumption $B5$: $\forall P \in WFF, \forall m \in TTA, m \in t_{PT}(P)$ iff $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P)$.

We will need to show that all of the above are true, given that practices of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA .

Proof

Proposition 41 *Given practices of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ which together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA , it is the case that truth medium PT satisfies assumption Set E with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$.*

1. Since $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA , we know that there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and
 - (a) There exists an interpretation $\mathcal{T}_{WFF}$ of WFF that satisfies SIM and COM with respect to f ,
 such that $\forall P \in \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)}(P) = (\text{if } P = \emptyset \text{ then } \mathcal{U} \text{ else } \bigcap \mathcal{T}_{WFF}[P]);$
 and
 - (b) $\mathcal{T}_{TTA}$ satisfies AMC with respect to f .
2. So, by Proposition 32 (the proof showing the correctness of the application for the WFF -language), we know that ST satisfies assumption Set E with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$.
3. We now consider each of the assumptions in Set E , and show that PT satisfies each of them with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$.

(A4 and A5)

Since ST satisfies assumptions A4 and A5 with respect to $\mathcal{T}_{TTA}$, and $M_{ST} = M_{PT}$, we know that PT satisfies assumptions A4 and A5 with respect to $\mathcal{T}_{TTA}$.

(B5)

[Show $\forall P \in \mathcal{P}(WFF) \forall m \in TTA \ (m \in t_{PT}(P) \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P))$]

The details of this proof are to be found at the end of the chapter in Section 3.7. It is a straightforward application of the definitions, and the fact that truth medium ST satisfies Set E with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$.

(B8)

$$\left[\begin{array}{c} \text{Show } \forall P \in \mathcal{P}(WFF) \\ \text{a. } \exists S \subseteq TTA \text{ such that } \bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{\mathcal{P}(WFF)}(P) \text{ and} \\ \text{b. } \exists R \subseteq TTA \text{ such that } \bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{\mathcal{P}(WFF)}(P) \end{array} \right]$$

For part a., there are two cases. If $P = \emptyset$, take $S = TTA$. If P is nonempty, take $S = \bigcap t_{ST}[P]$, that is those truth assignments which are models of every sentence in P .

For part b., there are again two cases. If $P = \emptyset$, take $R = \emptyset$. If P is nonempty, take $R = TTA - \bigcap t_{ST}[P]$, that is those truth assignments not a model of every sentence of P .

Details are found at the end of the chapter in Section 3.7. ■ Proposition.

Summary of the proof Given practices of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ which together satisfy the semantic conventions for practices interpreting $\mathcal{P}(WFF)$ and TTA , it is the case that truth medium PT satisfies assumption Set E with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$. As a consequence, for any such practices $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$; the relation MC_{PT} makes the representational relation of logical consequence $ALC_{\langle \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)} \rangle}$ explanatorily intelligible. This completes the application.

3.6 Feature logics application

3.6.1 Introduction

As we mentioned in the brief discussion of feature logics in Section 3.3.3, the most important attribute of this application is recognizing the need for further investigation. The account of logical consequence typically given for feature logics, i.e. that the consequence relation is given by the relation of preservation of truth-in-a-feature-structure across all feature structures, is incomplete from the representational viewpoint. The described relation may be (and this author expects is) the appropriate proxy relation, but a justification of that belief needs to be demonstrated. As we just saw in the case of propositional logic, there is a distinction between knowing what the proxy relationship is, and having a justification of the correctness of that proxy, and an understanding of why it works. Our general method for applying the representational schema outlines for us the steps we need to take in order to complete the feature logic account of consequence, and justify the use of the existing

proxy relation.

We will begin this section with a brief presentation of the main structural elements of a feature logic. We will then proceed to consider how the various model-theoretic representational techniques we have defined might be applied to the task of constructing a more complete theory of logical consequence for a feature logic. In the course of this consideration, we will see that there is a challenge in applying techniques which require disjoint models to feature logics, and further, that the Set *BE* technique, which has no such requirement, may be an appropriate means for carrying out the construction.

3.6.2 Feature logics

The sentences of feature logics talk about feature structures. The sentences of feature logics are interpreted modally, that is, they are interpreted from the point of view of objects within a feature structure itself. We will begin our introduction by reviewing some key definitions in one formulation of feature structures, from Rounds 1997.

Feature systems and feature structures

Feature structures are a specialization of a more general class of object, known as **feature systems**. A feature system is characterized by a domain of elements, a set of partial functions on that domain (implementing features), and a set of subsets of that domain (implementing types). Feature systems can be categorized by the features and types they support. Let L be a set of **feature names**, and A be a set of **sort (type) names**. Then $\langle L, A \rangle$ is a **feature signature** characterizing a particular type of feature system. A feature system $\mathbb{A}$ of signature $\langle L, A \rangle$ is a tuple $\mathbb{A} = \langle D, \{f_l \mid l \in L\}, \{D_a \mid a \in A\} \rangle$ where D is the domain of elements, the f_l are partial functions on D (one for each feature), and the D_a are subsets of D (one for each sort name).

The feature-functions are interpreted as follows: if f_l maps element x to element y , then the value of feature l for element x is element y . Function symbols for features are written on the right (and the f is left off), so $l(d)$ is written dl , with dl being the value of feature l for element d . If l is defined at d , we write $dl \downarrow$, else we write $dl \uparrow$. We use p, q to denote strings of feature names. Such strings are called **paths**. Equations of the form $dp = eq$ means that

$dp \downarrow$ and $eq \downarrow$ and the values of dp and eq are equal.

The sort-sets are interpreted as follows: if element x is a member of set D_a , then x is of the sort a .

As a simple example, consider

$$D = \{a, s, r, e, m\},$$

$$L = \{mother\}, \text{ and}$$

$$A = \{Alan, Steven, Ronnie, Estelle, Mirra, Male, Female\}.$$

A feature system of signature $\langle L, A \rangle$ could be defined as follows:

$$\mathbb{B} = \left\langle \begin{array}{l} D = \{a, s, r, e, m\}, \\ \{f_{mother} = \{\langle a, r \rangle, \langle s, r \rangle, \langle r, e \rangle\}\}, \\ \{D_{Alan} = \{a\}, D_{Steven} = \{s\}, D_{Ronnie} = \{r\}, D_{Estelle} = e, D_{Mirra} = \{m\}, \\ D_{Male} = \{a, s\}, D_{Female} = \{r, e, m\}\} \end{array} \right\rangle$$

We could display feature system $\mathbb{B}$ graphically in Figure 3-1.

We will now define three concepts: first, the concept of a subsystem of a feature system; then in terms of that, the concept of a principal subsystem; then in terms of that, a feature structure. A **subsystem of a feature system** $\mathbb{A}$ consists of (1) a subset E of $D^{\mathbb{A}}$ (the domain of $\mathbb{A}$) such that if f is a feature, $d \in E$, and $df \downarrow$, then $df \in E$ (that is, E is closed under features), and (2) subsets E_a of D_a such that $E_a \subseteq E$ (that is, all the sort-sets of the subsystem are subsets of E). Given some element $d \in D^{\mathbb{A}}$, the **principal subsystem** $P(d)$ is defined as follows. The domain $E(d)$ of this subsystem is the set $\{dp \mid p \in L^* \wedge dp \downarrow\}$, and $E_a = D_a \cap E(d)$. So for instance, $P(s)$ for the feature system $\mathbb{B}$ is shown in Figure 3-2. A feature system $\mathbb{A}$ is a **feature structure** if there is a $d \in D^{\mathbb{A}}$, such that $P(d) = D^{\mathbb{A}}$; that is, there is some element d in the domain of $\mathbb{A}$ such that the principal subsystem formed from d is equal to $\mathbb{A}$ itself. That is the case exactly when there is some element d in the domain of $\mathbb{A}$ such that every element in the domain of $\mathbb{A}$ can be reached via some finite path of features from d . The feature system $\mathbb{B}$ in Figure 3-1 is not a feature structure. The subsystem $P(s)$ in Figure 3-2 (considered as a feature system on its own) is, with $d = s$.

A **homomorphism** between feature systems $\mathbb{A}$ and $\mathbb{B}$ of the same signature, is a total map γ between $D^{\mathbb{A}}$ and $D^{\mathbb{B}}$, satisfying (1) for any $d \in D^{\mathbb{A}}$ and $f \in L$, if $d\gamma \downarrow$ and $df^{\mathbb{A}} \downarrow$, then $df^{\mathbb{A}}\gamma = d\gamma f^{\mathbb{B}}$. In particular, $d\gamma f^{\mathbb{B}}$ is defined; (2) whenever $d \in D_a^{\mathbb{A}}$ we have $d\gamma \in D_a^{\mathbb{B}}$. In

Feature system **B**

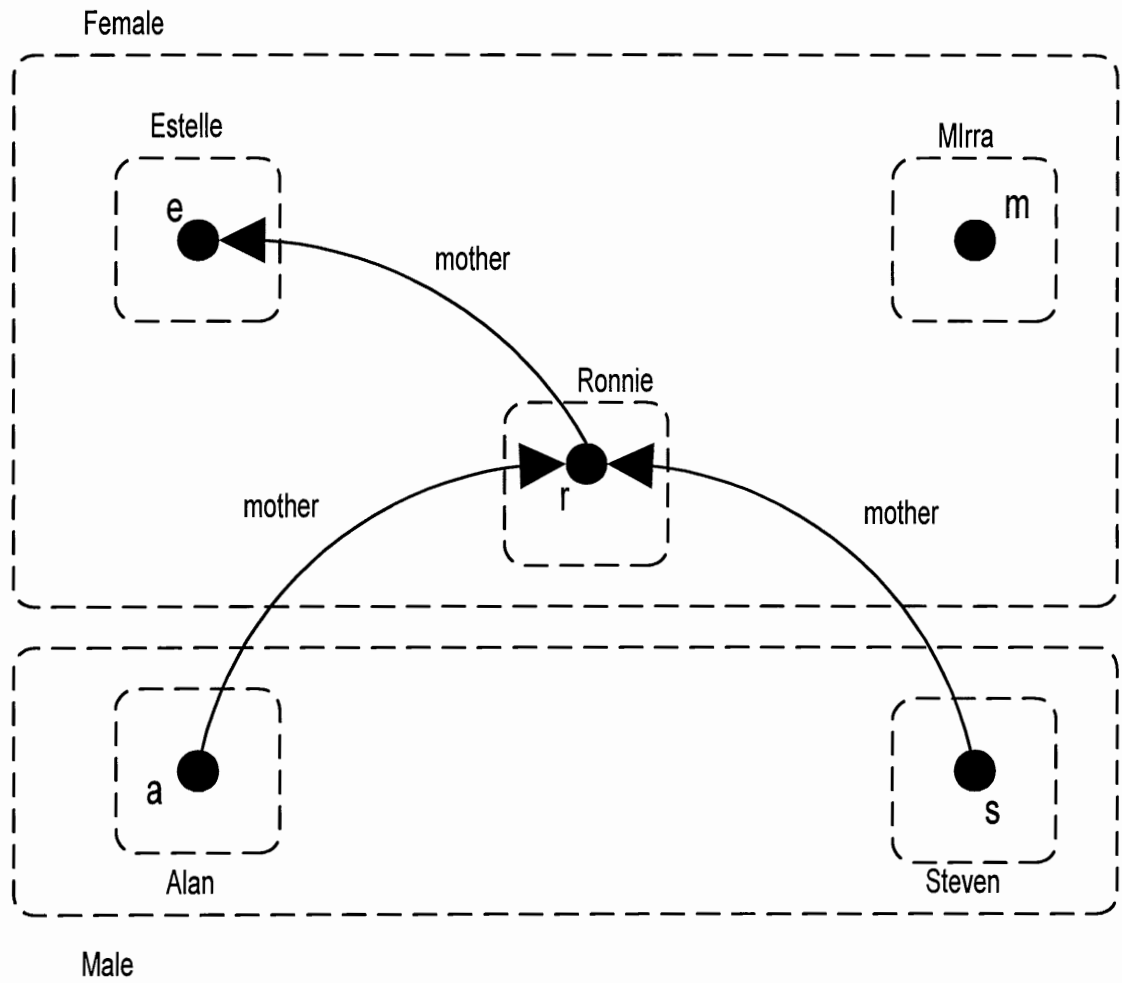


Figure 3-1: Feature system B

Principal subsystem P(s)

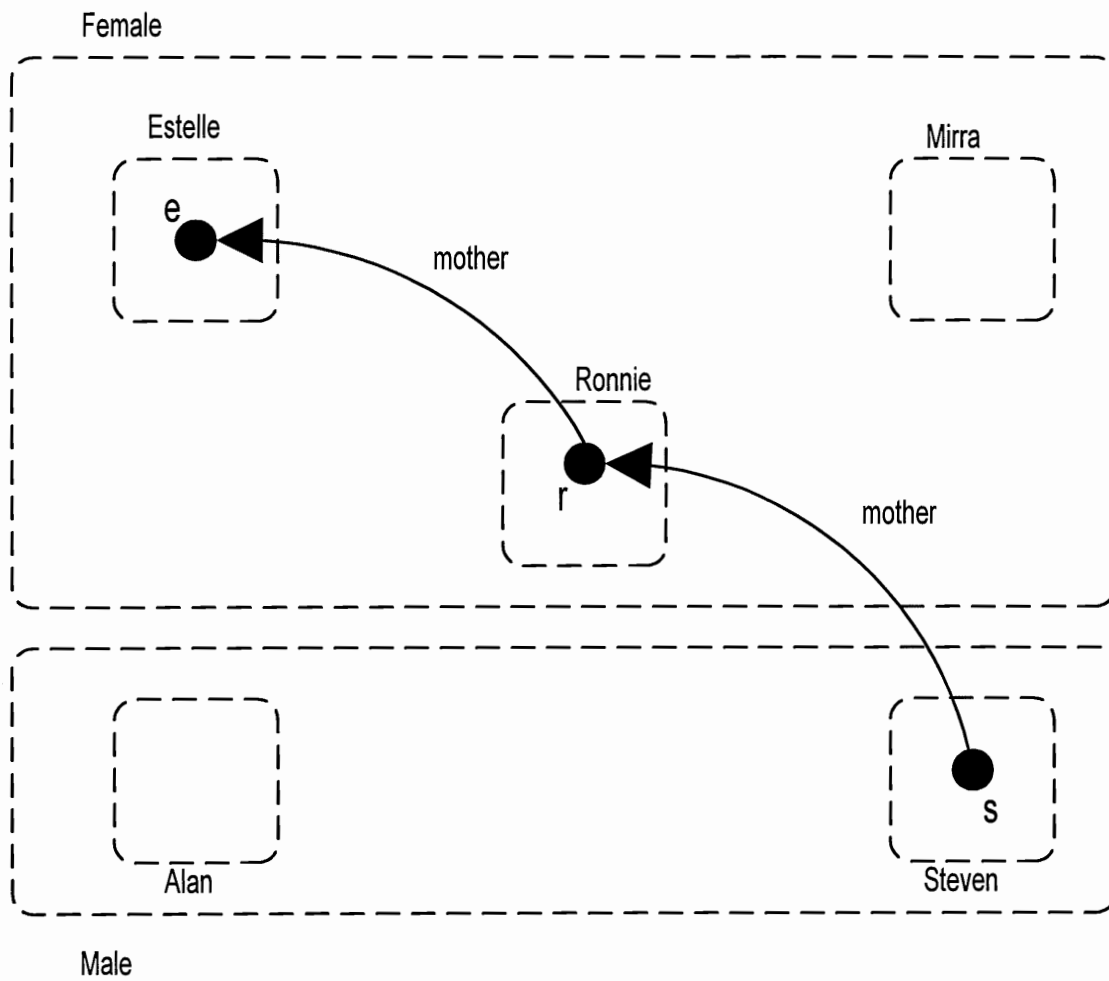


Figure 3-2: Principal subsystem P(s)

terms of the definition of a homomorphism, we can define the **subsumption preorder** $\sqsubseteq$ on $\mathbb{A}$ as follows: $d \sqsubseteq e$ iff there is a homomorphism $\gamma : P(d) \rightarrow P(e)$, with $d\gamma = e$. We say that d **subsumes** e . As Rounds puts it, “The intuitive idea behind the definition of subsumption is that of information content. If an element d subsumes an element e , then in some sense e bears at least as much information as d does” (Rounds 1997, 482). The concept of subsumption may be even easier to see between elements of different feature structures of the same signature. So for instance, element s in feature structure $\mathbb{C}$ in Figure 3-3 subsumes element s in feature structure $P(s)$ in Figure 3-2. In addition to all of the information borne by element s in feature structure $\mathbb{C}$, element s in feature structure $P(s)$ bears the information that e is the value of *mother* for r , and that e is of sort *Female*, and that e is of sort *Estelle*. We should note that there is a great diversity of feature-type systems and structures in the literature. For the purposes of our discussion, this one will do.

A language for feature structures

Here we review the language $L(KR)$ introduced in Rounds 1997 (p. 483). This is just one of many languages used to express information about feature structures. The language is parametrically defined in terms of the feature signature $\langle L, A \rangle$ of the feature structures under consideration. The basic formulas of the language are:

- (Constants) a for each $a \in A$;
- (Truth) The special formula **true**;
- (Path equations) $p \stackrel{\circ}{=} q$ for $p, q \in L^*$.

The compound formulas are defined inductively. If φ and ψ are formulas, then so are:

- $\varphi \wedge \psi$
- $\varphi \vee \psi$
- $l : \varphi$ for $l \in L$.

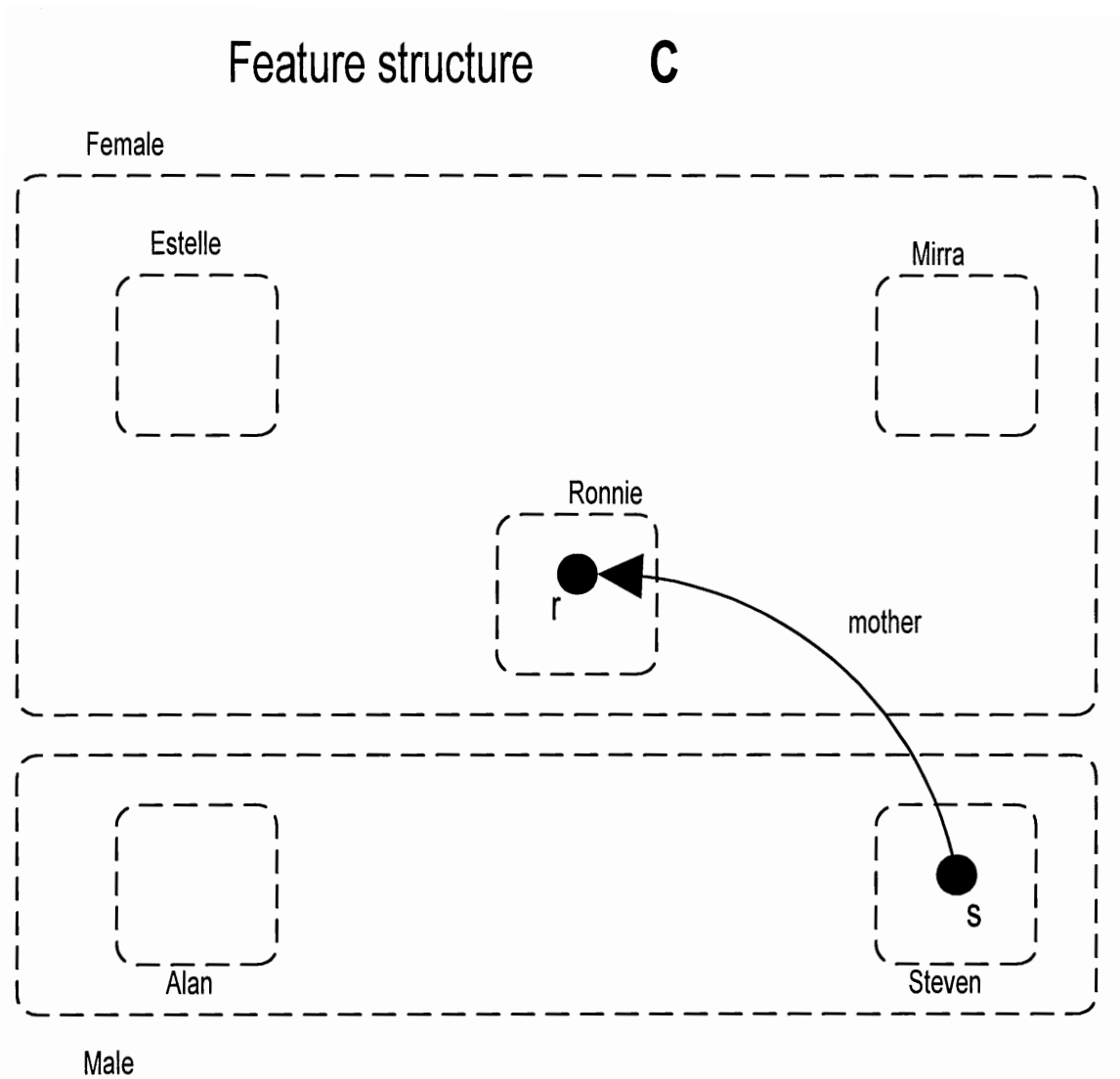


Figure 3-3: Element s of this feature structure subsumes element s of feature structure $P(s)$

The semantics of the logic are defined relative to a feature system. Let $\mathbb{A}$ be a feature system. Let d range over elements in $D^{\mathbb{A}}$. Then $\models$ is characterized relative to $\mathbb{A}$ as follows:

- $d \models a$ if $d \in D_a$;
- $d \models \mathbf{true}$ always;
- $d \models p \doteq q$ if $dp = dq$;
- $d \models \varphi \wedge \psi$ if $d \models \varphi$ and $d \models \psi$;
- $d \models \varphi \vee \psi$ if $d \models \varphi$ or $d \models \psi$;
- $d \models l : \varphi$ if $dl \models \varphi$ (implicitly dl is defined).

The above definition shows how $\models$ is dependent on a feature system and an element. We write $(\mathbb{A}, d) \models \varphi$ to indicate that element d satisfies φ in structure $\mathbb{A}$.

Consequence can be expressed as follows: Sentence φ is a consequence of a set of sentences Σ , just in case, for all feature structures $\mathbb{A}$ and elements d , if for all $\sigma \in \Sigma$, $(\mathbb{A}, d) \models \sigma$, then $(\mathbb{A}, d) \models \varphi$.⁴ We see how this is an instance of the model-theoretic category, with the assertion types being sets of sentences in the language, the models being feature structure/element pairs, truth-in-a-model being given by $\models$ (adjusted to conjoin nonempty sets of sentences, and be true in all models for the empty set). Taken together, the whole package ends up giving rise to a proxy for consequence relation of the usual model-theoretic form.

3.6.3 Applying the representational schema

What we would like to do is construct an application of the representational schema that is capable of justifying the picture of consequence just outlined. Our representational view has made it clear to us that the relation of consequence outlined above is not the consequence relation proper, but rather is a proxy for that relation. We would like to be able to articulate the assumptions we hold about feature structures and the language of the feature logic, that

⁴ A similar definition can be given for feature systems.

if true, are sufficient to show that the proxy relation we are using is indeed equivalent to the representational conception of logical consequence; and why.

The representational schema shows us the way to proceed. Our most basic choice is that of explanatory strategy. Since we are looking to justify a model-theoretic proxy relation, the natural strategy to start with is the model-theoretic one (e.g. vs. the order-consistency strategy). With that choice made, we will need to choose a particular model-theoretic representational technique. We have several to choose from (Set E , Set PP , Set BE); though note that if none of these do the trick, we could seek to create another one.

In addition to selecting the technique, we will need to identify the assertion types, form a truth medium, give a statement of the semantic conventions describing acceptable interpretations for the sentences and models, and then prove that if practices of categorization for the sentences and models satisfy those conventions, then the truth medium we have constructed satisfies the technique's technique-specific assumptions with respect to those practices. Then a full explanatory account will be in place.

Let us momentarily set aside the choice of technique, and begin by considering the set of assertion types, and formation of a truth medium. Our discussion above has already suggested candidates for an appropriate set of assertion types, and the way to go about forming a truth medium. We could take sets of sentences in the feature logic as the assertion types, pairs $\langle A, d \rangle$ of feature structures and elements as the models, and a version of the relation $\models$ defined above as the relation of truth-in-a-model, slightly adjusted to handle sets of sentences instead of single sentences. Call the medium formed this way FL . Then the proxy for consequence relation for any model-theoretic technique that we have defined will be MC_{FL} , which is preservation of truth-at-an-object-in-a-feature-structure across all feature structure/object combinations.

Seeing this structure, we can apply a simplification from our experience with the case of propositional logic, and factor the problem into two parts: applying a representational technique in the case where single sentences are the assertions, and then building the application in the case where the assertions are sets of sentences from the solution in the single sentence case. Since the latter should be trivial, and follow the outlines of our propositional logic solution, we will from this point on in the discussion focus on the former (the consequence

for single sentences problem).

So let us consider the case where the truth medium is composed as follows. The assertion types are single sentences in the feature logic. For this discussion, we will call this set WFF . Pairs $\langle A, d \rangle$ of feature structures and elements are the models. For this discussion, we will call this set FSE . The relation $\models$ as defined above (with no adjustment) is the relation of truth-in-a-model. For this discussion we use the name t for the function giving the models of any sentence via the relation $\models$. Call the medium formed this way SL . So $SL = \langle WFF, FSE, t \rangle$.

Now let us turn to the question of semantic conventions. We are going to need to describe acceptable interpretations for the set of sentences WFF and the models FSE . An interpretation for WFF would be a function $\mathcal{T}_{WFF} : WFF \rightarrow \mathcal{P}(U)$, assigning to each sentence a set of possibilities in which it would be true; and an interpretation for FSE would be a function $\mathcal{T}_{FSE} : FSE \rightarrow \mathcal{P}(U)$, assigning to each feature structure/element pair a set of possibilities in which it would be true. Our working out of the semantic conventions of propositional logic should give us some guidance in how to proceed here. There we saw how the semantic conventions of one form of representation can be coordinated with another; e.g. in the case of the WFF -language of propositional logic, the semantic conventions required the existence of an interpretation of the sentence symbols which was compatible with both the interpretation of the sentences and the interpretation of the total truth assignments. Here we will want to coordinate the semantic conventions of the interpretations of the sentences and the feature structure/element pairs.

The general shape of the solution we can see is something like this. We characterize the semantic conventions for the interpretations of the feature structure/element pairs in FSE , and then define the semantic conventions for the interpretations of the sentences in WFF in terms of those. We could express the semantic conventions for interpretations of WFF in the following form: An interpretation $\mathcal{T}_{WFF}$ of the set of sentences WFF is acceptable, provided that there exists an acceptable interpretation $\mathcal{T}_{FSE}$ of the feature structure/element combinations in FSE , and $\mathcal{T}_{WFF}$ bears the appropriate relation to $\mathcal{T}_{FSE}$ (that appropriate relation characterized by $\models$).

At this stage, we should step back from the details, and notice something important.

The elements of FSE , our putative models for this application of a model-theoretic representational technique, have an ordering defined upon them... the subsumption preorder... which intuitively implies the relation of relative information content; if d subsumes e ($d \sqsubseteq e$), then e bears the same or more information as d does. In the spirit of Dretske 1981, Barwise and Etchemendy 1995, and Barwise 1999, etc., we are using the inverse of possibilities represented as a measure of information. That is to say, e bears the same or more information than d , just in case the possibilities represented by e are a subset of the possibilities covered by d . Jon Barwise states this as “The Inverse Relationship Principle: Whenever there is an increase in available information there is a corresponding decrease in possibilities, and vice versa” (Barwise 1999, 5). So the semantic conventions for an interpretation $\mathcal{T}_{FSE}$ of FSE , will imply something like this:

For all $\langle \mathbb{A}, d \rangle, \langle \mathbb{B}, e \rangle \in FSE$, if $\langle \mathbb{A}, d \rangle \sqsubseteq \langle \mathbb{B}, e \rangle$ then $\mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$

Looking back to our earlier example, in Figures 3-2 and 3-3, $\langle \mathbb{C}, s \rangle$ subsumes $\langle P(s), s \rangle$. According to our intended interpretations $\langle P(s), s \rangle$ contains at least as much information as $\langle \mathbb{C}, s \rangle$ (and in this case actually more, since it carries information about the mother of element r , etc.) So in this case, for any acceptable interpretation $\mathcal{T}_{FSE}$ of FSE , we should have $\mathcal{T}_{FSE}(\langle P(s), s \rangle) \subseteq \mathcal{T}_{FSE}(\langle \mathbb{C}, s \rangle)$.

The existence of this ordering and its intended interpretation has a significant impact on our choice of model-theoretic representational technique. For if our models are represented by feature structure/element pairs in FSE , and the possibilities represented by those models are intended to overlap, then neither the Set E nor Set PP techniques will be appropriate for this application. Both of those techniques require (as a part of their technique-specific assumptions) that the sets of possibilities represented by the models be disjoint.

We have two choices in moving forward. Either we can change our choice of models, or we can use a different technique. Changing our choice of models, would require either

1. Identifying a subset of the feature structures to use as models (a subset for which the possibilities represented by the elements of that set would be disjoint, some kind of maximal feature structures); or
2. Identifying some other class of structures for which the possibilities represented by

those structures were disjoint, and coming up with a different way of characterizing truth-in-a-model.

Note however that neither of these alternatives would be able to justify the proxy for consequence relation as it has been developed in the literature, since neither follows the form of that account. In addition, both approaches seem likely to add complexity to the account. The first approach (a maximal subset of FSE) adds the complexity of defining that maximal subset, and dealing with its infinitely sized members. The second approach (a new class of structures) involves coming up with a new truth definition.

We believe that the best approach will be to try and use a different technique, namely the Set BE technique. This is a model-theoretic representational technique which allows models that overlap in the possibilities they represent. If we applied the Set BE technique directly to the truth medium SL (recall $SL = \langle WFF, FSE, t \rangle$), then the task would be to formulate appropriate (intuition matching) semantic conventions for interpretations of WFF and FSE such that, if a pair of interpretations $\mathcal{T}_{WFF}$ of the set of sentences and $\mathcal{T}_{FSE}$ of the class of feature structure/element pairs satisfied those conventions, then SL would satisfy the Set BE assumptions ($B1, B2, A7$) with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{FSE}$.

We have reason to expect that it should be possible to work out this account. We have not yet worked out the account. Here we will sketch one way in which it seems like it should be able to work.

Satisfying assumptions $B1$ and $B2$ depends on the correct specification of the t function, which should follow from the definition of $\models$. More interesting is the question of whether the semantic conventions for interpretations of the class of feature structure/element pairs would satisfy $A7$, with respect to the subsumption ordering. In this context, $A7$ would read: $\forall \langle \mathbb{A}, d \rangle, \langle \mathbb{B}, e \rangle \in FSE, \mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$ iff $\langle \mathbb{A}, d \rangle \sqsubseteq \langle \mathbb{B}, e \rangle$.

We already know (by our intuitive understanding of the relation of $\sqsubseteq$ and $\mathcal{T}_{FSE}$, that if $\langle \mathbb{A}, d \rangle \sqsubseteq \langle \mathbb{B}, e \rangle$ then $\mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$. So we have to ask: does our intuitive understanding also include the converse, that if $\mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$ then $\langle \mathbb{A}, d \rangle \sqsubseteq \langle \mathbb{B}, e \rangle$? Or looking at the equivalent contrapositive, if $\langle \mathbb{A}, d \rangle$ does not subsume $\langle \mathbb{B}, e \rangle$ then $\mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \not\subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$?

Consider the case where $\langle \mathbb{A}, d \rangle$ does not subsume $\langle \mathbb{B}, e \rangle$. That means there is no ho-

homomorphism γ from $P(\langle \mathbb{A}, d \rangle)$ to $P(\langle \mathbb{B}, e \rangle)$. That is, for every map γ from $P(\langle \mathbb{A}, d \rangle)$ to $P(\langle \mathbb{B}, e \rangle)$, such that $d\gamma = e$ there

1. exists a $d \in D^{\mathbb{A}}$ and $f \in L$, such that $d\gamma \downarrow$ and $df^{\mathbb{A}} \downarrow$, yet $df^{\mathbb{A}}\gamma \neq d\gamma f^{\mathbb{B}}$ (either the values are distinct or $d\gamma f^{\mathbb{B}}$ is not defined), or
2. there exists a $d \in D_a^{\mathbb{A}}$ such that $d\gamma \notin D_a^{\mathbb{B}}$.

We could restate this by saying the lack of homomorphism indicates that there is some attribute (either a feature assignment, or sort assignment) in the principal structure $P(\langle \mathbb{A}, d \rangle)$ either distinct from or not in the principal structure $P(\langle \mathbb{B}, e \rangle)$. If every attribute is logically independent of the others, and attributes are interpreted conjunctively, then such a difference between $P(\langle \mathbb{A}, d \rangle)$ and $P(\langle \mathbb{B}, e \rangle)$ would indicate that there were some possibilities represented by $P(\langle \mathbb{B}, e \rangle)$ not represented by $P(\langle \mathbb{A}, d \rangle)$, that is: $\mathcal{T}_{FSE}(\langle \mathbb{B}, e \rangle) \not\subseteq \mathcal{T}_{FSE}(\langle \mathbb{A}, d \rangle)$. That would give the other direction of A7. So we see that we have some reason to think that the Set *BE* technique may be applicable in this case.

The completion of this analysis will have to wait for future work on this project. For now, we can recognize two things: (1) that this additional analysis is important in making the account of logical consequence for feature logics more complete; and (2) that the Set *BE* technique may be a good way to carry it out.

We have now completed our presentation of examples of applications (and potential applications) of the model-theoretic representational techniques. The next chapter begins Part III, the discussion of the order-consistency strategy for representational semantics.

This concludes the linear text of the chapter. The remainder of the material in this chapter are the proofs of propositions mentioned earlier in the text.

3.7 Proofs of supporting propositions mentioned in the text

Proof of Proposition 33:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting *WFF* and *TTA*, it is the case that for every sentence, and every total truth assignment, if the possibilities represented by the sentence and the

total truth assignment overlap at all, then the possibilities represented by the total truth assignment are completely contained within the possibilities represented by the sentence. Formally, we state:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA ,

for all $p \in WFF, \forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) \neq \emptyset$ implies $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting WFF and TTA . Let f be the shared acceptable interpretation of the sentence symbols.

By induction on WFF .

1. Base case. $p \in SS$, so $p = A_n$ for some $n \in \mathcal{N}$.

(a) Let $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(A_n) \neq \emptyset$.

(b) Since m is a tta, either $m(A_n) = T$ or $m(A_n) = F$.

(c) Claim: $m(A_n) \neq F$.

i. Assume $m(A_n) = F$. Then $\mathcal{T}_{TTA}(m) \subseteq \mathcal{U} - f(A_n)$. This contradicts $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(A_n) \neq \emptyset$.

(d) So $m(A_n) = T$. Then $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(A_n)$. ■ Base.

2. Inductive case ($\neg$)

(a) Inductive hypothesis (I.H.): Assume that for some $p \in WFF, \forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) \neq \emptyset$ implies $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

(b)
$$\left[\begin{array}{l} \text{Show that } \forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) \neq \emptyset \\ \text{implies } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)). \end{array} \right]$$

(c) Let $m \in TTA$ such that $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) \neq \emptyset$.

(d) Then $\exists u$ such that $u \in \mathcal{T}_{TTA}(m)$ and $u \in \mathcal{U} - \mathcal{T}_{WFF}(p)$ (by *COM*). So $u \notin \mathcal{T}_{WFF}(p)$.

(e) So $\mathcal{T}_{TTA}(m) \not\subseteq \mathcal{T}_{WFF}(p)$.

(f) So $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) = \emptyset$. (by I.H.).

- (g) So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{U} - \mathcal{T}_{WFF}(p)$.
- (h) So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)). \blacksquare(\neg)$

3. Inductive case ($\wedge$)

- (a) (IH) Let $p \in WFF$, such that $\forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) \neq \emptyset$ implies $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.
Let $q \in WFF$, such that $\forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(q) \neq \emptyset$ implies $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(q)$.
- (b) $\left[\begin{array}{l} \text{Show } \forall m \in TTA, \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q)) \neq \emptyset \\ \text{implies } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q)). \end{array} \right]$
- (c) Let $m \in TTA$ such that $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q)) \neq \emptyset$.
- (d) Then $\exists u$ such that $u \in \mathcal{T}_{TTA}(m)$ and $u \in \mathcal{T}_{WFF}(p)$ and $u \in \mathcal{T}_{WFF}(q)$ (by *COM*).
- (e) So $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) \neq \emptyset$. By I.H., $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.
- (f) So $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(q) \neq \emptyset$. By I.H., $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(q)$.
- (g) So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\wedge}(p, q))$ (by *COM*). $\blacksquare(\wedge)$. $\blacksquare$ Proposition.

Proof of Proposition 34:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that every element of TTA is possible when interpreted with respect to $\mathcal{T}_{WFF}$, that is,
(A5 applied) $\forall m \in TTA, (\mathcal{T}_{TTA}(m) \neq \emptyset)$

1. Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting WFF and TTA , respectively. Let f be the shared acceptable interpretation of the sentence symbols.
2. Let $m \in TTA$ such that $\mathcal{T}_{TTA}(m) = \emptyset$. [Show contradiction.]
3. Let n be an element of TTA exactly like m , except in its assignment to A_0 . If $m(A_0) = T$, then $n(A_0) = F$; if $m(A_0) = F$, then $n(A_0) = T$. So $n \not\leq_{TA} m$. Yet $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{TTA}(n)$. This violates convention *LI*, and thus our assumption that f satisfies the semantic conventions for an interpretation of the sentence symbols.. Contradiction.

4. So $\mathcal{T}_{TTA}(m) \neq \emptyset$. ■ Proposition.

Proof of Proposition 35:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that no two distinct elements of TTA represent the same possibility (when interpreted with respect to $\mathcal{T}_{TTA}$), that is,

(A4 applied) $\forall m, n \in TTA, (m \neq n \text{ implies } \mathcal{T}_{TTA}(m) \cap \mathcal{T}_{TTA}(n) = \emptyset)$.

1. Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting WFF and TTA , respectively. Let f be the shared acceptable interpretation of the sentence symbols.
2. Let $m, n \in TTA$, such that $m \neq n$.
3. If $m \neq n$ then there exists an $A_i \in SS$ such that $m(A_i) \neq n(A_i)$.
4. There are two cases: (1) $m(A_i) = T$ and $n(A_i) = F$, or $m(A_i) = F$ and $n(A_i) = T$.
5. Case 1. $m(A_i) = T$ and $n(A_i) = F$.
 - (a) Then $\mathcal{T}_{TA(f)}(m \upharpoonright \{A_i\}) = f(A_i)$.⁵
 - (b) And $\mathcal{T}_{TA(f)}(n \upharpoonright \{A_i\}) = \mathcal{U} - f(A_i)$.
 - (c) So $\mathcal{T}_{TA(f)}(m \upharpoonright \{A_i\}) \cap \mathcal{T}_{TA(f)}(n \upharpoonright \{A_i\}) = \emptyset$.
 - (d) By Proposition 22, since $(m \upharpoonright \{A_i\}) \preceq_{TA} m$, $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{TA(f)}(m \upharpoonright \{A_i\})$.
 - (e) Similarly, since $(n \upharpoonright \{A_i\}) \preceq_{TA} n$, $\mathcal{T}_{TTA}(n) \subseteq \mathcal{T}_{TA(f)}(n \upharpoonright \{A_i\})$.
 - (f) So $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{TTA}(n) = \emptyset$. ■ Case 1.
6. Case 2. $m(A_i) = F$ and $n(A_i) = T$.

(a) Same as case 1 with m and n exchanged. ■ Case 2. ■ Proposition.

⁵We use the notation $\upharpoonright$ to indicate function restriction, so $m \upharpoonright \{A_i\}$ indicates the function m restricted to the domain $\{A_i\}$.

Proof of Proposition 36:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that

(B5 applied) For every $p \in WFF$, $\forall m \in TTA$, $(m \models p \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p))$.

Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting WFF and TTA . Let f be the shared acceptable interpretation of the sentence symbols.

By induction on WFF .

1. Base case. $p \in SS$.

(a) Let $m \in TTA$.

(b) $(\rightarrow)$

i. Assume $m \models p$.

ii. By definition of $\models$, we know that $m(p) = T$. So $\mathcal{T}_{TTA}(m) \subseteq f(p)$ by the definition of $\mathcal{T}_{TTA}$, and $f(p) = \mathcal{T}_{WFF}(p)$ $\blacksquare (\rightarrow)$

(c) $(\leftarrow)$

i. Assume $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

ii. Say $m \not\models p$. [Show contradiction.]

iii. Then $m(p) = F$, by definition of $\models$.

iv. Then $\mathcal{T}_{TTA}(m) \subseteq \mathcal{U} - \mathcal{T}_{WFF}(p)$.

v. Then $\mathcal{T}_{TTA}(m) = \emptyset$.

vi. That violates Proposition 34 $\blacksquare (\leftarrow)$ $\blacksquare$ Base.

2. Inductive case $(\neg)$

(a) Inductive hypothesis (I.H.):

Let $p \in WFF$ such that $\forall m \in TTA$, $(m \models p \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p))$

(b) [Show that $\forall m \in TTA$, $(m \models \mathcal{E}_{\neg}(p) \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)))$]

(c) Let $m \in TTA$.

(d) $(\rightarrow)$

- i. Let $m \models \mathcal{E}_-(p)$.
- ii. So by the definition of $\models$, $m \not\models p$.
- iii. By the I.H., $\mathcal{T}_{TTA}(m) \not\subseteq \mathcal{T}_{WFF}(p)$.
- iv. So $\exists u \in \mathcal{T}_{TTA}(m)$ such that $u \notin \mathcal{T}_{WFF}(p)$.
- v. So $u \in \mathcal{T}_{WFF}(\mathcal{E}_-(p))$ (by *COM*)
- vi. So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_-(p))$ by Proposition 33.■ $(\rightarrow)$

(e) $(\leftarrow)$

- i. Assume $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_-(p))$.
- ii. $\mathcal{T}_{WFF}(\mathcal{E}_-(p)) = \mathcal{U} - \mathcal{T}_{WFF}(p)$ by *COM*.
- iii. So $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(p) = \emptyset$.
- iv. Yet $\mathcal{T}_{TTA}(m) \neq \emptyset$ by Proposition 34.
- v. So $\mathcal{T}_{TTA}(m) \not\subseteq \mathcal{T}_{WFF}(p)$.
- vi. So $m \not\models p$ by I.H.
- vii. So $m \models \mathcal{E}_-(p)$ by the definition of $\models$.■ $(\leftarrow)$ ■ $(\neg)$

3. Inductive case $(\wedge)$

(a) (I.H.) Let $p \in WFF$ such that $\forall m \in TTA$, $(m \models p \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p))$.

Let $q \in WFF$ such that $\forall m \in TTA$, $(m \models q \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(q))$.

(b) [Show that $\forall m \in TTA$, $(m \models \mathcal{E}_\wedge(p, q) \text{ iff } \mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q)))$]

(c) Let $m \in TTA$.

(d) $(\rightarrow)$

- i. Assume that $m \models \mathcal{E}_\wedge(p, q)$.
- ii. So by the definition of $\models$, $m \models p$ and $m \models q$.
- iii. By the I.H., $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$ and $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(q)$.
- iv. So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(q)$.
- v. $\mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q)) = \mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(q)$ by *COM*.

- vi. Since $\mathcal{T}_{TTA}(m) \neq \emptyset$, $\mathcal{T}_{TTA}(m) \cap \mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q)) \neq \emptyset$.
- vii. So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q))$ by Proposition 33. ■ ($\rightarrow$)
- (e) ($\leftarrow$)
 - i. Assume $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q))$.
 - ii. $\mathcal{T}_{WFF}(\mathcal{E}_\wedge(p, q)) = \mathcal{T}_{WFF}(p) \cap \mathcal{T}_{WFF}(q)$ by *COM*.
 - iii. So $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$ and $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(q)$.
 - iv. So by the I.H., $m \models p$ and $m \models q$.
 - v. So by the definition of $\models$, $m \models \mathcal{E}_\wedge(p, q)$. ■ ($\leftarrow$) ■ ($\wedge$). ■ Proposition.

Proof of Proposition 37:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting *WFF* and *TTA*, it is the case that For every $p \in WFF$, the set $S = \{m \in TTA \mid m \models p\}$ is such that $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{WFF}(p)$. (This implies *B8a*).

1. Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting *WFF* and *TTA*. Let f be the shared acceptable interpretation of the sentence symbols.
2. Let $p \in WFF$.
3. Let $S = \{m \in TTA \mid m \models p\}$.
4. [Show $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{WFF}(p)$].
5. ($\subseteq$)
 - (a) Let $u \in \bigcup \mathcal{T}_{TTA}[S]$.
 - (b) Then $u \in \mathcal{T}_{TTA}(s)$ for some $s \in S$.
 - (c) Since $s \in S$, $s \models p$.
 - (d) So $\mathcal{T}_{TTA}(s) \subseteq \mathcal{T}_{WFF}(p)$, by Proposition 36.
 - (e) So $u \in \mathcal{T}_{WFF}(p)$. ■ ($\subseteq$)

6. ($\supseteq$)

(a) Let $u \in \mathcal{T}_{WFF}(p)$.

(b) [Show that there exists an $m \in TTA$ such that $m \models p$ and $u \in \mathcal{T}_{TTA}(m)$]

(c) We build a truth assignment m that represents u .

Define m as follows:

$\forall A_i \in SS$ if $u \in f(A_i)$ then $m(A_i) = T$ else $m(A_i) = F$.

(d) So $u \in \mathcal{T}_{TTA}(m)$.

(e) Since $u \in \mathcal{T}_{WFF}(p)$ and $u \in \mathcal{T}_{TTA}(m)$, we know by Proposition 33 that $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.

(f) By Proposition 36, $m \models p$.

(g) So $m \in S$, and $u \in \bigcup \mathcal{T}_{TTA}[S]$. ■ ($\supseteq$) ■ (B8a)

Proof of Proposition 38:

Given that practices of categorization $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ together satisfy the semantic conventions for practices interpreting WFF and TTA , it is the case that

For every $p \in WFF$, the set $R = \{m \in TTA \mid m \not\models p\}$ is such that $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{WFF}(p)$.

(This implies B8b).

1. Let $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$ be practices of categorization that together satisfy the semantic conventions for practices interpreting WFF and TTA . Let f be the shared acceptable interpretation of the sentence symbols.

2. Let $p \in WFF$.

3. Let $R = \{m \in TTA \mid m \not\models p\}$.

4. [Show $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{WFF}(p)$]

5. ($\subseteq$)

(a) Let $u \in \bigcup \mathcal{T}_{TTA}[R]$.

(b) Then $u \in \mathcal{T}_{TTA}(r)$ for some $r \in R$.

- (c) Since $r \in R$, $r \not\models p$.
- (d) By Proposition 36, $\mathcal{T}_{TTA}(r) \not\subseteq \mathcal{T}_{WFF}(p)$.
- (e) By Proposition 33, $\mathcal{T}_{TTA}(r) \cap \mathcal{T}_{WFF}(p) = \emptyset$.
- (f) So $u \notin \mathcal{T}_{WFF}(p)$, so $u \in \mathcal{U} - \mathcal{T}_{WFF}(p)$.
- (g) By *COM*, $\mathcal{U} - \mathcal{T}_{WFF}(p) = \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
- (h) So $u \in \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
- (i) By Proposition 33, since $\mathcal{T}_{TTA}(r) \cap \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) \neq \emptyset$, $\mathcal{T}_{TTA}(r) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
- (j) So $\mathcal{T}_{TTA}(r) \subseteq \mathcal{U} - \mathcal{T}_{WFF}(p)$. ■ ($\subseteq$)

6. ($\supseteq$)

- (a) Let $u \in \mathcal{U} - \mathcal{T}_{WFF}(p)$.
- (b) [Show $\exists m \in TTA$ such that $m \not\models p$ and $u \in \mathcal{T}_{TTA}(m)$]
- (c) We build a truth assignment m that represents u .
Define m as follows:
 $\forall A_i \in SS$ if $u \in f(A_i)$ then $m(A_i) = T$ else $m(A_i) = F$.
- (d) So $u \in \mathcal{T}_{TTA}(m)$.
- (e) By *COM*, $\mathcal{U} - \mathcal{T}_{WFF}(p) = \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
- (f) Since $u \in \mathcal{T}_{TTA}(m)$ and $u \in \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$, by Proposition 33, we know that
 $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
- (g) So by Proposition 36, $m \models \mathcal{E}_{\neg}(p)$. So $m \not\models p$ by the definition of $\models$.
- (h) So $m \in R$, and $u \in \bigcup \mathcal{T}_{TTA}[R]$. ■ ($\supseteq$) ■ Proposition.

Proof of Proposition 41 .

B5 portion of proof of Proposition 41

[Show $\forall P \in \mathcal{P}(WFF) \forall m \in TTA$ ($m \in t_{PT}(P)$ iff $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P)$)] .

1. Let P be an arbitrary element of $\mathcal{P}(WFF)$.
2. Let m be an arbitrary element of TTA .

3. (Case 1) $P = \emptyset$.

(a) $t_{PT}(P) = TTA$.

(b) $\mathcal{T}_{\mathcal{P}(WFF)}(\emptyset) = \mathcal{U}$.

(c) $m \in t_{PT}(P)$ is true, as is $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P)$. ■ (Case 1).

4. (Case 2) $P \neq \emptyset$.

(a) $t_{PT}(P) = \bigcap t_{ST}[P]$.

(b) $\mathcal{T}_{\mathcal{P}(WFF)}(P) = \bigcap \mathcal{T}_{WFF}[P]$.

(c) $m \in t_{PT}(P)$ iff

$m \in t_{ST}(p)$ for every $p \in P$ iff

$m \models p$ for every $p \in P$ (by definition of t_{ST}) iff

$\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$ for every $p \in P$ (since ST satisfies $B5$ with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$) iff

$\mathcal{T}_{TTA}(m) \subseteq \bigcap \mathcal{T}_{WFF}[P]$ iff

$\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P)$. ■ (Case 2). ■ ($B5$).

B8 portion of proof of Proposition 41

$$\left[\begin{array}{l} \text{Show } \forall P \in \mathcal{P}(WFF) \\ \text{a. } \exists S \subseteq TTA \text{ such that } \bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{\mathcal{P}(WFF)}(P) \text{ and} \\ \text{b. } \exists R \subseteq TTA \text{ such that } \bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{\mathcal{P}(WFF)}(P) \end{array} \right]$$

(Part a)

1. [Show $\forall P \in \mathcal{P}(WFF)$ ($\exists S \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[S] = \mathcal{T}_{\mathcal{P}(WFF)}(P)$)]

2. (Case 1) $P = \emptyset$.

(a) $\mathcal{T}_{\mathcal{P}(WFF)}(P) = \mathcal{U}$.

(b) Let $S = TTA$.

(c) By Proposition 31, $\bigcup \mathcal{T}_{TTA}[TTA] = \mathcal{U}$. ■ (Case 1.)

3. (Case 2) $P \neq \emptyset$.

- (a) $\mathcal{T}_{\mathcal{P}(WFF)}(P) = \bigcap \mathcal{T}_{WFF}[P]$.
- (b) Let $S = \bigcap t_{ST}[P]$.
- (c) [Show $\bigcup \mathcal{T}_{TTA}[S] = \bigcap \mathcal{T}_{WFF}[P]$]
- (d) ($\subseteq$)
 - i. Let $u \in \bigcup \mathcal{T}_{TTA}[S]$.
 - ii. Then there is an $m \in S$ such that $u \in \mathcal{T}_{TTA}(m)$.
 - iii. Since $m \in S$, we know that $m \in t_{ST}(p)$ for every $p \in P$.
 - iv. By the definition of $t_{ST}(p)$, we have $m \models p$ for every $p \in P$.
 - v. Since ST satisfies $B5$ with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$, $\forall p \in P$, $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.
 - vi. Therefore, $u \in \bigcap \mathcal{T}_{WFF}[P]$. ■ ($\subseteq$)
- (e) ($\supseteq$)
 - i. Let $u \in \bigcap \mathcal{T}_{WFF}[P]$.
 - ii. Then $\forall p \in P$, $u \in \mathcal{T}_{WFF}(p)$.
 - iii. By Proposition 31, $\bigcup \mathcal{T}_{TTA}[TTA] = \mathcal{U}$.
 - iv. So there exists an $m \in TTA$, such that $u \in \mathcal{T}_{TTA}(m)$.
 - v. [Show $\forall p \in P$, $m \in t_{ST}(p)$]
 - vi. Let $p \in P$.
 - vii. By the definition of $\models$, if $m \not\models \mathcal{E}_{\neg}(p)$, then $m \models p$.
 - viii. [Claim: $m \not\models \mathcal{E}_{\neg}(p)$]
 - A. Assume $m \models \mathcal{E}_{\neg}(p)$. [Show contradiction.]
 - B. Since ST satisfies $B5$ with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$, $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
 - C. By convention COM of the WFF -language of propositional logic,

$$\mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p)) = \mathcal{U} - \mathcal{T}_{WFF}(p).$$
 - D. So $u \notin \mathcal{T}_{WFF}(p)$. Contradiction.
 - E. So $m \not\models \mathcal{E}_{\neg}(p)$. ■ Claim.
 - ix. Since p was arbitrary, $\forall p \in P$, $m \models p$.

- x. So $\forall p \in P, m \in t_{ST}(p)$.
- xi. So $m \in S$.
- xii. So $u \in \bigcup \mathcal{T}_{TTA}[S]$. ■ ($\supseteq$) ■ (Case 2.) ■ (Part a).

(Part b)

[Show $\forall P \in \mathcal{P}(WFF)$ ($\exists R \subseteq TTA$ such that $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \mathcal{T}_{\mathcal{P}(WFF)}(P)$)]

1. (Case 1) $P = \emptyset$.

- (a) $\mathcal{U} - \mathcal{T}_{\mathcal{P}(WFF)}(P) = \mathcal{U} - \mathcal{U} = \emptyset$.
- (b) Let $R = \emptyset$.
- (c) $\bigcup \mathcal{T}_{TTA}[R] = \emptyset$. ■ (Case 1)

2. (Case 2) $P \neq \emptyset$.

- (a) $\mathcal{T}_{\mathcal{P}(WFF)}(P) = \bigcap \mathcal{T}_{WFF}[P]$.
- (b) Let $R = TTA - \bigcap t_{ST}[P]$.
- (c) [Show $\bigcup \mathcal{T}_{TTA}[R] = \mathcal{U} - \bigcap \mathcal{T}_{WFF}[P]$]
- (d) ($\subseteq$)
 - i. Let $u \in \bigcup \mathcal{T}_{TTA}[R]$.
 - ii. Then there is an $m \in R$ such that $u \in \mathcal{T}_{TTA}(m)$.
 - iii. Since $m \in R$, we know that there exists a $p \in P$ such that $m \notin t_{ST}(p)$.
 - iv. By the definition of $t_{ST}(p)$, we have $m \not\models p$.
 - v. So $m \models \mathcal{E}_{\neg}(p)$ (by definition of $\models$).
 - vi. Since ST satisfies $B5$ with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$,
 $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
 - vii. So $u \in \mathcal{T}_{WFF}(\mathcal{E}_{\neg}(p))$.
 - viii. So since $\mathcal{T}_{WFF}$ satisfies the conventions of the WFF -language of propositional logic, $u \notin \mathcal{T}_{WFF}(p)$.
 - ix. Therefore, $u \notin \bigcap \mathcal{T}_{WFF}[P]$. ■ ($\subseteq$)

(e) $(\supseteq)$

- i. Let $u \in \mathcal{U} - \bigcap \mathcal{T}_{WFF}[P]$.
- ii. Then there exists a $p \in P$, $u \notin \mathcal{T}_{WFF}(p)$.
- iii. By Proposition 31, $\bigcup \mathcal{T}_{TTA}[TTA] = \mathcal{U}$.
- iv. So there exists an $m \in TTA$, such that $u \in \mathcal{T}_{TTA}(m)$.
- v. [Show $m \notin t_{ST}(p)$]
- vi. Assume $m \in t_{ST}(p)$. [Show contradiction.]
- vii. Then $m \models p$, by definition of t_{ST} .
- viii. Since ST satisfies $B5$ with respect to $\mathcal{T}_{WFF}$ and $\mathcal{T}_{TTA}$, $\mathcal{T}_{TTA}(m) \subseteq \mathcal{T}_{WFF}(p)$.
- ix. But $u \in \mathcal{T}_{TTA}(m)$ and $u \notin \mathcal{T}_{WFF}(p)$. Contradiction.
- x. So $m \notin t_{ST}(p)$.
- xi. So $m \notin \bigcap t_{ST}[P]$.
- xii. So $m \in R$. ■ $(\supseteq)$ ■ (Case 2) ■ (Part b).

Part III

Assimilating Order-consistency Semantics to the Representational Schema

Chapter 4

Order-consistency

Representational Techniques

4.1 Introduction

The goal for this part of the dissertation is to motivate, define and investigate order-consistency representational techniques. These techniques are distinguished from the model-theoretic techniques we discussed in Part II in virtue of the different explanatory strategy they employ. Where a model-theoretic representational technique explains the consequence relation between assertion types in terms of a logically simpler form of representation (the models) and a defined relation of truth-in-a-model, an order-consistency representational technique explains the same relation in terms of the simpler concepts of ordering and consistency. In this chapter, we will define and consider the general properties of two order-consistency representational techniques. In the next chapter, we will explore applications of these techniques.

We can sketch the overall plan of this chapter as follows. You may recall that in Section 1.6.2, we located the fundamental intuition underlying order-consistency representational techniques in a corollary to Lindenbaum's Lemma, the one that says: A sentence q is deducible from a set of sentences P just in case q belongs to every maximal extension of P . Looking at this corollary, we can see that a proxy for consequence (deducibility) can be expressed in terms of consistency and ordering (since maximality is defined in those

terms). We will use that intuition as our starting point in creating a new representational technique, the Set *CL* technique. One of the technique-specific assumptions for the Set *CL* technique will be a representational version of Lindenbaum’s Lemma, requiring the existence of maximal extensions. Some interpreted languages may not have maximal extensions. For other interpreted languages, maximal extensions may be cumbersome to work with. So we will consider the question of whether Lindenbaum’s Lemma is necessary for an order-consistency semantics. That is to say, we will consider the question: is there some other order-consistency technique that does not require the assumption of maximal extensions?¹ In answer, we will create just such a technique, which we will call the Set *CG* technique. We will explore the relationships between the two techniques, and show that the Set *CG* technique is a generalization of the Set *CL* technique.

In this next section, we present an overview of all of the material in the chapter, and then proceed to work through the details.

4.2 Overview

4.2.1 Creating an order-consistency representational technique from the corollary to Lindenbaum’s Lemma

How will we get started? By definition, every representational technique is an instance of the representational schema. That schema gives a structure to use in constructing a new technique. There are four elements to techniques constructed using the schema: a foundational characterization, an intelligible medium, a proxy for consequence relation, and a set of technique-specific assumptions. Our first task will be to use these elements to create enough “representational context” in order to be able to express representational versions of Lindenbaum’s Lemma and the order-consistency condition specified in the corollary (i.e. “ q belongs to every maximal extension of P ”). That context will include the following:

1. We will use the same foundational characterizations of assertion types, possibilities, practices of categorization, and representational conception of logical consequence

¹Special thanks to Johan van Benthem for asking this question.

(*ALC*) as was used for the model-theoretic techniques.

2. We will construct a new form of intelligible medium, which we will call the **consistency medium**. Given an interpreted set of assertion types whose relation of logical consequence is to be modelled, the consistency medium will supplement that given set of assertion types with a partial order and a specification of the subset of those types which are taken to be consistent.
3. We will add a technique-specific assumption (called *D2*) which will say, in terms of the foundational characterization, that those assertion types which are specified to be consistent as a part of the consistency medium, are in fact consistent (that is, that an assertion type specified to be consistent in fact represents some possibility).

We will then be able to express representational versions of Lindenbaum's Lemma and the order-consistency condition specified in the corollary in terms of the consistency medium, given that medium's intended interpretation. Call the representational version of Lindenbaum's Lemma *LL*, and call the representational version of "*q* belongs to every maximal extension of *P*," *LLC*. We can then add *LL* and *LLC* to the "representational context," and be on the way to a representational technique. In the developing technique, *LL* will play the role of a technique-specific assumption. Meanwhile, *LLC* will play the role of the proxy for consequence relation.

Let us take a moment and take stock. At this point in the development, what will we have? We will have a foundational characterization. We will have an intelligible medium (the consistency medium). We will have a proxy for consequence relation (*LLC*). And we have some technique-specific assumptions, specifically *D2* and *LL*.

So now we can ask the question: is this enough? Have we defined a new representational technique? And the answer at that point will be "no." Even if a consistency medium satisfies the technique specific assumptions specified so far (*D2* and *LL*), we can construct cases in which *LLC* is not equivalent to the representational conception of logical consequence. We will need to do more.

One way to complete the definition in progress and reach the goal of a new representational technique will be to add more technique-specific assumptions to the ones defined so

far. And we have a natural place to look for these assumptions: the language of propositional logic. We already know that in the syntactic context, Lindenbaum’s Lemma applies to the language of propositional logic, and the order-consistency condition specified in the corollary is equivalent to deducibility for propositional logic (the syntactic proxy for consequence). So we will look for semantic conditions, expressed in representational terms, which characterize the language of propositional logic, in the hopes that if we add these conditions to the technique-specific assumptions we already have, those collected assumptions will be sufficient to imply the equivalence of *LLC* and the representational conception of consequence. (Note that the version we are interested in will be what we have called the $\mathcal{P}(WFF)$ -version of the language of propositional logic.)

We will identify two such conditions: conditions we call “**monotonic extension of commitment**” and “**weak extensibility**.” We will motivate and define these conditions, and later prove that conventional usages of propositional logic satisfy them both. We will show how to express these conditions with respect to the consistency medium and the practice of categorization that interprets its assertion types. In that form, we shall know these conditions by the names *D1* and *D3* respectively. We will be able to show that if a consistency medium *D* satisfies *D1*, *D2*, *D3* and *LL* with respect to the practice of categorization that interprets the assertion types of *D*, then *LLC* is equivalent to the representational conception of consequence for those types as interpreted. And importantly, the consistency medium (because of its structure and intended interpretation) will permit an explanation of the consequence relation made intelligible by *LLC* in terms of the simpler concepts of ordering and consistency. So at that point, we will have constructed a full representational technique. We will call this technique the Set *CL* technique.

4.2.2 Creating an order-consistency technique that does not require maximal extensions

One of the most salient features of the Set *CL* technique, is, not surprisingly, its dependence on the representational version of Lindenbaum’s Lemma (*LL*). *LL* is one of the technique-specific assumptions for the Set *CL* technique. That means that every interpreted set of assertion types to which the Set *CL* technique is applied must satisfy assumption *LL*,

that is, it must guarantee maximal extensions. Some languages may not have maximal extensions, and for others, working with maximal extensions may be cumbersome. So, as we mentioned in the Introduction to this chapter, we will consider the question: is it possible to construct an order-consistency technique that does not require the representational version of Lindenbaum's Lemma (LL)? We will show that the answer to this question is “yes” and in the process define a new technique, the Set CG technique.

In our discussion of model-theoretic representational techniques, we saw that we could generate new techniques by changing the set of technique-specific assumptions used. In the techniques we looked at, the only difference between techniques were different sets of technique-specific assumptions. The foundational characterization, intelligible medium, and proxy for consequence relation were the same for all. Here, in considering order-consistency techniques, we are going to see something new. We are going to generate a new technique from the Set CL technique by dropping assumption LL from the set of technique-specific assumptions, and then changing the proxy for consequence relation “to make up the difference.” (For if there were no difference, Set CL would not have needed assumption LL in the first place).

The new proxy for consequence relation for use in the Set CG technique will be defined in terms of a binary relation of compossibility between assertion types. We will call the new proxy LC . The relation of compossibility is derived from the primitive partial order relation and primitive unary consistency predicate present in the consistency medium. Stated in a line, for assertion types p and q , $LC(p, q)$ iff every assertion type compossible with p is compossible with q . It can be shown that if a consistency medium D satisfies $D1$, $D2$, and $D3$ with respect to the practice of categorization interpreting its assertion types; then LC is equivalent to the representational conception of logical consequence for those types as interpreted. So we can form a new technique consisting of the shared foundational characterization; the consistency medium, the new proxy for consequence relation LC , and the technique-specific assumptions $D1$, $D2$, and $D3$. This is the Set CG technique.

We will prove that LC , the proxy for consequence relation for the Set CG technique, bears a special relation to LLC , the proxy for consequence relation for the Set CL technique. Given that a consistency medium satisfies LL , then LC is equivalent to LLC . We can

show this result purely order-theoretically. This result has the following implication for the relationship between the two techniques. If a consistency medium satisfies assumptions *D1*, *D2*, and *D3* with respect to the practice of categorization by which its assertion types are interpreted, the *LC* is equivalent to the representational conception of logical consequence for those types. If it further satisfies condition *LL*, then *LLC* is equivalent to the representational conception of logical consequence for those types as well. (We will actually be able to characterize the range of the *LLC* proxy relation even more accurately than this). The upshot of all this is that every application of the Set *CL* technique just is an application of the Set *CG* technique, so the range of applicability of the Set *CL* technique is a subset of the range of applicability of the Set *CG* technique.

After the exposition described above has been completed, we will give the proof that Set *CG* technique makes the representational relation of logical consequence intelligible, the corollary that the Set *CL* technique makes the representational relation of logical consequence intelligible, and then give the proofs which characterize the cases in which the *LLC* proxy relation is capable of making consequence intelligible.

4.2.3 Proving that any conventional interpretation of propositional logic satisfies the conditions of monotonic extension of commitment and weak extensibility

Section 4.8 will take up a matter mentioned earlier, but set aside to keep the flow of argument going. This section will prove that any conventional interpretation of the language of propositional logic does in fact satisfy the conditions of monotonic extension of commitment and weak extensibility. These results support the claims made earlier, help make the conditions proven more salient, but also, and importantly, introduce a number of new concepts which will be important for the proofs in Part IV. The concepts to be introduced include: **partially ordered truth media**, **operators on interpreted truth media**, a property of truth media **negation-completeness**, and two properties of partially ordered truth media **monotonic extension of commitment with respect to models** and **weak extensibility with respect to models**.

4.3 Creating an order-consistency technique from the corollary to Lindenbaum’s Lemma: the Set CL technique

We have located the fundamental intuition underlying order-consistency representational techniques in a corollary to Lindenbaum’s Lemma, the one that says: A sentence q is deducible from a set of sentences P just in case q belongs to every maximal extension of P . Looking at this corollary, we can see that a proxy for consequence (deducibility) can be expressed in terms of consistency and ordering (since maximality is defined in those terms). We will use that intuition as our starting point in creating a new representational technique, the Set *CL* technique.

4.3.1 Constructing a “representational context”

We have set ourselves the task of creating a new representational technique embodying the concepts present in the corollary to Lindenbaum’s Lemma. The representational schema is our guide to constructing new techniques. There are four elements to techniques constructed using the schema: a foundational characterization, an intelligible medium, a proxy for consequence relation, and a set of technique-specific assumptions. We will begin our solution by using these elements to create enough “representational context” to enable us to express representational versions of Lindenbaum’s Lemma and the order-consistency condition expressed in the corollary (i.e. “ q belongs to every maximal extension of P .”). That expression will require that the context support notions of ordering and consistency.

Foundational characterization

We will use the same foundational characterizations of assertion types, possibilities, practices of categorization, and representational conception of logical consequence (*ALC*) as was used for the model-theoretic techniques. We repeat the assumptions here for clarity.

1. We assume a conceptual space of possibility;
2. We assume the existence of a practice by which each assertion type categorizes possibilities into those of which the assertion type is true, and those of which it is false;

3. We assume that the conclusion of an argument is a logical consequence of its premise if and only if that argument preserves truth across all possibilities.
4. We assume that an assertion type is consistent just in case it represents some possibility.

We formally model these assumptions as follows:

1. We model the set of assertion types by a set Σ ;
2. We model arguments as pairs of assertion types, the first being the premise and the second the conclusion;
3. We model the space of possibility by the set of possibilities $\mathcal{U}$;
4. We model the practice of “categorizing a possibility as being one in which an assertion type is true” by the function: $\mathcal{T}_\Sigma : \Sigma \rightarrow \mathcal{P}(\mathcal{U})$, such that, for any $\sigma \in \Sigma$, $\mathcal{T}_\Sigma(\sigma)$ is the set of possibilities in which σ is true. We say that $\mathcal{T}_\Sigma(\sigma)$ is the set of possibilities **represented by σ** .
5. In relation to a set of assertion types Σ interpreted via a practice of categorization $\mathcal{T}_\Sigma$, we model the assumed concept of logical consequence with the relation *ALC* (Assumed Logical Consequence) defined on $\Sigma \times \Sigma$ as follows:
 $\forall p, q \in \Sigma, ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p, q)$ iff $\mathcal{T}_\Sigma(p) \subseteq \mathcal{T}_\Sigma(q)$.
6. In relation to a set of assertion types Σ interpreted via a practice of categorization $\mathcal{T}_\Sigma$, we model the assumed concept of consistency as follows:
 $\forall p \in \Sigma, p$ is consistent iff $\mathcal{T}_\Sigma(p) \neq \emptyset$.

New form of intelligible medium: the consistency medium

We now construct a new form of intelligible medium. Recall that an intelligible medium is a mathematical structure supplementing and including the original set of interpreted assertion types for which the relation of consequence is to be modelled. We want to be able to express versions of Lindenbaum’s Lemma and the order-consistency condition expressed in its corollary with respect to the structure we create here. So the constructed intelligible

medium needs to support both an ordering on assertion types, and a specification of which assertion types are consistent.

We begin with the set Σ of assertion types for which the relation of logical consequence is to be modelled. We add a partial order $\preceq$ on those types. We also add a subset C of Σ , with the intention that the assertion types in C are all, and only, the assertion types in Σ which are held to be consistent. So the complete intelligible medium will be a triple $D = \langle \Sigma, \preceq, C \rangle$. We will call it a **consistency medium**. We will refer to the elements of a consistency medium D , as follows: $\Sigma_D, \preceq_D, C_D$.

The consistency medium has a surface similarity to “**information systems**” of the type defined by Dana Scott (Scott 1970, 169-176; Scott 1982, 577-613; Davey and Priestley 1990, 63-71). There, an information system is a triple $\mathbf{A} = \langle A, Con, \vdash \rangle$, where A is a set of tokens, Con is a nonempty set of finite subsets of A , and $\vdash$ is a relation of entailment. The relation between consistency media and information systems is subtle, since in a consistency medium we specify order and consistency and consequence is implied; whereas in an information system, we specify consistency and consequence and order is implied. The connection between consistency media and information systems will probably go through the concept of algebraic $\bigcap$ -structures, since those structures play a role more like consistency media, in that they determine information systems, and hence consequence relations. Working out the relation between consistency media and information systems is one of the next steps in this project, and the development of the concept of minimal media (in Chapter 9) is a stage toward this end.

Visualizing consistency media

We can adapt a standard diagramming technique to illustrate particular **finite consistency media** (consistency media with a finite number of assertion types). The Hasse diagramming technique is a standard for illustrating finite partial orders. For more background on Hasse diagrams see Davey and Priestley, p. 7ff. We repeat in the note below the key points needed to read such diagrams.² For any finite consistency medium D , the Hasse technique

²Begin with the definition of the **covering** relation. “Let P be an ordered set and let $x, y \in P$. We say that x is covered by y (or y covers x), ... if $x \prec y$ and $x \preceq z \prec y$ implies $z = x$. The latter condition is

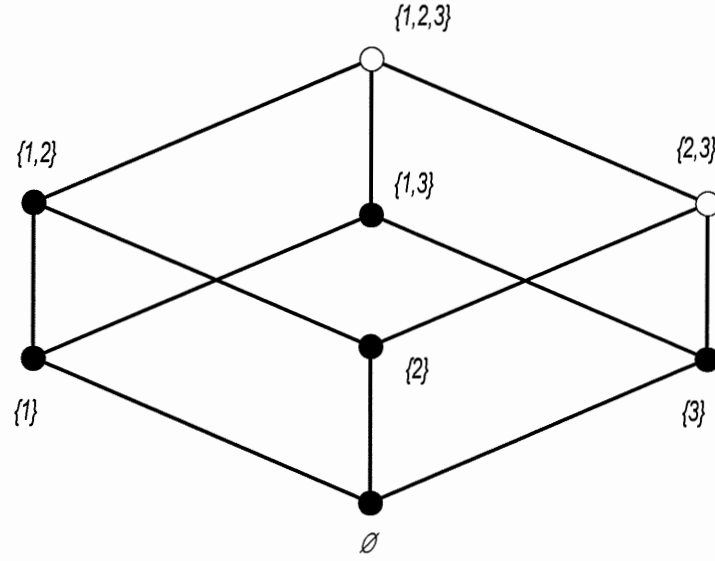


Figure 4-1: Modified Hasse diagram showing a complete finite consistency medium P .

is sufficient to illustrate the set of assertion types Σ_D and the ordering $\preceq_D$. We need only add a way to see the subset C_D . We indicate extension of C_D on the ordering diagram by filling in the circles of those assertion types in C_D . This modified version of the Hasse diagram is capable of showing an entire (finite) consistency medium in a single diagram.

For example, see Figure 4-1. This diagram shows a consistency medium P in which the

demanding that there be no element z of P with $x \prec z \prec y$." As Davey and Priestley point out, in the finite case, the covering relation determines, and is determined by, the order relation. The definition of the Hasse diagram is given in terms of the covering relation. "Let P be a finite ordered set. We can represent P by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation). The construction goes as follows.

1. To each point $x \in P$,
associate a point $P(x)$ of the euclidean plane $\mathcal{R}^2$, depicted by a small circle with centre at $P(x)$.
2. For each covering pair
 (x, y) such that y covers x in P , take a line segment $l(x, y)$ joining the circle at $P(x)$ to the circle at $P(y)$.
3. Carry out (1) and (2) in such a way that
 - (a) if y covers x , then $P(x)$ is 'lower' than $P(y)$ (that is, in standard cartesian coordinates, has a strictly smaller second coordinate);
 - (b) the circle at $P(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$ "
(Davey and Priestley 1990, p. 7).

set of assertion types is $\Sigma = \mathcal{P}(\{1, 2, 3\})$, where the ordering $\preceq$ is given by set inclusion, and every assertion type except those containing both 2 and 3 is in C .

First technique-specific assumption: D2

As of this point in the discussion, we have expressed our intention that in a given consistency medium D , the elements of the set C_D be all, and only, the elements of Σ_D which are consistent. Now we need to back that up with our first technique-specific assumption. We will define, as an technique-specific assumption, the condition that for any assertion type $p \in \Sigma_D$, $p \in C_D$ just in case p is actually consistent. This assumption will apply to every consistency medium D with respect to the practice of categorization $\mathcal{T}_{\Sigma_D}$ by which the elements of Σ_D (the assertion types for D) are interpreted.

D2: (Consistency) $\forall p \in \Sigma_D$ ($p \in C_D$ iff $\mathcal{T}_{\Sigma_D}(p) \neq \emptyset$)

D2 paraphrase: We assume that C_D contains all, and only, the elements of Σ_D which are consistent when those elements are interpreted with respect to $\mathcal{T}_{\Sigma_D}$.

We have now introduced enough structure to make it possible to express representational versions of Lindenbaum's Lemma and the order-consistency proxy for consequence relation from the corollary to Lindenbaum's Lemma.

4.3.2 Expressing the representational forms of Lindenbaum's Lemma and the order-consistency condition from the corollary to Lindenbaum's Lemma

Informal definition

In keeping with our foundational characterizations, we can informally re-express Lindenbaum's Lemma and the order-consistency condition in the corollary as follows:

Lindenbaum's Lemma:

We replace “set of sentences” with the more general “assertion type” and the result is “Every consistent assertion type has a maximal extension.”

Order-consistency condition in the Corollary to Lindenbaum's Lemma:

We begin with “ q belongs to every maximal extension of P .” Since extension in the set of sentences context is given by set inclusion, this is equivalent to “every maximal extension of P extends $\{q\}$.” Since according to our foundational characterization, we are using assertion types instead of sets of sentences, this becomes: “every maximal extension of p is an extension of q .” We can further rephrase the condition, since if x is a maximal extension of p and x extends q , then x is a maximal extension of q . So we can say: “**every maximal extension of p is a maximal extension of q .**”

Formal definition of the representational form of Lindenbaum's Lemma (LL)

Recall that a maximal assertion type is one that is consistent, and has only inconsistent proper extensions. Relative to a consistency medium D , we can express the **set of maximal extensions** in D as follows:

$$Max_D = \{x \in C_D \mid \forall y \in \Sigma_D (y \succ_D x \rightarrow y \notin C_D)\}$$

Max_D is the set of all maximal extensions for the consistency medium D .

For example, in the consistency medium P shown in Figure 4-1 on page 136, $Max_P = \{\{1, 2\}, \{1, 3\}\}$.

We can now define a function $Max_D : \Sigma_D \rightarrow Max_D$, such that for any assertion type p , the value of the $Max_D(p)$ is **the set of maximal extensions of p** . We define the function Max_D as follows:

$$\forall p \in \Sigma_D, Max_D(p) = \{x \in Max_D \mid x \succeq_D p\}.$$

For example, in the consistency medium P shown in Figure 4-1 on page 136, $Max_P(\{1\}) = \{\{1, 2\}, \{1, 3\}\}$, while $Max_P(\{2\}) = \{\{1, 2\}\}$.

Above we informally characterized the representational form of Lindenbaum's Lemma as: Every consistent assertion type has a maximal extension. Using the concepts just defined, we can express **the representational form of Lindenbaum's Lemma** formally as follows:

$$(LL) \forall p \in C_D, Max_D(p) \neq \emptyset.$$

We will take LL to be a technique-specific assumption in the representational technique we are developing.

The example consistency medium P shown in Figure 4-1 on page 136 satisfies LL .

Formal definition of the representational form of the order-consistency condition in the corollary to Lindenbaum's Lemma

Above we informally characterized the representational form of the order-consistency condition in the corollary to Lindenbaum's Lemma as: "every maximal extension of p is a maximal extension of q ." Using the concepts just defined, we can express the representational form of the order-consistency condition in the corollary to Lindenbaum's Lemma formally using a defined predicate LLC . LLC is a predicate to be applied to pairs of assertion types. We can state the definition of LLC formally as follows:

$$\forall p, q \in \Sigma_D, LLC_D(p, q) \text{ iff } Max_D(p) \subseteq Max_D(q).$$

We will use LLC as the proxy for consequence relation in the representational technique we are developing.

4.3.3 Taking stock: an incomplete representational technique

Let us review what we have at this point in the development. We have a foundational characterization, identical to that used for the model-theoretic techniques. We have an intelligible medium, the consistency medium. We have a proxy for consequence relation LLC . And we have two technique-specific assumptions: $D2$ and LL .

We can consider what we have developed, and ask "is this enough? have we developed a complete representational technique?" The answer is "no." The reason is that our technique-specific assumptions are not strong enough to imply the intelligibility of consequence. That is, given that some consistency medium D satisfies $D2$ and LL with respect to $\mathcal{T}_{\Sigma_D}$, it is not the case that the proxy for consequence relation LLC is equivalent to the representational concept of logical consequence ALC .

Consider the following example: Say that the set of assertion types $\Sigma = \{ p, q \}$, that the universe of relevant possibility $\mathcal{U} = \{1, 2\}$, and that the practice of categorization $\mathcal{T}_{\Sigma}$ maps p to $\{1\}$ and q to $\{1, 2\}$. Form a consistency medium D , such that $\Sigma_D = \{p, q\}$, $\preceq_D = \{\langle p, p \rangle, \langle p, q \rangle, \langle q, q \rangle\}$, and $C_D = \{p, q\}$. Then $Max_D = \{q\}$, $Max_D(p) = \{q\}$, and $Max_D(q) = \{q\}$. So D satisfies $D2$ and LL with respect to $\mathcal{T}_{\Sigma_D}$. Yet in this case, $LLC_D(q, p)$ is true but $ALC_{\langle \Sigma, \mathcal{T}_{\Sigma} \rangle}(q, p)$ is false.

So $D2$ and LL are not sufficient to imply the intelligibility of consequence.

We need to do more. One way to complete the definition in progress and reach the goal of a new representational technique will be to add more technique-specific assumptions to the ones defined so far. We begin the search for such assumptions in the next section.

4.3.4 Looking for additional technique-specific assumptions

A place to look: the language of propositional logic

We have a natural place to look for these additional assumptions: the language of propositional logic. We already know that in the syntactic context, Lindenbaum’s Lemma applies to the language of propositional logic, and the order-consistency condition specified in the corollary is equivalent to deducibility for propositional logic (the syntactic proxy for consequence). So we will look for semantic conditions, expressed in representational terms, which characterize the language of propositional logic, in the hopes that if we add these conditions to the technique-specific assumptions we already have, those collected assumptions will be sufficient to imply the equivalence of *LLC* and the representational conception of consequence. (Note that the version of propositional logic we are interested in will be what we have called the $\mathcal{P}(WFF)$ -language of propositional logic.)

Properties of the $\mathcal{P}(WFF)$ -language of propositional logic

We can note the following properties of the $\mathcal{P}(WFF)$ -language of propositional logic:

1. The set of assertion types for the language is **structurally defined**. That is to say, there is a mathematical construction (in this case the powerset operation) by which the set of assertion types for the system is generated. Structural definition is a common way to characterize a set of assertion types, as it makes it possible to give a finite description of an infinite set of assertion types.
2. The set of assertion types for the $\mathcal{P}(WFF)$ -language of propositional logic supports an ordering which corresponds to a notion of **structural elaboration**. By structural elaboration, we mean a relation between two assertion types such that the structure of the first assertion type is somehow included in the structure of the second. We could equivalently say that the structure of the first is “**elaborated**” by the second, or

the second “**structurally elaborates**” the first. The flexibility in this “definition” is due to the great variety in the possible structural forms for assertion types, and the methods by which they could be interpreted. Like the notion of “satisfaction” the best we can do is to gesture in the general direction of the concept intended, and provide specific examples. See Etchemendy 1990, 33ff, for a discussion of the challenge of providing a general account of the satisfaction relation.

- (a) In the case of the $\mathcal{P}(WFF)$ -language of propositional logic, the inclusion ordering corresponds to structural elaboration. For any sets of propositional sentences P and Q , Q structurally elaborates P just in case the structure of P is included within the structure of Q , that is, just in case $P \subseteq Q$.
- (b) In the case of feature structure/element combinations from feature logic, the subsumption relation corresponds to structural elaboration.
For feature structure/element combinations $\langle \mathbb{A}, d \rangle$ and $\langle \mathbb{B}, e \rangle$, $\langle \mathbb{B}, e \rangle$ structurally elaborates $\langle \mathbb{A}, d \rangle$ just in case $\langle \mathbb{A}, d \rangle$ subsumes $\langle \mathbb{B}, e \rangle$.

The concept of structural elaboration can only apply to sets of assertion types which are structurally defined (and so have articulable internal structure).

We see from the above that the set of assertion types for the $\mathcal{P}(WFF)$ -language of propositional logic (i.e. the set $\mathcal{P}(WFF)$ itself) can be thought of as a partially ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ in which the ordering $\subseteq$ corresponds to structural elaboration among the assertion types. In the following sections, we will look at two conditions which apply to partially ordered sets of assertion types $\langle \Sigma, \preceq \rangle$, and the practices of categorization $\mathcal{T}_\Sigma$ interpreting those types. These two conditions are called “monotonic extension of commitment” and “weak extensibility.” These are both conditions satisfied by the assertion types of the $\mathcal{P}(WFF)$ -language of propositional logic with respect to any conventional interpretation of those types. In other words, the partially ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies both conditions with respect to any practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ that meets the semantic conventions of the $\mathcal{P}(WFF)$ -language of propositional logic. We will prove this result in Section 4.8 below.

These two conditions: monotonic extension of commitment and weak extensibility, when adapted to the context of consistency media, will provide the technique-specific assumptions we need to complete the representational technique under construction.

4.3.5 The condition of monotonic extension of commitment

The concept of extension of commitment

Extension of commitment is a relation between assertion types. Simply put, extension of commitment is the inverse of the assumed relation of logical consequence. That is to say, p extends the commitments of q iff q is a consequence of p . Formally, given a set of assertion types Σ and a practice of categorization $\mathcal{T}_\Sigma$, for any $p, q \in \Sigma$, p extends the commitments of q iff $\mathcal{T}_\Sigma(p) \subseteq \mathcal{T}_\Sigma(q)$.

Extension of commitment in relation to elaboration of structure

The $\mathcal{P}(WFF)$ -language of propositional logic exhibits the following property.

Given two assertion types P, Q , if Q elaborates the structure of P , then Q extends the commitments of P .

We call this property “**monotonic extension of commitment with respect to successive elaboration of structure.**” Consider the following sequence of assertion types drawn from the $\mathcal{P}(WFF)$ -language of propositional logic: $\{A_1\}, \{A_1, A_2\}, \{A_1, A_2, A_3\}$. These assertion types are ordered by $\subseteq$ as follows: $\{A_1\} \subseteq \{A_1, A_2\} \subseteq \{A_1, A_2, A_3\}$. We know from the semantic conventions of propositional logic that the possibilities represented by each member in this sequence is a subset of the possibilities represented by the previous member, regardless of how the sentence symbols are interpreted. We can state this relationship formally as follows. For all $P, Q \in \mathcal{P}(WFF)$; if $P \subseteq Q$ then $\mathcal{T}_{\mathcal{P}(WFF)}(Q) \subseteq \mathcal{T}_{\mathcal{P}(WFF)}(P)$. We will show how this property follows from the semantic conventions of the $\mathcal{P}(WFF)$ -language of propositional logic below (in Section 4.8).

Let us here consider the value of using a language of assertion which obeys such a property. (Note: when we say “**language of assertion**” we mean a set of assertion types together with the semantic conventions constraining acceptable interpretations of those types). This property is very useful for beings like ourselves (or computers) which have fi-

nite processing capability and which produce and consume the expressions of propositional claims over time. The processing of a complex propositional claim, e.g. $\{A_1, A_2, A_3\}$ can be modelled as the sequential processing of three assertion-types $\{A_1\}, \{A_1, A_2\}, \{A_1, A_2, A_3\}$. The idea behind this modelling technique is as follows. At each step, there is a single assertion type that models the current commitments of the processor at that point in time. The monotonicity property increases the efficiency of processing compound propositional claims by guaranteeing that as long as the processing follows some sequence of elaboration of structure, every step in processing either reduces the possibilities under consideration or leaves them unchanged. Given the assumption of monotonicity, once a possibility has been excluded, there is never some future processing step which will bring it back into consideration. This means that the complex expressions of those languages of assertion which obey this assumption can be processed very efficiently. For this reason, many languages of assertion (though not all) are assumed to obey this property as a consequence of their semantic conventions.

The constraint defined

Monotonic extension of commitment (*MEC*) is a constraint defined relatively to an ordered set of assertion types $\langle \Sigma, \preceq \rangle$ and a practice of categorization $\mathcal{T}_\Sigma : \Sigma \rightarrow \mathcal{P}(\mathcal{U})$.

$$MEC : \text{For all } p, q \in \Sigma, p \preceq q \text{ implies } \mathcal{T}_\Sigma(q) \subseteq \mathcal{T}_\Sigma(p)$$

It is important to note that the specification of the constraint does not require that the ordering $\preceq$ actually correspond to the elaboration of structure within the set of assertion types. Such a correspondence helps explain why the constraint is frequently observed, but it is not part of the constraint itself.

4.3.6 The condition of weak extensibility

The constraint introduced in context

Weak extensibility is a constraint which supports the usefulness of monotonic extension of commitment via the successive elaboration of structure. To explain what weak extensibility is, and to see why it is important, let us consider the following example. Consider a subject (person/machine/etc.) who is extending their propositional commitments over time, and

is using a single assertion type drawn from the $\mathcal{P}(WFF)$ -language of propositional logic to represent their current commitments. Say that the current commitments of the subject are represented at some specific time by the assertion type $P = \{A_1, A_2, A_3\}$. Another commitment which the subject could express using the $\mathcal{P}(WFF)$ -language of propositional logic would be $Q = \{A_4, A_5\}$. With respect to Q , it would be useful for the subject to be able to extend their current commitments in either of two directions. They could choose to commit to the conjunction of P and Q , or they could choose to commit to the conjunction of P and the negation of Q . Given their processing style, it is desirable for the subject to be able to express these extended commitments through an elaboration of the structure of the expression of their current commitments, namely through an elaboration of the structure of $\dot{P}$. It would be a limitation on the usefulness of the language of assertion in use, if in the process of extending their commitments through successive elaboration, the subject reached a point where they could not add some expressible commitment (or its negation) to their currently expressed commitment.

The $\mathcal{P}(WFF)$ -language of propositional logic gives the subject various ways to extend their commitments along paths of successive elaboration of structure. Let us consider how the $\mathcal{P}(WFF)$ -language of propositional logic adapts to our example.

To represent the conjunction of P and Q , the subject need simply form the union of P and Q . That union, as a subset of WFF , will be an element of $\mathcal{P}(WFF)$. Call it R . The assertion type $R = P \cup Q = \{A_1, A_2, A_3, A_4, A_5\}$ exactly represents the possibilities in which the conjunction of P and Q would be true, regardless of the interpretations of the sentence symbols involved. Further, $P \subseteq R$, that is to say, R is a structural elaboration of P .

To represent the conjunction of P and the negation of Q , if Q is finite, the subject could form the union of P and a singleton containing the negation of the conjunction of the elements of Q . That union, as a subset of WFF , will be an element of $\mathcal{P}(WFF)$. Call it S . The assertion type $S = \{A_1, A_2, A_3, \neg(A_4 \wedge A_5)\}$ exactly represents the possibilities in which the conjunction of P and the negation of Q would be true, regardless of the interpretations of the sentence symbols involved. Further, $P \subseteq S$, that is to say, S is a structural elaboration of P .

When Q is an infinite set, the method just described does not work, since we have not

assumed infinitary expressions. A different method is available to the subject. To represent the conjunction of P and the negation of Q , when Q is infinite, the subject has available the following set of sets of assertion types $\Gamma = \{P \cup \{\neg q\} \mid q \in Q\}$. In the specific case of our example, $\Gamma = \{\{A_1, A_2, A_3, \neg A_4\}, \{A_1, A_2, A_3, \neg A_5\}\}$. Collectively, that set exactly represents the possibilities in which the conjunction of P and the negation of Q would be true, regardless of the interpretation of the sentence symbols involved. Further, $P \subseteq \{A_1, A_2, A_3, \neg A_4\}$, and $P \subseteq \{A_1, A_2, A_3, \neg A_5\}$. That is to say, each of the elements of Γ is a structural elaboration of P .

The elements of Γ give two alternative ways in which the subject could extend the commitments represented by P , which together exhaust the possibilities in which the conjunction of P and the negation of Q would be true. On the method by which we constructed Γ , the subject is not necessarily able to extend their commitments via structural elaboration to represent in a single assertion type all and only the possibilities in which the conjunction of P and the negation of Q would be true. What the subject is guaranteed by this method is that they will be able to extend their commitments via structural elaboration to represent any possibility in which the conjunction of P and the negation of Q would be true, and furthermore, that all possibilities represented by such an extension would be possibilities in which the conjunction of P and the negation of Q would be true.

We can use these examples to motivate the definitions of two related properties: strong extensibility and weak extensibility. Both properties are defined relatively to a partially ordered set of assertion types $\langle \Sigma, \preceq \rangle$ and a practice of categorization $\mathcal{T}_\Sigma$ by which the elements of that set are interpreted. Both properties can be thought of as “richness” constraints... if met, they mean that the partially ordered set of assertion types is rich enough to permit a certain kind of extension of commitment to continue without halt.

Strong extensibility

Given a partially ordered set of assertion types $\langle \Sigma, \preceq \rangle$ and a practice of categorization $\mathcal{T}_\Sigma$ by which the elements of that set are interpreted, we say that $\langle \Sigma, \preceq \rangle$ is **strongly extensible** with respect to $\mathcal{T}_\Sigma$ if and only if

for every pair of assertion types $p, q \in \Sigma$,

- (a) there exists some assertion type $r \in \Sigma$ such that $r \succeq p$ and $r \succeq q$ and r exactly represents the possibilities in which the conjunction of p and q would be true (i.e. $\mathcal{T}_\Sigma(r) = \mathcal{T}_\Sigma(p) \cap \mathcal{T}_\Sigma(q)$); and
- (b) there exists some assertion type $s \in \Sigma$ such that $s \succeq p$ and s exactly represents the possibilities in which the conjunction of p and the negation of q would be true (i.e. $\mathcal{T}_\Sigma(s) = \mathcal{T}_\Sigma(p) - \mathcal{T}_\Sigma(q)$).

In the examples we considered above, if $P = \{A_1, A_2, A_3\}$, and $Q = \{A_4, A_5\}$, are interpreted as assertion types in the $\mathcal{P}(WFF)$ -language of propositional logic, then for P, Q , the member $\{A_1, A_2, A_3, A_4, A_5\}$ of $\mathcal{P}(WFF)$ is an assertion type satisfying part (a) of the definition of strong extensibility with respect to any acceptable interpretation $\mathcal{T}_{P(WFF)}$; and the member $\{A_1, A_2, A_3, \neg(A_4 \wedge A_5)\}$ of $\mathcal{P}(WFF)$ is an assertion type satisfying part (b) of the definition with respect to any acceptable interpretation $\mathcal{T}_{P(WFF)}$. However, the method by which $\{A_1, A_2, A_3, \neg(A_4 \wedge A_5)\}$ was constructed fails in the case where the negated assertion type is an infinite set of sentences; and therefore that method is insufficient to show the strong extensibility of the $\mathcal{P}(WFF)$ -language of propositional logic.

An example of a partially ordered set of assertion types which is strongly extensible with respect to the practice of categorization by which it is interpreted is given in Section 5.4.

Notation for talking about ordering relations

Before discussing weak extensibility, let us introduce some notation we will use for talking about ordering relations.

$\uparrow p$ is the up-set of p with respect to the ordered set $\langle \Sigma, \preceq \rangle$, that is, $\uparrow p = \{x \in \Sigma \mid x \succeq p\}$.

P^u is the set of upper bounds of P with respect to the ordered set $\langle \Sigma, \preceq \rangle$, that is, $P^u = \{x \in \Sigma \mid \text{for all } p \in P, x \succeq p\}$. Note that $\{p, q\}^u = \uparrow p \cap \uparrow q$.

If we need to make clear which ordered set we are referring to, we use a subscript naming the set ordered, e.g. $\uparrow_\Sigma(p)$ is the up-set of p with respect to the ordered set $\langle \Sigma, \preceq_\Sigma \rangle$.

Weak extensibility

Given a partially ordered set of assertion types $\langle \Sigma, \preceq \rangle$ and a practice of categorization $\mathcal{T}_\Sigma$ by which the elements of that set are interpreted, we say that $\langle \Sigma, \preceq \rangle$ is **weakly extensible** with respect to $\mathcal{T}_\Sigma$ if and only if

for every pair of assertion types $p, q \in \Sigma$,

- (a) there exists some set of assertion types $R \subseteq \{p, q\}^u$ such that the members of R , collectively, exactly represent the possibilities in which the conjunction of p and q would be true (i.e. $\bigcup \mathcal{T}_\Sigma[R] = \mathcal{T}_\Sigma(p) \cap \mathcal{T}_\Sigma(q)$); and
- (b) there exists some set of assertion types $S \subseteq \uparrow p$ such that the members of S , collectively, exactly represent the possibilities in which the conjunction of p and the negation of q would be true (i.e. $\bigcup \mathcal{T}_\Sigma[S] = \mathcal{T}_\Sigma(p) - \mathcal{T}_\Sigma(q)$).

In the examples above, if $P = \{A_1, A_2, A_3\}$, and $Q = \{A_4, A_5\}$, are interpreted as assertion types of the $\mathcal{P}(WFF)$ -language of propositional logic, then for P, Q , the set $R = \{P \cup Q\} = \{\{A_1, A_2, A_3, A_4, A_5\}\}$ is an example of a set of assertion types satisfying part (a) of the definition of weak extensibility with respect to any acceptable interpretation $\mathcal{T}_{P(WFF)}$; and the set $S = \{\{A_1, A_2, A_3, \neg A_4\}, \{A_1, A_2, A_3, \neg A_5\}\}$ is an example of a set of assertion types satisfying part (b) of the definition with respect to any acceptable interpretation $\mathcal{T}_{P(WFF)}$. The methods by which we constructed these sets work in general, and we use them in a proof to follow (Section 4.8).

To summarise:

Given a partially ordered set of assertion types $\langle \Sigma, \preceq \rangle$ and a practice of categorization $\mathcal{T}_\Sigma$ by which the elements of that set are interpreted, we say that $\langle \Sigma, \preceq \rangle$ is **weakly extensible** with respect to $\mathcal{T}_\Sigma$ if and only if

$$\forall p, q \in \Sigma \ (\exists R \subseteq \{p, q\}^u \text{ such that } \bigcup \mathcal{T}_\Sigma[R] = \mathcal{T}_\Sigma(p) \cap \mathcal{T}_\Sigma(q)) \text{ and}$$

$$\forall p, q \in \Sigma \ (\exists S \subseteq \uparrow p \text{ such that } \bigcup \mathcal{T}_\Sigma[S] = \mathcal{T}_\Sigma(p) - \mathcal{T}_\Sigma(q))$$

4.3.7 Recapitulation: monotonic extension of commitment and weak extensibility

Let us consider what we have seen to this point. We have presented two conditions: “monotonic extension of commitment” and “weak extensibility” which are defined in relation to a partially ordered set of assertion types and a practice of categorization which interprets those types. We have seen reasons why it could be valuable to use languages of assertion whose assertion types are assumed to satisfy these conditions relative to the practices of categorization by which they are interpreted. We have seen reasons to expect (and will prove below in Section 4.8) that the semantic conventions which underlie our use of a standard language of assertion, the $\mathcal{P}(WFF)$ -language of propositional logic, are sufficient to imply that conventional uses of that language satisfy both conditions. That is, if a practice $\mathcal{T}_{\mathcal{P}(WFF)}$ interpreting $\mathcal{P}(WFF)$ is assumed to meet the semantic conventions of the $\mathcal{P}(WFF)$ -language of propositional logic, then the partially ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies monotonic extension of commitment and weak extensibility with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$. Similar semantic conventions which support this result in the case of the $\mathcal{P}(WFF)$ -language of propositional logic are assumed by many logical languages, and thus, the conventional applications of those languages can also be shown to satisfy monotonic extension of commitment and weak extensibility.

A property of the assumptions of monotonic extension of commitment and weak extensibility

One property we should note before continuing. The assumptions of monotonic extension of commitment and weak extensibility are independent of any of the technique-specific assumptions we used in the model-theoretic representational techniques. To see that this is the case, we can recognize that every assumption used in those techniques is dependent on the set of models, and all are independent of any ordering on the set of assertion types. On the other hand, both monotonic extension of commitment and weak extensibility are dependent on some ordering on the set of assertion types, and independent of any set of models. So one can easily construct cases in which one of the sets of technique-specific assumptions used by a model-theoretic technique are satisfied, yet monotonic extension of

commitment and weak extensibility fail; or vice versa.

4.3.8 A complete order-consistency representational technique: the Set *CL* technique

Adding technique-specific assumptions requiring monotonic extension of commitment and weak extensibility to the incomplete representational technique we had in Section 4.3.3 will give us a complete order-consistency representational technique, the Set *CL* technique. In order to express monotonic extension of commitment and weak extensibility with respect to a consistency medium, we introduce some new notation.

Notation for talking about the ordering relation used in consistency media

We will use a slight abbreviation of the notation above for talking about ordering relations with respect to a consistency medium. Every consistency medium includes an ordered set $\langle \Sigma_D, \preceq_D \rangle$. In relation to a consistency medium D , we use $\uparrow_D$ and P_D^u in place of $\uparrow_{\Sigma_D}$ and $P_{\Sigma_D}^u$.

The result of this substitution is as follows:

$\uparrow_D(p)$ is the up-set of p with respect to medium D (and hence the ordered set $\langle \Sigma_D, \preceq_D \rangle$), that is, $\uparrow_D(p) = \{x \in \Sigma_D \mid x \succeq_D p\}$

P_D^u is the set of upper bounds of P with respect to medium D (and hence the ordered set $\langle \Sigma_D, \preceq_D \rangle$), that is, $P_D^u = \{x \in \Sigma_D \mid \text{for all } p \in P, x \succeq_D p\}$

The Set *CL* representational technique

We can now give the full definition of the Set *CL* order-consistency representational technique, an instance of the representational schema. This technique abstracts the key concepts in the corollary to Lindenbaum's Lemma.

- Foundational assumptions. The foundational assumptions are the same as for all of the model-theoretic representational techniques (and are detailed in Section 4.3.1 above).
- Intelligible medium: The consistency medium.

- Proxy for consequence relation: The relation *LLC* (the representational version of the order-consistency condition in the corollary to Lindenbaum's Lemma).
- Technique-specific assumptions. There are four, and together we call them **the Set CL assumptions**

1. Monotonic extension of commitment

We assume that the ordered set $\langle \Sigma_D, \preceq_D \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_D}$.

D1: (Monotonic extension of commitment) $\forall p, q \in \Sigma_D$ ($p \preceq_D q$ implies $\mathcal{T}_{\Sigma_D}(q) \subseteq \mathcal{T}_{\Sigma_D}(p)$)

2. Consistency

We assume that C_D contains all, and only, the elements of Σ_D which are consistent with respect to $\mathcal{T}_{\Sigma_D}$.

D2: (Consistency) $\forall p \in \Sigma_D$ ($p \in C_D$ iff $\mathcal{T}_{\Sigma_D}(p) \neq \emptyset$)

3. Weak extensibility

We assume that the ordered set $\langle \Sigma_D, \preceq_D \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_D}$.

D3: (Weak extensibility)

D3a : $\forall p, q \in \Sigma_D$ ($\exists R \subseteq \{p, q\}_D^u$ such that $\bigcup \mathcal{T}_{\Sigma_D}[R] = \mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q)$)

D3b : $\forall p, q \in \Sigma_D$ ($\exists S \subseteq \uparrow_D(p)$ such that $\bigcup \mathcal{T}_{\Sigma_D}[S] = \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$)

4. Representational version of Lindenbaum's Lemma

Every consistent assertion type has a maximal extension.

(LL) $\forall p \in C_D, \text{Max}_D(p) \neq \emptyset$.

As we will show in Section 4.6 below, if a consistency medium D satisfies the Set CL assumptions with respect to a practice of categorization $\mathcal{T}_{\Sigma_D}$, then the proxy for consequence relation *LLC* is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$. Furthermore, because of the definition of the consistency medium and technique-specific assumptions, theories of consequence generated using the technique explain the relation of logical consequence in terms of two simpler concepts: a partial order representing primitive consequence, and a unary consistency predicate. These results show

that the Set *CL* technique is an instance of the representational schema, and as such, our first example of an order-consistency representational technique.

An important note

One thing which should be very clear from the above is that this technique does not do without the concepts of conjunction and negation. These concepts are built into the assumption of weak extensibility. Yet also note that what is required for an interpreted language to satisfy that assumption is weaker than other possible assumptions based on conjunction and negation (for example, strong extensibility).

4.4 Creating an order-consistency technique that does not require maximal extensions: the Set *CG* technique

4.4.1 Setting out the question: are maximal extensions necessary for an order-consistency technique?

We have just completed setting out our first order-consistency technique, the Set *CL* technique. That technique includes the assumption of the representational version of Lindenbaum's Lemma (*LL*). That means that every interpreted set of assertion types to which the Set *CL* technique is applied must satisfy assumption *LL*, that is, it must guarantee that every consistent assertion type has a maximal extension. Some languages may not have maximal extensions, and for others, working with maximal extensions may be cumbersome. So, we here consider the question: is it possible to construct an order-consistency representational technique that does not require the assumption of the representational version of Lindenbaum's Lemma (*LL*)? We will answer this question affirmatively by constructing just such a technique: the Set *CG* technique.

4.4.2 Plan for generating a new technique

As we mentioned in the Overview, we are going to construct the Set *CG* technique in a different way than the various model-theoretic variants we created from the Set *E* technique.

There, we created new techniques simply by changing the set of technique-specific assumptions. Here, we will create a new technique by changing two elements of the representational schema, the set of technique-specific assumptions and the proxy for consequence relation. We are going to create the Set *CG* technique from the Set *CL* technique by deleting the assumption *LL* (leaving *D1*, *D2*, and *D3*), and replacing the proxy for consequence relation *LLC* with a new proxy for consequence relation *LC* which we will describe below. We will see that the relation *LC* has a special relationship to the relation *LLC*, namely that in the context of *LL*, the two are equivalent! This result will enable us to see that the Set *CG* technique is a generalization of the Set *CL* technique.

We will begin by describing the new proxy for consequence relation *LC*. This in hand, we can give a definition of the Set *CG* technique. We will then prove the relationship between *LC* and *LLC*, and then consider its consequences. The proofs that the Set *CG* and *CL* techniques do indeed make consequence intelligible, and the proofs of the conditions under which the proxy for consequence relation *LLC* makes consequence intelligible, will follow in Sections 4.5, 4.6, and 4.7, respectively.

4.4.3 Defining the proxy for consequence relation *LC*

Motivation for, and structure of, the definition of *LC*

The fundamental concept underneath the proxy for consequence relation *LC* is the notion of “compossibility.” This is a concept we can model with respect to our representational framework. Given some set of assertion types Σ interpreted by a practice of categorization $\mathcal{T}_\Sigma$, we can choose to view two assertion types $p, q \in \Sigma$, as **compossible** just in case there is some possibility in which both p and q are true, that is, if $\mathcal{T}_\Sigma(p) \cap \mathcal{T}_\Sigma(q) \neq \emptyset$. What we are going to do is build a proxy for consequence (the relation *LC*) from a “proxy for compossibility” (the relation *CP*). **The key insight is that under certain conditions (which the technique-specific assumptions *D1*, *D2*, and *D3* are sufficient to guarantee) preservation of compossibility across all assertion types will be equivalent to preservation of truth across all possibilities.**

That’s a little dense, so let’s unpack it. Preservation of compossibility across all assertion types is a relation defined on pairs of assertion types p, q drawn from some set of assertion

types Σ interpreted by a practice $\mathcal{T}_\Sigma$. A pair p, q of assertion types is in that relation just in case, for every assertion type x in Σ , if p is compossible with x , then q is compossible with x . Preservation of truth across all possibilities is just *ALC*, the representational concept of logical consequence we have been using all along.

So now let's begin. First we will define a technique for making the relation of compossibility intelligible. That will involve defining a relation *CP* on pairs of assertion types in a consistency medium. *CP* will be a proxy for compossibility. If a consistency medium satisfies technique-specific assumptions *D1*, *D2*, and *D3a* with respect to the practice of categorization by which its assertion types are interpreted, then *CP* will be equivalent to the relation of compossibility between those types (that is, $CP(p, q)$ just in case $\mathcal{T}_\Sigma(p) \cap \mathcal{T}_\Sigma(q) \neq \emptyset$).

Then we will use this “proxy for compossibility” relation *CP* to build a relation *LC* on assertion types in a consistency medium. *LC* will be the relation of preservation of *CP* across all assertion types. What that means is that a pair of assertion types p, q will be in the relation *LC*, just in case, for every assertion type x in the medium, if $CP(p, x)$ then $CP(q, x)$. By itself, this may not look like much! But above we just saw that if a consistency medium satisfies technique-specific assumptions *D1*, *D2*, and *D3a* with respect to the practice of categorization by which its assertion types are interpreted, that *CP* was equivalent to compossibility. So under those conditions, *LC* is equivalent to preservation of compossibility across all assertion types. If we add the condition that the consistency medium in question satisfies technique-specific assumption *D3b* with respect to its interpretive practice, then we can apply the key insight and show that, under those extended conditions, *LC* will be equivalent to the representational consequence.

The definition of *CP* (the “proxy for compossibility”)

Now we are ready to begin the actual definitions. We will begin with definition of *CP* and work our way toward *LC*.

Given an arbitrary consistency medium *D*, we can define a relation CP_D on $\Sigma_D \times \Sigma_D$ as follows:

$$\forall p, q \in \Sigma_D, CP_D(p, q) \text{ iff } \exists y \in C_D ((y \succeq_D p) \text{ and } (y \succeq_D q))$$

We will prove below, that, for any consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2, D3a$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p, q \in \Sigma_D, CP_D(p, q)$ iff $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$ (i.e. p and q are compossible). This proof (Proposition 53) will demonstrate that under those conditions, the relation CP is a proxy for compossibility.

To see an example, consider the consistency medium P shown in Figure 4-1 on page 136. There, $CP_P(\{1\}, \{3\})$ because of $\{1, 3\}$; but it is not the case that $CP_P(\{2\}, \{3\})$.

The next step on the way to LC is to define a function (CC) , which, for any assertion type p , gives the set of all assertion types to which p bears the relation CP . That is, given an arbitrary consistency medium D , the function $CC_D : \Sigma_D \rightarrow \mathcal{P}(\Sigma_D)$ is defined as follows:

$$\forall p \in \Sigma_D, CC_D(p) = \{x \in \Sigma_D \mid CP_D(p, x)\}$$

Since we will use it more often than the relation CP , we give an expanded form of the definition of the function CC below. This is simply plugging the definition of CP into the definition of CC .

$$\forall p \in \Sigma_D, CC_D(p) = \{x \in \Sigma_D \mid \exists y \in C_D ((y \succeq_D p) \text{ and } (y \succeq_D x))\}$$

To see an example, consider the consistency medium P shown in Figure 4-1 on page 136. There we see that $CC_P(\{1\}) = C_P$; and $CC_P(\{3\}) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$.

Given the relationship of CP and CC , we also know that, for any consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2, D3a$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p, q \in \Sigma_D, q \in CC_D(p)$ iff $CP_D(p, q)$ iff $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$. So if D satisfies $D1, D2, D3a$ with respect to $\mathcal{T}_{\Sigma_D}$, then for any $p \in \Sigma_D$, membership in $CC_D(p)$ is equivalent to compossibility with p .

The relation LC : the Set CG proxy for consequence relation

Recall that we characterized the relation LC as “preservation of CP across all assertion types.” What that means is that a pair of assertion types p, q will be in the relation LC , just in case, for every assertion type x in the medium, if $CP(p, x)$ then $CP(q, x)$.

We formally define the relation LC on $\Sigma_D \times \Sigma_D$ as follows:

$$\forall p, q \in \Sigma_D, LC_D(p, q) \text{ iff } \forall x \in \Sigma_D, (CP_D(p, x) \rightarrow CP_D(q, x)).$$

We define an equivalent form of LC using CC_D as follows:

$$\forall p, q \in \Sigma_D, LC_D(p, q) \text{ iff } CC_D(p) \subseteq CC_D(q)$$

(This latter form is the one we will use in most proofs).

For any consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2, D3a$ with respect to $\mathcal{T}_{\Sigma_D}$ then $LC_D(p, q)$ is equivalent to preservation of compossibility across all assertion types. Further, if D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, LC_D is equivalent to $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ (the representational conception of logical consequence). We prove these results in Section 4.5 below. These proofs demonstrate the suitability of the relation LC as the proxy for consequence relation for the Set CG technique.

A definition: If, for two assertion types $p, q \in \Sigma_D$, $LC_D(p, q)$ then we will say that q is an LC_D -consequence of p .

4.4.4 An order-consistency representational technique that does not require maximal extensions: the Set CG technique

Now we can define the Set CG technique, an order-consistency representational technique that does not require maximal extensions. The Set CG technique is an instance of the representational schema, and as such, has the usual four components.

- Foundational assumptions. The foundational assumptions are the same as for all of the model-theoretic representational techniques and the Set CL technique (and are detailed in Section 4.3.1 above).
- Intelligible medium: The consistency medium.
- Proxy for consequence relation: The relation LC .
- Technique-specific assumptions. There are three, and together we call them **the Set CG assumptions**
 1. Monotonic extension of commitment
 $D1$: (Monotonic extension of commitment) Same as for the Set CL technique.
 2. Consistency
 $D2$: (Consistency) Same as for the Set CL technique.
 3. Weak extensibility
 $D3$: (Weak extensibility) Same as for the Set CL technique.

As we will show in Section 4.5 below, if a consistency medium D satisfies the Set CG assumptions with respect to a practice of categorization $\mathcal{T}_{\Sigma_D}$, then the proxy for consequence relation LC is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$. Furthermore, because of the definition of the consistency medium and technique-specific assumptions, theories of consequence generated using the technique explain the relation of logical consequence in terms of two simpler concepts: a partial order representing primitive consequence, and a unary consistency predicate (as with the Set LC technique). These results show that the Set CG technique is an instance of the representational schema.

4.4.5 The relationship between the Set CL and Set CG techniques

There is an important relationship between the proxy for consequence relation LLC used by the Set CL technique and the proxy for consequence relation LC used by the Set CG technique. If the representational version of Lindenbaum's Lemma (the assumption LL) is true of a consistency medium, then LLC is equivalent to LC . We show this purely order-theoretic fact below.

What does this relationship mean? If a consistency medium D satisfies assumption Set CL with respect to the practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting its assertion types, then D satisfies assumption Set CG with respect to $\mathcal{T}_{\Sigma_D}$, and further, D satisfies assumption LL . Under those conditions, the relation LLC_D is equivalent to LC_D , and LC_D is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$. So LLC_D is equivalent to the representational conception of logical consequence. This constitutes a proof that the Set CL technique makes the representational conception of logical consequence intelligible. It also shows that the Set CG technique is a generalization of the Set CL technique, since every application of the Set CL technique just is an application of the Set CG technique.

It turns out that the proxy for consequence relation LLC used by the Set CL technique is capable of making consequence intelligible in some cases which satisfy Set CG but do not satisfy LL . It gives a better understanding of the workings of the LLC relation to consider these cases (which we do in Section 4.7 below).

Proposition 42 *Given that a consistency medium D satisfies LL , then LC_D is equivalent to LLC_D .*

The proof takes two parts. First we show that for any consistency medium D , LC_D implies LLC_D . Then we show that for any consistency medium D satisfying LL , it is the case that LLC_D implies LC_D .

Proposition 43 *For any consistency medium D , $\forall p, q \in \Sigma_D$, $LC_D(p, q)$ implies $LLC_D(p, q)$.*

1. Let p, q be elements of Σ_D , such that $LC_D(p, q)$, that is, $CC_D(p) \subseteq CC_D(q)$.
2. [Show $LLC_D(p, q)$ that is, $Max_D(p) \subseteq Max_D(q)$]
3. Let $x \in Max_D(p)$.
4. So $x \in Max_D$ and $x \succeq_D p$.
5. [Show $x \succeq_D q$]
6. Note that $x \in CC_D(p)$, since $x \in C_D$, $x \succeq_D x$, and $x \succeq_D p$.
7. So $x \in CC_D(q)$.
8. So $\exists y \in C_D$, such that $y \succeq_D x$ and $y \succeq_D q$.
9. Since $x \in Max_D$, $\forall y \in \Sigma_D (y \succ_D x \rightarrow y \notin C_D)$.
10. Since $y \in C_D$, $y \not\succ_D x$.
11. So $y = x$.
12. So $x \succeq_D q$. ■ Proposition.

Proposition 44 *For any consistency medium D satisfying LL , for any $p, q \in \Sigma_D$, $LLC_D(p, q)$ implies $LC_D(p, q)$.*

1. Let D be a consistency medium satisfying LL .
2. Let p, q be elements of Σ_D , such that $LLC_D(p, q)$ that is, $Max_D(p) \subseteq Max_D(q)$.

3. [Show $LC_D(p, q)$, that is, $CC_D(p) \subseteq CC_D(q)$]
4. Let $x \in CC_D(p)$, i.e. $\exists y \in C_D$ such that $y \succeq_D x$ and $y \succeq_D p$.
5. Since $y \in C_D$, by LL , $\exists z \in Max_D(y)$.
6. So $z \in Max_D$ and $z \succeq_D y$.
7. By transitivity, $z \succeq_D x$ and $z \succeq_D p$.
8. So $z \in Max_D(p)$, and therefore, $z \in Max_D(q)$.
9. So $z \in C_D$ such that $z \succeq_D x$ and $z \succeq_D q$.
10. So $x \in CC_D(q)$. ■ Proposition.

4.5 Proof that the Set CG technique makes consequence intelligible

4.5.1 Basic propositions about $\preceq$, CC , and LC

Proposition 45 *For any consistency medium $\langle \Sigma, \preceq, C \rangle$, for all $p, q \in \Sigma$, if $p \preceq q$, then $CC(q) \subseteq CC(p)$.*

By the transitivity of $\preceq$ and the definition of CC .

Note that the converse of this relation is a condition often found in lattice theory: “for all $p, q \in \Sigma$, if $p \not\preceq q$, then there exists an r which is compatible with p but incompatible with q .” (in a Boolean logic: $r = p \wedge \neg q$).

Proposition 46 *For any consistency medium $\langle \Sigma, \preceq, C \rangle$, for all $p, q \in \Sigma$, if $p \preceq q$, then $LC(q, p)$.*

Apply the definition of LC to Proposition 45 above.

Proposition 47 *For any consistency medium $\langle \Sigma, \preceq, C \rangle$, for all $p \in \Sigma$, $p \in C \rightarrow p \in CC(p)$.*

By the definition of CC and reflexivity of $\preceq$.

Proposition 48 *LC is reflexive.*

By the definition of LC and the reflexivity of $\subseteq$.

Proposition 49 *LC is transitive.*

By the definition of LC and the transitivity of $\subseteq$.

Proposition 50 *LC obeys dilution. Dilution can be defined relative to the ordering $\preceq$. The intuitive idea we are seeking to model is that increasing commitments in the premise or decreasing commitments in the conclusion preserves consequence. For arbitrary consistency medium $\langle \Sigma, \preceq, C \rangle$, we define dilution as follows. If $LC(p, q)$ then for any $p^+ \in \Sigma$ such that $p \preceq p^+$, and for any $q^- \in \Sigma$ such that $q^- \preceq q$, $LC(p^+, q^-)$.*

By Proposition 46 and the transitivity of LC .

Proposition 51 *For any consistency medium D satisfying assumption Set CG with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, for all $e, f \in \Sigma_D$, $e \preceq_D f$ and $f \in C_D$ implies $e \in C_D$.*

This proposition shows that if D satisfies assumption Set CG with respect to $\mathcal{T}_{\Sigma_D}$, then consistency is conserved as one moves down the ordering $\preceq_D$.

Proof:

1. Let D be an arbitrary consistency medium satisfying assumption Set CG with respect to some practice of categorization $\mathcal{T}_{\Sigma_D}$. Let $e, f \in \Sigma_D$.
2. Assume $e \preceq_D f$ and $f \in C_D$.
3. By $D1$, $e \preceq_D f$ implies $\mathcal{T}_{\Sigma_D}(f) \subseteq \mathcal{T}_{\Sigma_D}(e)$.
4. By $D2$, $f \in C_D$ implies that $\mathcal{T}_{\Sigma_D}(f) \neq \emptyset$. So $\mathcal{T}_{\Sigma_D}(e) \neq \emptyset$.
5. By $D2$, $\mathcal{T}_{\Sigma_D}(e) \neq \emptyset$ implies $e \in C_D$. ■ Proposition.

Proposition 52 *For any consistency medium D satisfying assumption Set CG with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, for all $p, q \in \Sigma_D$, if $p \in CC_D(q)$, then $p \in C_D$.*

By the definition of CC and Proposition 51 above.

4.5.2 Proof of the “intelligibility of compossibility”

Here we show that, under certain conditions, the relation CP is a proxy for compossibility.

Proposition 53 *For any consistency medium D satisfying assumptions $D1$, $D2$, and $D3a$ with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, for all $p, q \in \Sigma_D$, $CP_D(p, q)$ iff $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.*

This proposition shows that, if consistency medium D satisfies assumptions $D1$, $D2$, and $D3a$ with respect to $\mathcal{T}_{\Sigma_D}$, then the structural relationship expressed by $CP_D(p, q)$ is equivalent to the unintelligible semantic relationship of compossibility expressed by $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.

1. Let D be an arbitrary consistency medium satisfying assumptions $D1$, $D2$, and $D3a$ with respect to some practice of categorization $\mathcal{T}_{\Sigma_D}$.

($\rightarrow$)

1. Let p, q be elements of Σ_D such that $CP_D(p, q)$.
2. [Show $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.]
3. Since $CP_D(p, q)$, $\exists y \in C_D$ such that $y \succeq_D p$ and $y \succeq_D q$.
4. $y \succeq_D p$ implies $\mathcal{T}_{\Sigma_D}(y) \subseteq \mathcal{T}_{\Sigma_D}(p)$, by $D1$.
5. $y \succeq_D q$ implies $\mathcal{T}_{\Sigma_D}(y) \subseteq \mathcal{T}_{\Sigma_D}(q)$, by $D1$.
6. $y \in C_D$ implies $\mathcal{T}_{\Sigma_D}(y) \neq \emptyset$, by $D2$.
7. So $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$. ■($\rightarrow$)

($\leftarrow$)

1. Let p, q be elements of Σ_D such that $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.
2. [Show $CP_D(p, q)$, that is, $\exists y \in C_D$ such that $y \succeq_D p$ and $y \succeq_D q$].
3. Let $x \in \mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q)$.

4. By *D3a*, we know that $\exists R \subseteq \{p, q\}_D^u$ such that $\bigcup \mathcal{T}_{\Sigma_D} [R] = \mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q)$.
5. So there is an $r \in R$ such that $x \in \mathcal{T}_{\Sigma_D}(r)$.
6. By *D2*, $r \in C_D$.
7. Since $r \in \{p, q\}_D^u$, we know that $r \succeq_D p$ and $r \succeq_D q$. So $CP_D(p, q)$. ■($\leftarrow$)■ Proposition.

Corollary 54 *For any consistency medium D satisfying assumptions *D1*, *D2*, and *D3a* with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, for all $p, q \in \Sigma_D$, $p \in CC_D(q)$ iff $\mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.*

This corollary claims that, if consistency medium D satisfies assumptions *D1*, *D2*, and *D3a* with respect to $\mathcal{T}_{\Sigma_D}$, then for any assertion type p , membership in $CC_D(p)$ is equivalent to compossibility with p . Immediate from Proposition 53 and the definition of CC .

4.5.3 Proof of the “Intelligibility of Consequence for the Set CG technique” theorem

Here we prove the intelligibility of consequence theorem for the Set *CG* technique.

Theorem 55 *For any consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$ such that D satisfies assumption Set *CG* with respect to $\mathcal{T}_{\Sigma_D}$,*

it is the case that the proxy for consequence relation LC_D is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ (that is, $\forall p, q \in \Sigma_D$, $CC_D(p) \subseteq CC_D(q)$ iff $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$).

Let D be an arbitrary consistency medium satisfying assumption Set *CG* with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$. Let p, q be arbitrary elements of Σ_D .

($\rightarrow$)

1. [Show $CC_D(p) \subseteq CC_D(q)$ implies $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$.]
2. Assume $\mathcal{T}_{\Sigma_D}(p) \not\subseteq \mathcal{T}_{\Sigma_D}(q)$. [Show $CC_D(p) \not\subseteq CC_D(q)$.]

3. So $\exists u \in \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 4. By $D3b$, we know $\exists S \subseteq \uparrow_D(p)$ such that $\bigcup \mathcal{T}_{\Sigma_D}[S] = \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 5. So there is some $s \in S$ such that $u \in \mathcal{T}_{\Sigma_D}(s)$, further, $s \succeq_D p$, and $\mathcal{T}_{\Sigma_D}(s) \subseteq \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 6. $u \in \mathcal{T}_{\Sigma_D}(s)$ implies $s \in C_D$ (by $D2$).
 7. Since $s \succeq_D s$ (reflexivity of $\preceq$), $s \succeq_D p$, and $s \in C_D$, we know that $s \in CC_D(p)$, by the definition of CC .
 8. [Claim: $s \notin CC_D(q)$]
 - (a) Assume $s \in CC_D(q)$. [Show a contradiction.]
 - (b) By Corollary 54, $\mathcal{T}_{\Sigma_D}(s) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.
 - (c) This contradicts $\mathcal{T}_{\Sigma_D}(s) \subseteq \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$. ■ Claim.
 9. So $s \notin CC_D(q)$. ■ ($\rightarrow$)
- ($\leftarrow$)
1. [Show $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$ implies $CC_D(p) \subseteq CC_D(q)$.]
 2. Assume $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$.
 3. Case 1. $\mathcal{T}_{\Sigma_D}(p) = \emptyset$.
 - (a) [Show that $CC_D(p) = \emptyset$.]
 - (b) Assume $\exists x \in CC_D(p)$. [Show contradiction.]
 - (c) Then by the definition of CC , $\exists y \in C_D$ such that $y \succeq_D x$, and $y \succeq_D p$.
 - (d) Since $y \in C_D$, and $y \succeq_D p$, then $p \in C_D$ by Proposition 51.
 - (e) Then $\mathcal{T}_{\Sigma_D}(p) \neq \emptyset$ (by $D2$). Contradiction.
 - (f) So $CC_D(p) = \emptyset$, so $CC_D(p) \subseteq CC_D(q)$ trivially. ■ (Case 1).
 4. Case 2. $\mathcal{T}_{\Sigma_D}(p) \neq \emptyset$.

- (a) Let $x \in CC_D(p)$.
- (b) Then $\mathcal{T}_{\Sigma_D}(x) \cap \mathcal{T}_{\Sigma_D}(p) \neq \emptyset$, by Corollary 54 above.
- (c) Since $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$, we know $\mathcal{T}_{\Sigma_D}(x) \cap \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$ as well.
- (d) Then $x \in CC_D(q)$ by Corollary 54 above. ■(Case 2)■($\leftarrow$) ■Theorem.

4.6 Corollary showing that the Set CL technique makes consequence intelligible

Here we prove the intelligibility of consequence corollary for the Set CL technique.

Corollary 56 *For any consistency medium D*

and practice of categorization $\mathcal{T}_{\Sigma_D}$ such that D satisfies assumption Set CL with respect to $\mathcal{T}_{\Sigma_D}$, it is the case that the proxy for consequence relation LLC_D is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$

1. Let D be an arbitrary consistency medium satisfying assumption Set CL with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$.
2. Then D satisfies assumption Set CG with respect to $\mathcal{T}_{\Sigma_D}$, and further, D satisfies assumption LL .
3. Under those conditions, (a) by Proposition 42, the relation LLC_D is equivalent to LC_D ; and (b) by Theorem 55 LC_D is equivalent to the representational conception of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$.
4. So LLC_D is equivalent to $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$. ■Corollary.

4.7 Exploring the cases in which the proxy relation LLC makes consequence intelligible

4.7.1 Discussion

From Corollary 56, we know that when a consistency medium D satisfies assumption Set CL with respect to the practice $\mathcal{T}_{\Sigma_D}$, the proxy for consequence relation LLC is equivalent to the representational conception of logical consequence. In order to understand LLC more fully, it is instructive to characterize more accurately the cases in which that equivalence (between LLC and representational consequence) holds. We carry out that characterization in this section. We will focus our attention on cases in which the consistency medium under consideration satisfies assumption Set CG with respect to the practice of categorization interpreting its assertion types, but does not satisfy LL (that is, there are consistent assertion types without maximal extensions). In all those cases, the Set CG proxy for consequence relation LC is equivalent to representational consequence. We wish to ask the question: in which of those cases is the Set CL proxy for consequence relation LLC also equivalent to representational consequence?

The key to the characterization is to recognize the interaction between an assertion type with no maximal extensions and the relation LLC . If an assertion type p has no maximal extensions, then $Max_D(p)$ will be a subset of $Max_D(q)$ for all $q \in \Sigma_D$. So unless every assertion type in Σ_D is a representational consequence (as expressed via $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$) of p , the proxy relation LLC will not be equivalent to consequence. We can express this condition formally as follows:

$$(LLX) \forall c \in C_D, Max_D(c) = \emptyset \rightarrow (\forall d \in \Sigma_D, \mathcal{T}_{\Sigma_D}(c) \subseteq \mathcal{T}_{\Sigma_D}(d))$$

It turns out that LLX exactly characterizes the cases in which

1. some consistency medium D satisfies the Set CG assumptions with respect to $\mathcal{T}_{\Sigma_D}$,
2. D does not satisfy LL , and
3. LLC is equivalent to consequence ($ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$).

We prove this claim below.

4.7.2 Propositions

We have already proved the claim that if D satisfies LL , then $LLC_D \equiv LC_D$. (Proposition 42).

Now we shall prove the claim:

Given that consistency medium D satisfies CG with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, and D does not satisfy LL , LLC_D is equivalent to $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ just in case D satisfies LLX with respect to $\mathcal{T}_{\Sigma_D}$.

Preparation

In preparation, we take some theorems from modal logic and define and prove their representational correspondents.

Interaction of consequence and extension From modal logic: If $\Gamma \vdash_{\Sigma} A$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\Sigma} A$. This proposition is given as 2.16(b) in Chellas 1980, 47.

We translate this into the following representational correspondent:

If q is a consequence of p , and r extends p , then q is a consequence of r .

Proposition 57 $\forall p, q, r \in \Sigma_D$, if $LC_D(p, q)$ and $p \preceq_D r$ then $LC_D(r, q)$

By Proposition 46, and the transitivity of LC (Proposition 49).

Interaction of consequence and maximal extension From modal logic: Let Γ be a Σ -maximal set of sentences. Then $A \in \Gamma$ iff $\Gamma \vdash_{\Sigma} A$. This proposition is given as 2.18.1 in Chellas 1980, 53.

We translate this into the following representational correspondent.

Let x be a maximal extension. Then x is a maximal extension of p just in case p is a consequence of x .

Proposition 58 Let $x \in Max_D$. Then $p \preceq_D x$ iff $LC_D(x, p)$.

The proof of this proposition is found in Section 4.9 at the end of this chapter.

Proof of scope of LLC

Proposition 59 *Given that D satisfies CG with respect to $\mathcal{T}_{\Sigma_D}$, and D does not satisfy LL , LLC_D is equivalent to $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ iff D satisfies LLX with respect to $\mathcal{T}_{\Sigma_D}$.*

The full proof of this proposition is found in Section 4.9 at the end of this chapter.

The left to right direction of the proposition is quick. If D does not satisfy LLX , then the statement of LLX provides a counterexample to representational consequence.

The right to left direction of the proposition is more involved. We assume LLX , and then show both directions of the “intelligibility equivalence.”

The left to right direction of the “intelligibility equivalence” uses ideas from the proof of Theorem 2.20(1) in Chellas 1980, 57. The proof is of the contrapositive, (e.g. assuming that q is not a representational consequence of p and showing that $LLC_D(p, q)$ is false). The assumption asserts the existence of a possibility in $\mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$, which by $D3b$ guarantees an assertion type s whose possibilities are within $\mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$. Given LLX , s will have a maximal extension; but that maximal extension cannot extend q .

The right to left direction of the “intelligibility equivalence” is easy, since we know that D satisfies CG with respect to $\mathcal{T}_{\Sigma_D}$, so $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ is equivalent to LC_D which implies LLC_D .

4.8 Proofs that the ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies the conditions of monotonic extension of commitment and weak extensibility with respect to any conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic

4.8.1 Overview

We begin this section with a presentation of a series of new concepts: **partially ordered truth media**, **operators on interpreted truth media**, a property of truth media **negation complete with respect to models**, and two properties of partially ordered truth media **monotonic extension of commitment with respect to models** and **weak ex-**

tensibility with respect to models. As a part of the discussion, we will introduce some general propositions with respect to those concepts. **These concepts and propositions will be generally useful beyond the proofs in this chapter.** In particular, we will use them in the proofs in Part IV. Following presentation of the new concepts and propositions, we will use them in giving the proof that the ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies the conditions of monotonic extension of commitment and weak extensibility with respect to any conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic.

4.8.2 The concept of partially ordered truth media

Above we defined the structure of a truth medium as the triple $\langle \Sigma, M, t \rangle$ where Σ is a set of assertion types, M is a set of models, and t is a function from Σ to $\mathcal{P}(M)$.

We can extend this concept to that of a **partially ordered truth medium**, which is a truth medium where the set Σ of assertion types has a partial order $\preceq$ defined upon it. For partially ordered truth medium A , we refer to the partial order on Σ_A by $\preceq_A$. (This is an abbreviation of the more formal $\preceq_{\Sigma_A}$).

4.8.3 Operators on interpreted truth media

A **unary operator uop on interpreted truth media**, is one which takes an arbitrary truth medium A and the practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ by which it is interpreted, and from them, constructs a new truth medium $uop(A)$, and the practices $\mathcal{T}_{\Sigma_{uop(A)}}$ and $\mathcal{T}_{M_{uop(A)}}$ by which $uop(A)$ is interpreted.

A **binary operator bop on interpreted truth media**, is one which takes an arbitrary truth medium A and the practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ by which it is interpreted, and an arbitrary truth medium B and the practices $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$ by which it is interpreted, and from them, constructs a new truth medium $bop(A, B)$ and the practices $\mathcal{T}_{\Sigma_{bop(A, B)}}$ and $\mathcal{T}_{M_{bop(A, B)}}$ by which $bop(A, B)$ is interpreted.

We can see that the results of these operators are interpreted truth media as well.

This concept of operators on interpreted media was inspired by the general approach in

Barwise and Seligman 1997. It will enable us to move relations of consequence from one context to another.

4.8.4 The operator pow

In this section, we define a unary operator on interpreted truth media pow .

Given an arbitrary truth medium A used with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,

the operator pow constructs a partially ordered truth medium $pow(A)$,

and derived practices of categorization $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$ defined as follows:

Truth medium $pow(A)$:

- $\Sigma_{pow(A)} = \mathcal{P}(\Sigma_A)$
- $\preceq_{pow(A)}$ is defined as the inclusion order on $\Sigma_{pow(A)}$
- $M_{pow(A)} = M_A$
- $t_{pow(A)} : \Sigma_{pow(A)} \rightarrow \mathcal{P}(M_{pow(A)})$ is defined as follows:
for any $S \in \Sigma_{pow(A)}$, $t_{pow(A)}(S) =$ if $S = \emptyset$ then $\bigcup t_A[\Sigma_A]$ else $\bigcap t_A[S]$

Derived practices $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$:

- $\mathcal{T}_{\Sigma_{pow(A)}} : \Sigma_{pow(A)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:
for any $S \in \Sigma_{pow(A)}$, $\mathcal{T}_{\Sigma_{pow(A)}}(S) =$ if $S = \emptyset$ then $\bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$ else $\bigcap \mathcal{T}_{\Sigma_A}[S]$
- $\mathcal{T}_{M_{pow(A)}} : M_{pow(A)} \rightarrow \mathcal{P}(\mathcal{U}) = \mathcal{T}_{M_A}$

4.8.5 A property of truth media: negation completeness with respect to models

Negation completeness is a property of truth media. Given an arbitrary truth medium A , we say that A is **negation complete with respect to models** iff $\forall p \in \Sigma_A, \exists R \subseteq \Sigma_A$

such that $\bigcup t_A[R] = \bigcup t_A[\Sigma_A] - t_A(p)$. Let us explore what this property means. Given a truth medium A , there is the set $\bigcup t_A[\Sigma_A]$. This set is the set of those models which are models of some assertion type of A . $\bigcup t_A[\Sigma_A]$ is a subset of the set M_A of all models for the truth medium A . For any particular assertion type $p \in \Sigma_A$, we can form the expression $\bigcup t_A[\Sigma_A] - t_A(p)$. This is the set of models which are models of some assertion type, but which are not models of p . What the negation complete with respect to models condition tells us is that for every assertion type $p \in \Sigma_A$, there is a set of assertion types $R \subseteq \Sigma_A$, such that the models of the elements of R (taken together) are exactly equal to $\bigcup t_A[\Sigma_A] - t_A(p)$. What that means is that if a disjunction d of the elements of R could be formed, the models of d and the models of p would be disjoint, and the models of d and the models of p would together exhaust the set of models in which some assertion type is true.

Example 1: A truth medium which is not negation complete with respect to models

Consider the truth medium *Roll* from Section 3.4. In that case:

$$\Sigma_{Roll} = \{odd, greaterThanFour, five\},$$

$$M_{Roll} = \{1, 2, 3, 4, 5, 6\}, \text{ and}$$

t_{Roll} was given by the following table:

$p \in \Sigma_{Roll}$	$t_{Roll}(p)$
<i>odd</i>	$\{1, 3, 5\}$
<i>greaterThanFour</i>	$\{5, 6\}$
<i>five</i>	$\{5\}$

Roll is not negation complete with respect to models. Consider for example, the assertion type *odd* $\in \Sigma_{Roll}$. There is no set of assertion types $R \subseteq \Sigma_{Roll}$ for which $\bigcup t_{Roll}[R] = \bigcup t_{Roll}[\Sigma_{Roll}] - t_A(odd) = \{1, 3, 5, 6\} - \{1, 3, 5\} = \{6\}$.

Example 2: A truth medium which is negation complete with respect to models

Now let us look at an example of a truth medium that is negation complete with respect to models. Consider the truth medium *R2*, defined as follows:

$$\Sigma_{R2} = \{odd, greaterThanFour, five, one, three, six\},$$

$M_{R2} = \{1, 2, 3, 4, 5, 6\}$, and

t_{R2} is given by the following table:

$p \in \Sigma_{R2}$	$t_{R2}(p)$
<i>odd</i>	$\{1, 3, 5\}$
<i>greaterThanFour</i>	$\{5, 6\}$
<i>five</i>	$\{5\}$
<i>one</i>	$\{1\}$
<i>three</i>	$\{3\}$
<i>six</i>	$\{6\}$

So $\bigcup t_{R2} [\Sigma_{R2}] = \{1, 3, 5, 6\}$. Now we can see that, for every element p of Σ_{R2} , there is an $R \subseteq \Sigma_{R2}$ such that $\bigcup t_{R2} [R] = \bigcup t_{R2} [\Sigma_{R2}] - t_{R2}(p)$.

$p \in \Sigma_{R2}$	$t_{R2}(p)$	$R \subseteq \Sigma_{R2}$ such that $\bigcup t_{R2} [R] = \bigcup t_{R2} [\Sigma_{R2}] - t_{R2}(p)$
<i>odd</i>	$\{1, 3, 5\}$	$\{six\}$
<i>greaterThanFour</i>	$\{5, 6\}$	$\{one, three\}$
<i>five</i>	$\{5\}$	$\{one, three, six\}$
<i>one</i>	$\{1\}$	$\{three, five, six\}$
<i>three</i>	$\{3\}$	$\{one, five, six\}$
<i>six</i>	$\{6\}$	$\{one, three, five\}$

4.8.6 A property of partially ordered truth media: monotonic extension of commitment with respect to models

We have defined monotonic extension of commitment as a property of an ordered set of assertion types $\langle \Sigma, \preceq \rangle$ with respect to a practice of categorization $\mathcal{T}_\Sigma$. Truth media support a closely related property which we call “**monotonic extension of commitment with respect to models.**” Given a partially ordered truth medium A , A satisfies monotonic extension of commitment with respect to models iff $\forall p, q \in \Sigma_A$, if $p \preceq_A q$ then $t_A(q) \subseteq t_A(p)$.

Monotonic extension of commitment with respect to models is a property that a truth medium has independently of its interpretation. There is a proposition which links the property of monotonic extension of commitment with respect to models with the property of monotonic extension of commitment.

Proposition 60 *For any partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that A satisfies assumption B1 with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models then $\langle \Sigma_A, \preceq_A \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_A}$.*

The proof is in Section 4.9 below.

In a way, one can see the property of monotonic extension of commitment for the ordered set of assertion types in a partially ordered truth medium (considered relative to the practice of categorization that interprets them) as factored into two aspects: that portion borne by assumption B1; and another part which we have characterized as monotonic extension of commitment with respect to models.

4.8.7 The property of weak extensibility with respect to models

We have defined weak extensibility as a property of an ordered set of assertion types $\langle \Sigma, \preceq \rangle$ with respect to a practice of categorization $\mathcal{T}_\Sigma$. Truth media support a closely related property which we call “**weak extensibility with respect to models.**”

A partially ordered truth medium A is weakly extensible with respect to models iff

- a) For every $p, q \in \Sigma_A$, $\exists R \subseteq \{p, q\}_A^u$ such that $\bigcup t_A[R] = t_A(p) \cap t_A(q)$
- b) For every $p, q \in \Sigma_A$, $\exists S \subseteq \uparrow_A p$ such that $\bigcup t_A[S] = t_A(p) - t_A(q)$

Note: $\{p, q\}_A^u = (\uparrow_A p \cap \uparrow_A q)$

Weak extensibility with respect to models is a property that a truth medium has independently of its interpretation. There is a proposition which links the property of weak extensibility with respect to models with the property of weak extensibility. There are two lemmas which support the proposition, we present them first. Proofs of the lemmas and the proposition can be found in Section 4.9.

Lemma 61 *Given arbitrary truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for any $R \subseteq \Sigma_A$, for all $p, q \in \Sigma_A$, if $\bigcup t_A[R] = t_A(p) \cap t_A(q)$, then $\bigcup \mathcal{T}_{\Sigma_A}[R] = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_A}(q)$.*

Lemma 62 *Given arbitrary truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for any $S \subseteq \Sigma_A$, for all $p, q \in \Sigma_A$, if $\bigcup t_A[S] = t_A(p) - t_A(q)$, then $\bigcup \mathcal{T}_{\Sigma_A}[S] = \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)$.*

Proposition 63 *Given an arbitrary partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then if A is weakly extensible with respect to models, then the partially ordered set $\langle \Sigma_A, \preceq_A \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_A}$.*

In a way, one can see the property of weak extensibility for the ordered set of assertion types in a partially ordered truth medium (considered relative to the practice of categorization that interprets them) as factored into two aspects: that portion borne by the technique-specific assumptions in Set PP ; and another part which we have characterized as weak extensibility with respect to models.

4.8.8 Constructing partially ordered truth media which satisfy monotonic extension of commitment and weak extensibility using the operator pow

These two propositions show how applying the operator pow to an interpreted truth medium, yields a partially ordered interpreted truth medium which satisfies both monotonic extension of commitment and weak extensibility.

Proposition 64 *For any truth medium A , $pow(A)$ satisfies monotonic extension of commitment with respect to models.*

The proof of this proposition can be found in Section 4.9. The proof follows from the definition of the function t_{pow} .

Proposition 65 *Given that A is a truth medium that is negation complete with respect to models, $pow(A)$ is weakly extensible with respect to models.*

The proof of this proposition can be found in Section 4.9. It is detailed, but the essential ideas are present in the discussion of weak extensibility in Section 4.3.6.

4.8.9 Proof that the ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies the conditions of monotonic extension of commitment and weak extensibility with respect to any conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic

Our task in this section is to show the following proposition.

Proposition 66 *The ordered set $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies the conditions of monotonic extension of commitment and weak extensibility with respect to any conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic.*

The proof of this proposition is slightly tricky... but only because of navigating some terminological details. Most of the work has already been done for us in the concepts and propositions above.

1. Our first question is: what counts as a “conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic”? You will recall that in Section 3.5.3, when we were using the Set E technique to model the relation of logical consequence for the $\mathcal{P}(WFF)$ -language of propositional logic, we gave semantic conventions which described acceptable *pairs* of interpretations, one for the elements of $\mathcal{P}(WFF)$ and one for TTA . (See Definition 40). A conventional interpretation of the $\mathcal{P}(WFF)$ -language of propositional logic just is the $\mathcal{P}(WFF)$ portion of those conventions. To be clear, we state that portion here.

- A practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)} : \mathcal{P}(WFF) \rightarrow \mathcal{P}(\mathcal{U})$ is an acceptable interpretation of $\mathcal{P}(WFF)$, provided that
 - (a) there exists a function $f : SS \rightarrow \mathcal{P}(\mathcal{U})$ that is an acceptable interpretation of the sentence symbols, and
 - i. ($\mathcal{P}(WFF)$ interpretation conditions)
 - There exists an interpretation $\mathcal{T}_{WFF}$ of WFF that satisfies SIM and COM with respect to f ,

such that $\forall P \in \mathcal{P}(WFF)$,

$$\mathcal{T}_{\mathcal{P}(WFF)}(P) = (\text{if } P = \emptyset \text{ then } \mathcal{U} \text{ else } \bigcap \mathcal{T}_{WFF}[P]).$$

2. The next step is to form a partially ordered truth medium PTO , which is PT with an inclusion ordering added on the set of assertion types $\mathcal{P}(WFF)$. That is, $PTO = \langle \langle \mathcal{P}(WFF), \subseteq \rangle, TTA, t_{PT} \rangle$.

3. Recall that $ST = \langle WFF, TTA, t_{ST} \rangle$. (It was defined in Section 3.5.2). We can now show that $PTO = pow(ST)$.

(a) Applying pow to ST (and arbitrary practices of interpretation) we get:

- $\Sigma_{pow(ST)} = \mathcal{P}(WFF)$
- $\preceq_{pow(ST)}$ is the inclusion order on $\mathcal{P}(WFF)$
- $M_{pow(ST)} = TTA$
- $t_{pow(ST)} : \mathcal{P}(WFF) \rightarrow \mathcal{P}(TTA)$ is defined as follows:
for any $S \in \mathcal{P}(WFF)$, $t_{pow(ST)}(S) = \text{if } S = \emptyset \text{ then } \bigcup t_{ST}[WFF] \text{ else } \bigcap t_{ST}[S]$

(a) Since $\bigcup t_{ST}[WFF] = TTA$ (the models of any sentence and its negation together exhaust the set of total truth assignments, by the definition of $\models$), $t_{pow(ST)} = t_{PT}$

(b) Together the above show that $PTO = pow(ST)$.

4. Now we can apply Proposition 64, and know that $pow(ST)$, and therefore PTO satisfies monotonic extension of commitment with respect to models.

5. Then we have to show that ST is negation complete with respect to models (in order to apply Proposition 65).

(a) Claim: The truth medium $ST = \langle WFF, TTA, t_{ST} \rangle$ as constructed is negation complete with respect to models.

(b) Since $\bigcup t_{ST}[WFF] = TTA$, for any arbitrary p , $R = \{\mathcal{E}_{\neg}(p)\}$ satisfies the negation completeness condition.

6. Then we can apply Proposition 65, and know that $pow(ST)$, and therefore PTO is weakly extensible with respect to models.
7. To complete the proof, we want to use Propositions 60 and 63. These propositions allow us to bridge from a truth medium satisfying properties of monotonic extension of commitment with respect to models and weak extensibility with respect to models; to interpreted ordered sets of assertion types satisfying monotonic extension of commitment and weak extensibility proper. But to use those propositions, we need to show that PTO satisfies Set PP with respect to practices of categorization interpreting both $\mathcal{P}(WFF)$ and TTA . As of yet, we have no interpretation of TTA to work with.
8. Here is where we bring in the given practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$. From $\mathcal{T}_{\mathcal{P}(WFF)}$ we can create an interpretation $\mathcal{T}_{TTA}$ such that together the interpretations $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ satisfy the conventions for mutually acceptable interpretations of $\mathcal{P}(WFF)$ and TTA as specified in Definition 40. This construction is obvious...
we just use the interpretations of the sentence-symbol singletons (e.g. $\{A_0\}$) in $\mathcal{P}(WFF)$ by $\mathcal{T}_{\mathcal{P}(WFF)}$ as the basis.
9. Given that $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ satisfy those conventions, we can apply Proposition 41, and know that truth medium PT satisfies assumption Set E with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$.
10. Applying Proposition 13, we know that PT satisfies assumption Set PP with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$. Since $PTO = PT$ plus ordering, and nothing in Set PP refers to ordering, we know that partially ordered truth medium PTO satisfies assumption Set PP with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$ as well.
11. Applying Proposition 60 to PTO , $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$, we get $\langle \mathcal{P}(WFF), \subseteq \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$.
12. Applying Proposition 63 to PTO , $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$, we get $\langle \mathcal{P}(WFF), \subseteq \rangle$ is weakly extensible with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$. ■

This concludes the linear text of the chapter. The remainder of the material in this chapter are the proofs of propositions mentioned earlier in the text.

4.9 Proofs of propositions from the text

Proof of Proposition 58:

(Proposition 58) Let $x \in Max_D$. Then $p \preceq_D x$ iff $LC_D(x, p)$.

Let $x \in Max_D$.

($\rightarrow$)

1. Assume $p \preceq_D x$. Then $LC(x, p)$ by Proposition 46. ■ ($\rightarrow$)

($\leftarrow$)

1. Assume $LC_D(x, p)$. [Show $p \preceq_D x$]
2. Assume $p \not\preceq_D x$. Show contradiction.
3. Now $x \in CC_D(x)$ since $x \in C_D$ and $x \succeq_D x$.
4. [Claim $x \notin CC_D(p)$]
 - (a) Assume $x \in CC_D(p)$. So $\exists y \in C_D$ such that $y \succeq_D x$ and $y \succeq_D p$.
 - (b) $y \neq x$ since otherwise $x \succeq_D p$.
 - (c) So $y \succ_D x$. Since $x \in Max_D$, $y \notin C_D$. Contradiction. ■ Claim.
5. So $x \notin CC_D(p)$. So $\neg LC_D(x, p)$. Contradiction. ■ ($\leftarrow$) ■ Proposition.

Proof of Proposition 59:

(Proposition 59) Given that D satisfies CG with respect to $\mathcal{T}_{\Sigma_D}$, and D does not satisfy LL , LLC_D is equivalent to $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ iff D satisfies LLX with respect to $\mathcal{T}_{\Sigma_D}$.

Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$ a practice of categorization, such that D satisfies CG with respect to $\mathcal{T}_{\Sigma_D}$, and D does not satisfy LL .

($\rightarrow$)

1. Assume that D does not satisfy LLX with respect to $\mathcal{T}_{\Sigma_D}$.
 2. Then $\exists c \in C_D$, such that $Max_D(c) = \emptyset$ and $\exists d \in \Sigma_D$, $\mathcal{T}_{\Sigma_D}(c) \not\subseteq \mathcal{T}_{\Sigma_D}(d)$.
 3. Since $Max_D(c) = \emptyset$, then $Max_D(c) \subseteq Max_D(d)$, so $LLC_D(c, d)$.
 4. But $\neg ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(c, d)$. ■ ($\rightarrow$)
- ($\leftarrow$)

This direction of the proof uses ideas from the proof of Theorem 2.20(1) in Chellas 1980, 57.

1. Assume that D satisfies LLX with respect to $\mathcal{T}_{\Sigma_D}$.
2. $\left[\text{Show that } \forall p, q \in \Sigma_D, LLC_D(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \right]$
 - (a) Let $p, q \in \Sigma_D$.
 - (b) ($\rightarrow$)
 - i. Assume $\neg ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q)$, that is, $\mathcal{T}_{\Sigma_D}(p) \not\subseteq \mathcal{T}_{\Sigma_D}(q)$.
 - ii. $\left[\begin{array}{l} \text{Show } \neg LLC_D(p, q), \text{ that is, } Max_D(p) \not\subseteq Max_D(q), \\ \text{i.e. } \exists x \in Max_D \text{ such that } x \succeq_D p \text{ and } x \not\succeq_D q. \end{array} \right]$
 - iii. Since $\mathcal{T}_{\Sigma_D}(p) \not\subseteq \mathcal{T}_{\Sigma_D}(q)$, $\mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q) \neq \emptyset$.
 - iv. Since D satisfies $D3b$ with respect to $\mathcal{T}_{\Sigma_D}$, $\exists S \subseteq \uparrow_D(p)$ such that $\bigcup \mathcal{T}_{\Sigma_D}[S] = \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 - v. Let $u \in \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 - vi. Then $\exists s \in S$ such that $u \in \mathcal{T}_{\Sigma_D}(s)$.
 - vii. Since D satisfies $D2$ with respect to $\mathcal{T}_{\Sigma_D}$, $s \in C_D$.
 - viii. Further, $\mathcal{T}_{\Sigma_D}(s) \subseteq \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$; and $\mathcal{T}_{\Sigma_D}(s) \not\subseteq \mathcal{T}_{\Sigma_D}(q)$ since $u \in \mathcal{T}_{\Sigma_D}(s)$ and $u \notin \mathcal{T}_{\Sigma_D}(q)$.
 - ix. Given that D satisfies LLX with respect to $\mathcal{T}_{\Sigma_D}$, we know that $Max_D(s) = \emptyset \rightarrow \mathcal{T}_{\Sigma_D}(s) \subseteq \mathcal{T}_{\Sigma_D}(q)$.
 - x. So $Max_D(s) \neq \emptyset$.
 - xi. Let $x \in Max_D(s)$. So $x \in Max_D$, and $x \succeq_D s$.

- xii. Since $s \in S$, and $S \subseteq \uparrow_D(p)$, $s \succeq_D p$.
- xiii. By transitivity, $x \succeq_D p$.
- xiv. So $x \in \text{Max}_D(p)$.
- xv. [Claim: $x \not\succeq_D q$]
 - A. Assume $x \succeq_D q$.
 - B. By Proposition 58, $LC_D(x, q)$.
 - C. Since D satisfies Set *CG* with respect to $\mathcal{T}_{\Sigma_D}$, $\mathcal{T}_{\Sigma_D}(x) \subseteq \mathcal{T}_{\Sigma_D}(q)$.
 - D. Recall that $x \succeq_D s$. By Proposition 58, $LC_D(x, s)$.
 - E. Since D satisfies Set *CG* with respect to $\mathcal{T}_{\Sigma_D}$, $\mathcal{T}_{\Sigma_D}(x) \subseteq \mathcal{T}_{\Sigma_D}(s)$.
 - F. Since $x \in \text{Max}_D$, $x \in C_D$; so since D satisfies *D2* with respect to $\mathcal{T}_{\Sigma_D}$, $\mathcal{T}_{\Sigma_D}(x) \neq \emptyset$.
 - G. Let $v \in \mathcal{T}_{\Sigma_D}(x)$.
 - H. Then $v \in \mathcal{T}_{\Sigma_D}(q)$ and $v \in \mathcal{T}_{\Sigma_D}(s)$.
 - I. But $\mathcal{T}_{\Sigma_D}(s) \subseteq \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$, so $v \notin \mathcal{T}_{\Sigma_D}(q)$. Contradiction. ■ Claim.
- xvi. So $x \not\succeq_D q$. ■ ($\rightarrow$)

(c) ($\leftarrow$)

- i. [Show $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q)$ implies $LLC_D(p, q)$]
- ii. Assume $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q)$.
- iii. Since D satisfies Set *CG* with respect to $\mathcal{T}_{\Sigma_D}$, $LC_D(p, q)$.
- iv. By Proposition 43. ■ ($\leftarrow$) ■ Proposition.

Proof of Proposition 60:

(Proposition 60) For any partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that A satisfies assumption *B1* with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models then $\langle \Sigma_A, \preceq_A \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_A}$.

1. Let A be a partially ordered truth medium A , and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies assumption *B1* with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ and A satisfies monotonic extension of commitment with respect to models.

2. [Show $\forall p, q \in \Sigma_A$ if $p \preceq_A q$ then $\mathcal{T}_{\Sigma_A}(q) \subseteq \mathcal{T}_{\Sigma_A}(p)$.]
3. Let p, q be arbitrary elements of Σ_A such that $p \preceq_A q$.
4. Since A satisfies monotonic extension of commitment with respect to models we know that $t_A(q) \subseteq t_A(p)$.
5. Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know $(\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p))$ and $(\bigcup \mathcal{T}_{M_A}[t_A(q)] = \mathcal{T}_{\Sigma_A}(q))$.
6. Let $u \in \mathcal{T}_{\Sigma_A}(q)$.
7. Then there is an $m \in t_A(q)$ such that $u \in \mathcal{T}_{M_A}(m)$.
8. Since $t_A(q) \subseteq t_A(p)$, $m \in t_A(p)$.
9. So $u \in \bigcup \mathcal{T}_{M_A}[t_A(p)]$.
10. So $u \in \mathcal{T}_{\Sigma_A}(p)$. ■ Proposition.

Proof of Lemma 61:

(Lemma 61) Given arbitrary truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for any $R \subseteq \Sigma_A$, for all $p, q \in \Sigma_A$, if $\bigcup t_A[R] = t_A(p) \cap t_A(q)$, then $\bigcup \mathcal{T}_{\Sigma_A}[R] = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_A}(q)$.

1. Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Further, let $R \subseteq \Sigma_A$, and p, q arbitrary elements of Σ_A , such that $\bigcup t_A[R] = t_A(p) \cap t_A(q)$. [Show $\bigcup \mathcal{T}_{\Sigma_A}[R] = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_A}(q)$]
2. ($\subseteq$)
 - (a) Let $u \in \bigcup \mathcal{T}_{\Sigma_A}[R]$.
 - (b) Then there is some $r \in R$ such that $u \in \mathcal{T}_{\Sigma_A}(r)$.
 - (c) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $r \in \Sigma_A$, $\bigcup \mathcal{T}_{M_A}[t_A(r)] = \mathcal{T}_{\Sigma_A}(r)$. So there is some $m \in t_A(r)$ such that $u \in \mathcal{T}_{M_A}(m)$.

- (d) So $m \in \bigcup t_A[R]$.
- (e) So $m \in t_A(p)$ and $m \in t_A(q)$
- (f) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $p, q \in \Sigma_A$, we have $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$ and $\bigcup \mathcal{T}_{M_A}[t_A(q)] = \mathcal{T}_{\Sigma_A}(q)$.
- (g) So $u \in \mathcal{T}_{\Sigma_A}(p)$ and $u \in \mathcal{T}_{\Sigma_A}(q)$. ■ $\subseteq$

3. ($\supseteq$)

- (a) Let $u \in \mathcal{T}_{\Sigma_A}(p)$ and $u \in \mathcal{T}_{\Sigma_A}(q)$.
- (b) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $p, q \in \Sigma_A$, we have $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$ and $\bigcup \mathcal{T}_{M_A}[t_A(q)] = \mathcal{T}_{\Sigma_A}(q)$.
- (c) So there exists $x \in t_A(p)$ such that $u \in \mathcal{T}_{M_A}(x)$ and $y \in t_A(q)$ such that $u \in \mathcal{T}_{M_A}(y)$.
- (d) Since A satisfies assumption $A4$ with respect to $\mathcal{T}_{M_A}$, $x = y$.
- (e) So $x \in t_A(p) \cap t_A(q)$.
- (f) So $x \in t_A(r)$ for some $r \in R$.
- (g) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $r \in \Sigma_A$, $\bigcup \mathcal{T}_{M_A}[t_A(r)] = \mathcal{T}_{\Sigma_A}(r)$.
- (h) Since $u \in \mathcal{T}_{M_A}(x)$ and $x \in t_A(r)$ we have $u \in \mathcal{T}_{\Sigma_A}(r)$.
- (i) Since $r \in R$, $u \in \bigcup \mathcal{T}_{\Sigma_A}[R]$. ■ ($\supseteq$) ■ Lemma.

Proof of Lemma 62:

(Lemma 62) Given arbitrary truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for any $S \subseteq \Sigma_A$, for all $p, q \in \Sigma_A$, if $\bigcup t_A[S] = t_A(p) - t_A(q)$, then $\bigcup \mathcal{T}_{\Sigma_A}[S] = \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)$.

1. Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Further, let $S \subseteq \Sigma_A$, and p, q arbitrary elements of Σ_A , such that $\bigcup t_A[S] = t_A(p) - t_A(q)$. [Show $\bigcup \mathcal{T}_{\Sigma_A}[S] = \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)$]

2. ($\subseteq$)

- (a) Let $u \in \bigcup \mathcal{T}_{\Sigma_A} [S]$.
- (b) Then there is some $s \in S$ such that $u \in \mathcal{T}_{\Sigma_A}(s)$.
- (c) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $s \in \Sigma_A$,
 $\bigcup \mathcal{T}_{M_A} [t_A(s)] = \mathcal{T}_{\Sigma_A}(s)$.
- (d) So there is some $m \in t_A(s)$ such that $u \in \mathcal{T}_{M_A}(m)$.
- (e) Since $s \in S$, $m \in \bigcup t_A [S]$.
- (f) Since $\bigcup t_A [S] = t_A(p) - t_A(q)$, $m \in t_A(p)$ and $m \notin t_A(q)$.
- (g) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $p \in \Sigma_A$, we
have $\bigcup \mathcal{T}_{M_A} [t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$.
- (h) So $u \in \mathcal{T}_{\Sigma_A}(p)$.
- (i) [Claim: $u \notin \mathcal{T}_{\Sigma_A}(q)$.]
 - i. Assume $u \in \mathcal{T}_{\Sigma_A}(q)$. [Show contradiction]
 - ii. Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $q \in \Sigma_A$,
we have $\bigcup \mathcal{T}_{M_A} [t_A(q)] = \mathcal{T}_{\Sigma_A}(q)$.
 - iii. Then there exists an $n \in t_A(q)$ such that $u \in \mathcal{T}_{M_A}(n)$.
 - iv. Since A satisfies assumption $A4$ with respect to $\mathcal{T}_{M_A}$, and $u \in \mathcal{T}_{M_A}(m)$ and
 $u \in \mathcal{T}_{M_A}(n)$ we know $m = n$.
 - v. So $m \in t_A(q)$. Contradiction. ■ Claim.
- (j) So $u \in \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)$. ■ ($\subseteq$)

3. ($\supseteq$)

- (a) Let $u \in \mathcal{T}_{\Sigma_A}(p)$ and $u \notin \mathcal{T}_{\Sigma_A}(q)$.
- (b) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $p, q \in \Sigma_A$, we
have $\bigcup \mathcal{T}_{M_A} [t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$ and $\bigcup \mathcal{T}_{M_A} [t_A(q)] = \mathcal{T}_{\Sigma_A}(q)$.
- (c) So there exists $x \in t_A(p)$ such that $u \in \mathcal{T}_{M_A}(x)$.

- (d) There does not exist a $y \in t_A(q)$ such that $u \in \mathcal{T}_{M_A}(y)$, for if there was such a y , then $u \in \mathcal{T}_{\Sigma_A}(q)$.
- (e) So in particular, $x \notin t_A(q)$.
- (f) So $x \in t_A(p) - t_A(q)$.
- (g) Since $\bigcup t_A[S] = t_A(p) - t_A(q)$, $x \in \bigcup t_A[S]$.
- (h) So $x \in t_A(s)$ for some $s \in S$.
- (i) Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $s \in \Sigma_A$, we have $\bigcup \mathcal{T}_{M_A}[t_A(s)] = \mathcal{T}_{\Sigma_A}(s)$.
- (j) Since $u \in \mathcal{T}_{M_A}(x)$ and $x \in t_A(s)$ we have $u \in \mathcal{T}_{\Sigma_A}(s)$.
- (k) Since $s \in S$, we have $u \in \bigcup \mathcal{T}_{\Sigma_A}[S]$. ■ ($\supseteq$) ■ Lemma.

Now we can present the proposition linking weak extensibility with respect to models and weak extensibility.

Proof of Proposition 63:

(Proposition 63) Given an arbitrary partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then if A is weakly extensible with respect to models, then the partially ordered set $\langle \Sigma_A, \preceq_A \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_A}$.

1. Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and A is weakly extensible with respect to models.

$$2. \left[\begin{array}{c} \text{Show:} \\ \text{(Part a)} \\ \forall p, q \in \Sigma_A \ (\exists R \subseteq \{p, q\}_A^u \text{ such that } \bigcup \mathcal{T}_{\Sigma_A}[R] = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_A}(q)) \\ \text{(Part b)} \\ \forall p, q \in \Sigma_A \ (\exists S \subseteq \uparrow_A(p) \text{ such that } \bigcup \mathcal{T}_{\Sigma_A}[S] = \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)) \end{array} \right]$$

3. (Part a)

- (a) Let p, q be arbitrary elements of Σ_A .

(b) Since A is weakly extensible with respect to models, we know that

$$\exists R \subseteq \{p, q\}_A^u \text{ such that } \bigcup t_A [R] = t_A(p) \cap t_A(q).$$

(c) By Lemma 61 we have $\bigcup \mathcal{T}_{\Sigma_A} [R] = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_A}(q)$.

(d) So R satisfies the existential claim for Part a. ■(Part a.)

4. (Part b.)

(a) Let p, q be arbitrary elements of Σ_A .

(b) Since A is weakly extensible with respect to models, we know that

$$\exists S \subseteq (\uparrow_A p) \text{ such that } \bigcup t_A [S] = t_A(p) - t_A(q)$$

(c) By Lemma 62 we have $\bigcup \mathcal{T}_{\Sigma_A} [S] = \mathcal{T}_{\Sigma_A}(p) - \mathcal{T}_{\Sigma_A}(q)$.

(d) So S satisfies the existential claim for Part b. ■(Part b.)■Proposition.

Proof of Proposition 64:

(Proposition 64) For any truth medium A , $pow(A)$ satisfies monotonic extension of commitment with respect to models.

Let A be an arbitrary truth medium. Let P, Q be arbitrary elements of $\Sigma_{pow(A)}$, such that $P \preceq_{pow(A)} Q$.

[Show $t_{pow(A)}(Q) \subseteq t_{pow(A)}(P)$]

By the definition of $\preceq_{pow(A)}$, $P \subseteq Q$.

We split into four cases: 1) $P = \emptyset, Q = \emptyset$; 2) $P = \emptyset, Q \neq \emptyset$; 3) $P \neq \emptyset, Q = \emptyset$; 4) $P \neq \emptyset, Q \neq \emptyset$.

(Case 1) $P = \emptyset, Q = \emptyset$

1. $t_{pow(A)}(P) = t_{pow(A)}(Q) = \bigcup t_A [\Sigma_A]$ by the definition of pow .

2. So $t_{pow(A)}(Q) \subseteq t_{pow(A)}(P)$. ■(Case 1)

(Case 2) $P = \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcup t_A [\Sigma_A]$ by the definition of pow .

2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .

3. Since $Q \subseteq \Sigma_A$, $t_{pow(A)}(Q) \subseteq t_{pow(A)}(P)$ (Case 2)

(Case 3) $P \neq \emptyset, Q = \emptyset$

This case is not possible, since $P \subseteq Q$ (Case 3)

(Case 4) $P \neq \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcap t_A [P]$ by the definition of pow .

2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .

3. Since $P \subseteq Q$, $t_{pow(A)}(Q) \subseteq t_{pow(A)}(P)$. (Case 4) Proposition.

Proof of Proposition 65:

(Proposition 65) Given that A is a truth medium that is negation complete with respect to models, $pow(A)$ is weakly extensible with respect to models.

Let A be an arbitrary truth medium that is negation complete with respect to models.

(Part a)

$$\left[\begin{array}{l} \text{Show for every } P, Q \in \Sigma_{pow(A)}, \\ \exists \Gamma \subseteq \{P, Q\}_{pow(A)}^u \text{ such that } \bigcup t_{pow(A)} [\Gamma] = t_{pow(A)}(P) \cap t_{pow(A)}(Q) \end{array} \right]$$

Claim: $\Gamma = \{P \cup Q\}$ satisfies the existential claim above. Note that $P \preceq_{pow(A)} P \cup Q$, and $Q \preceq_{pow(A)} P \cup Q$, so $\{P \cup Q\} \subseteq \{P, Q\}_{pow(A)}^u$.

$$[\text{Show } t_{pow(A)}(P \cup Q) = t_{pow(A)}(P) \cap t_{pow(A)}(Q)]$$

We split into four cases: 1) $P = \emptyset, Q = \emptyset$; 2) $P = \emptyset, Q \neq \emptyset$; 3) $P \neq \emptyset, Q = \emptyset$; 4) $P \neq \emptyset, Q \neq \emptyset$

(Case 1) $P = \emptyset, Q = \emptyset$

1. $t_{pow(A)}(P) = t_{pow(A)}(Q) = \bigcup t_A [\Sigma_A]$ by the definition of pow .

2. $P \cup Q = \emptyset$ so $t_{pow(A)}(P \cup Q) = \bigcup t_A [\Sigma_A]$ by the definition of pow .

3. So $t_{pow(A)}(P \cup Q) = t_{pow(A)}(P) \cap t_{pow(A)}(Q)$. (Case 1)

(Case 2) $P = \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcup t_A [\Sigma_A]$ by the definition of pow .

2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .
3. Since $Q \subseteq \Sigma_A$ and $Q \neq \emptyset$, $t_{pow(A)}(Q) \subseteq t_{pow(A)}(P)$
4. So $t_{pow(A)}(P) \cap t_{pow(A)}(Q) = t_{pow(A)}(Q)$
5. Since $P = \emptyset$, $P \cup Q = Q$
6. So $t_{pow(A)}(P \cup Q) = t_{pow(A)}(Q) = t_{pow(A)}(P) \cap t_{pow(A)}(Q)$ ■ (Case 2)

(Case 3) $P \neq \emptyset, Q = \emptyset$

Same as Case 2, but with P, Q reversed. ■ (Case 3)

(Case 4) $P \neq \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcap t_A [P]$ by the definition of pow .
2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .
3. So $t_{pow(A)}(P) \cap t_{pow(A)}(Q) = \bigcap t_A [P \cup Q] = t_{pow(A)}(P \cup Q)$, since $P \cup Q$ is non-empty in this case. ■ (Case 4)

■ (Part a)

(Part b)

Show for every $P, Q \in \Sigma_{pow(A)}$,

$$\left[\begin{array}{l} \exists \Delta \subseteq (\uparrow_{pow(A)} P) \text{ such that } \bigcup t_{pow(A)} [\Delta] = t_{pow(A)}(P) - t_{pow(A)}(Q) \end{array} \right]$$

Let P, Q be arbitrary elements of $\Sigma_{pow(A)}$. That is, $P, Q \subseteq \Sigma_A$.

We split into four cases: 1) $P = \emptyset, Q = \emptyset$; 2) $P = \emptyset, Q \neq \emptyset$; 3) $P \neq \emptyset, Q = \emptyset$; 4) $P \neq \emptyset, Q \neq \emptyset$

(Case 1) $P = \emptyset, Q = \emptyset$

1. $t_{pow(A)}(P) = t_{pow(A)}(Q) = \bigcup t_A [\Sigma_A]$ by the definition of pow .
2. So $t_{pow(A)}(P) - t_{pow(A)}(Q) = \emptyset$
3. If we take $\Delta = \emptyset$, then the existential claim for part b is satisfied. ■ (Case 1)

(Case 2) $P = \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcup t_A [\Sigma_A]$ by the definition of pow .
2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .
3. So $t_{pow(A)}(P) - t_{pow(A)}(Q) = \bigcup t_A [\Sigma_A] - \bigcap t_A [Q]$
4. Let $Y = \{Z \in \Sigma_{pow(A)} \mid \bigcup t_A [Z] = \bigcup t_A [\Sigma_A] - t_A(q) \text{ for some } q \in Q\}$
5. Form the set $\Delta = \{\{y\} \mid y \in \bigcup Y\}$.
6. Note that since $P = \emptyset$, every element of Δ is in $\uparrow_{pow(A)} P$, so $\Delta \subseteq (\uparrow_{pow(A)} P)$.
7. [Claim: $\bigcup t_{pow(A)} [\Delta] = \bigcup t_A [\Sigma_A] - \bigcap t_A [Q]$]
8. Showing this claim will show that Δ satisfies the existential claim for part b.
9. The claim is identical to: $\left[\text{Claim: } \bigcup_{y \in \bigcup Y} t_{pow(A)}(\{y\}) = \bigcup t_A [\Sigma_A] - \bigcap t_A [Q] \right]$
10. Let y be an arbitrary element of $\bigcup Y$. Then $\{y\}$ is nonempty. Then $t_{pow(A)}(\{y\}) = \bigcap t_A [\{y\}] = t_A(y)$.
11. So the claim is identical to: $\left[\text{Claim: } \bigcup_{y \in \bigcup Y} t_A(y) = \bigcup t_A [\Sigma_A] - \bigcap t_A [Q] \right]$
12. ($\subseteq$)
 - (a) Let $m \in \bigcup_{y \in \bigcup Y} t_A(y)$
 - (b) So there is a $y \in \bigcup Y$ such that $m \in t_A(y)$.
 - (c) So there is a $Z \in Y$ such that $y \in Z$.
 - (d) Z has the property that $\bigcup t_A [Z] = \bigcup t_A [\Sigma_A] - t_A(q)$ for some $q \in Q$.
 - (e) Since $y \in Z$, and $m \in t_A(y)$, $m \in \bigcup t_A [Z]$.
 - (f) So $m \in \bigcup t_A [\Sigma_A]$, and there is a $q \in Q$ such that $m \notin t_A(q)$.
 - (g) So $m \in \bigcup t_A [\Sigma_A] - \bigcap t_A [Q]$ ■ ($\subseteq$)
13. ($\supseteq$)
 - (a) Let $m \in \bigcup t_A [\Sigma_A] - \bigcap t_A [Q]$.

- (b) So $m \in \bigcup t_A [\Sigma_A]$ and $m \notin \bigcap t_A [Q]$.
- (c) Since Q is nonempty in this case (Case 2) there is some $q \in Q$ such that $m \notin t_A(q)$.
- (d) Since A is negation complete with respect to models, and $q \in \Sigma_A$, $\exists R \subseteq \Sigma_A$ such that

$$\bigcup t_A [R] = \bigcup t_A [\Sigma_A] - t_A(q)$$
. Pick some such R .
- (e) So $R \in Y$.
- (f) Further, $m \in \bigcup t_A [R]$, since $m \in \bigcup t_A [\Sigma_A]$ and $m \notin t_A(q)$.
- (g) So $m \in t_A(r)$ for some $r \in R$.
- (h) Since $r \in R$ and $R \in Y$, $r \in \bigcup Y$.
- (i) So $m \in \bigcup_{y \in \bigcup Y} t_A(y)$ ■ ($\supseteq$) ■ (Claim) ■ (Case 2)

(Case 3) $P \neq \emptyset, Q = \emptyset$

1. $t_{pow(A)}(P) = \bigcap t_A [P]$ by the definition of pow .
2. $t_{pow(A)}(Q) = \bigcup t_A [\Sigma_A]$ by the definition of pow .
3. Since $P \subseteq \Sigma_A$ and $P \neq \emptyset$, $t_{pow(A)}(P) \subseteq t_{pow(A)}(Q)$ so $t_{pow(A)}(P) - t_{pow(A)}(Q) = \emptyset$.
4. If we take $\Delta = \emptyset$, then the existential claim for part b is satisfied. ■ (Case 3)

(Case 4) $P \neq \emptyset, Q \neq \emptyset$

1. $t_{pow(A)}(P) = \bigcap t_A [P]$ by the definition of pow .
2. $t_{pow(A)}(Q) = \bigcap t_A [Q]$ by the definition of pow .
3. So $t_{pow(A)}(P) - t_{pow(A)}(Q) = \bigcap t_A [P] - \bigcap t_A [Q]$.
4. Consider the set Y from Case 2.
5. Form the set $\Delta = \{P \cup \{y\} \mid y \in \bigcup Y\}$.
6. Note that for all $S \in \Delta$, $P \subseteq S$, so $S \succeq_{pow(A)} P$, so $S \in \uparrow_{pow(A)} P$, so $\Delta \subseteq \uparrow_{pow(A)} P$.

7. [Claim: $\bigcup t_{pow(A)}[\Delta] = \bigcap t_A[P] - \bigcap t_A[Q]$]. By showing this claim, we show that Δ satisfies the existential claim for part b.
8. Let S be an arbitrary element of Δ . Since $S = P \cup \{y\}$ for some $y \in \bigcup Y$, we know that S is nonempty.
9. So $t_{pow(A)}(S) = \bigcap t_A[S]$ by the definition of pow .
10. [Claim restated: $\bigcup_{S \in \Delta} (\bigcap t_A[S]) = \bigcap t_A[P] - \bigcap t_A[Q]$]
11. ($\subseteq$)
 - (a) Let $m \in \bigcup_{S \in \Delta} (\bigcap t_A[S])$.
 - (b) So there is an $S \in \Delta$ such that, for every $s \in S$, $m \in t_A(s)$.
 - (c) Since $S \in \Delta$, we know $P \subseteq S$ (by definition of Δ).
 - (d) So for every $p \in P$, $m \in t_A(p)$.
 - (e) Since $P \neq \emptyset$, $m \in \bigcap t_A[P]$.
 - (f) Since $S \in \Delta$, there is some $y \in \bigcup Y$, such that $y \in S$.
 - (g) So $m \in t_A(y)$.
 - (h) Further, there is a $Z \in Y$ such that $y \in Z$.
 - (i) Z has the property that $\bigcup t_A[Z] = \bigcup t_A[\Sigma_A] - t_A(q)$ for some $q \in Q$.
 - (j) Since $y \in Z$ and $m \in t_A(y)$, $m \in \bigcup t_A[Z]$.
 - (k) So there is a $q \in Q$ such that $m \notin t_A(q)$. ■ ($\subseteq$)
12. ($\supseteq$)
 - (a) Let $m \in \bigcap t_A[P] - \bigcap t_A[Q]$
 - (b) So for every $p \in P$, $m \in t_A(p)$; and for some $q \in Q$, $m \notin t_A(q)$. Recall that $Q \neq \emptyset$ in this case (Case 4).
 - (c) Pick some $p \in P$. We know one exists since P is nonempty in this case (Case 4).
 - (d) Since $P \subseteq \Sigma_A$, $p \in \Sigma_A$.

- (e) So $m \in \bigcup t_A [\Sigma_A] - \bigcap t_A [Q]$.
- (f) By argument in Case 2, $\supseteq$, we know that $m \in \bigcup_{y \in \bigcup Y} t_A(y)$.
- (g) So there is a $y \in \bigcup Y$ such that $m \in t_A(y)$.
- (h) So there is an $S \in \Delta$ specifically $S = P \cup \{y\}$, such that $m \in \bigcap t_A [S]$. ■ ($\supseteq$)
 ■(Case 4)■(Part b)■Proposition.

Chapter 5

Applications of Order-consistency Representational Techniques

5.1 Introduction

In this dissertation we explore the application of order-consistency representational techniques to four different interpreted languages: two simple languages meant to illustrate the basic concepts involved; the language of propositional logic; and languages in which feature structures are considered as assertions in their own right. We present, as we did in the case of model-theoretic techniques, a general methodology for applying either of the defined order-consistency representational techniques. Our next step will be to give a high-level overview of the example applications to be presented. For each application, we will explain (1) what the language(s) in question are; (2) which techniques we plan to use in building theories of consequence for those languages; and (3) why those applications are of interest. We will then proceed to apply the techniques. We will carry out complete applications in the case of the simple examples and the language of propositional logic; and sketch the application in the case of languages in which feature structures are considered as assertions in their own right. (Note that the application in the case of propositional logic will be deferred to the end of Chapter 6, as it will take advantage of the results of the proofs presented there).

As a postscript to the chapter, and in preparation for the proof in Chapter 6, we will

consider issues related to the applicability of order-consistency techniques in general.

5.2 A general methodology for applying order-consistency representational techniques

In Section 1.5.1, we introduced a general methodology for applying instances of the representational schema. We can now specialize that methodology in the context of order-consistency techniques.

As of now, we have defined two such techniques: the Set *CL* technique, and the Set *CG* technique. These techniques vary in terms of the proxy for consequence relation and the set of technique-specific assumptions which they employ. Since the intelligible proxy for consequence relation is only used once an application of a representational technique is complete, the only salient difference between the techniques for our purposes here is in the sets of technique-specific assumptions used. This means that we can, as we did in the model-theoretic case, describe a common methodology for applying either of these techniques. (Remember that these “steps” are really “aspects” and can be thought of as occurring simultaneously.)

Given a selected technique,

Step 1: Identify a set of assertion types for the language.

Step 2: Form a consistency medium by supplementing the set of assertion types with a partial order, and a subset of the original set of assertion types considered consistent.

Step 3: We have only one kind of representative element in a consistency medium: the set of assertion types. Define semantic conventions describing acceptable interpretations for the assertion types.

Step 4: Prove that if a practice of categorization for the assertion types satisfies the semantic conventions described for it in Step 3, then the consistency medium we have constructed in Step 2 satisfies the technique-specific assumptions with respect to that practice.

Given that proof, we know that for every acceptable interpretation of the set of assertion types, the proxy relation used by the selected technique is equivalent to the representational conception of logical consequence (preservation of truth across all possibilities). Further,

because of the technique used, we know that the theory so constructed explains the consequence relation between assertion types in terms of two simpler concepts: a partial order representing primitive consequence, and a unary consistency predicate.

5.3 Overview of the applications

5.3.1 First simple application: the “odd and even” language *OE*

In this section we build a model of the relation of logical consequence for a simple language which we call *OE*. This example, like the one presented in Section 3.4 concerns a primitive language used to describe the numerical value of a roll of a single six-sided die. This language will have four assertion types: *odd*, *even*, *oddOrEven*, and *oddAndEven*.

There are several objectives from this example. The simplicity of the example will let us make a tutorial pass through the methodology of applying an order-consistency technique, and help us to see some of its key features, in particular, the relative roles of semantic conventions and technique-specific assumptions. We will be able to give a concrete presentation of the conditions of monotonic extension of commitment and weak extensibility.

We will use the Set *CL* technique in this case. Since every application of Set *CL* is also an application of Set *CG*, this is also an application of Set *CG*. We will see how both the Set *CL* and Set *CG* proxy relations (*LLC* and *LC*) work in this case.

5.3.2 Second simple application: the “powerset of odd and even” language *POE*

We build the language of this section from the language of the previous section. The method of construction is simple. The set of assertion types for the language of this section (which we call *POE*) is the powerset of the set of consistent assertion types from the language *OE* of the previous example. The relationship between *OE* and *POE*, is similar to that between the two versions of the language of propositional logic *WFF* and $\mathcal{P}(WFF)$ which we have considered previously. The relationship is not exactly the same, but the difference will be noted.

Applying the representational schema to the task of modelling the relation of logical

consequence for *POE* will enable us to see several important ideas in action. *POE* is an example of structural definition (*POE* is defined by the powerset operator) and an ordering corresponding to structural elaboration (in that the assertion types of *POE* are ordered by set inclusion). We will also see that it is possible for one consistency medium to be capable of modelling the consequence relation of another. This aspect of the example will set up our discussion of minimal media in Chapter 9.

We will use the Set *CL* technique in this case. Since every application of Set *CL* is also an application of Set *CG*, this is also an application of Set *CG*. As in the case of the *OE* language, we will see how both the Set *CL* and Set *CG* proxy relations (*LLC* and *LC*) work in this case.

5.3.3 The language of propositional logic

An important example we consider is the language of propositional logic. In carrying out the application of order-consistency techniques to the language of propositional logic we demonstrate an example of a language which is capable of having its consequence relation modelled from the perspectives of multiple techniques. That the consequence relation of propositional logic is capable of both model-theoretic and order-consistency understanding was already known (that understanding is embedded in the corollary to Lindenbaum's Lemma). This example will show how that knowledge fits into the framework defined by the representational schema.

This example will be deferred until Section 6.11. There we will be able to show how the application can be carried out as an instance of the general methodology for constructing applications of the Set *CG* technique from applications of the Set *PP* technique. That will show that we can model the representational relation of logical consequence for propositional logic using the Set *CG* technique. In Chapter 7, we will show that every application generated by the methodology of Chapter 6 is in fact an application of the Set *CL* technique as well. So we will have shown that the Set *CL* technique is also capable of modelling the relation of logical consequence for propositional logic.

5.3.4 Languages in which feature structures are considered as assertions in their own right

We have introduced and discussed feature structures in the context of feature logics (Section 3.6). There feature structures were considered as the models for the sentential language of a feature logic. But the role of feature structures is not limited to being models for some other language. Feature structures can be used as assertion types in their own right (and are, in many computing systems). For example, expressions in the language *XML*, which is a relatively new standard for carrying information on the Internet, can be assimilated to the mathematical model of feature structures.

The way to see this additional role is to take the representational perspective. On the representational view, the models in a model-theoretic semantics are full-fledged representations; making propositional claims, and assumed to have a practice of categorization relating them to the possibilities they represent in the same way that assertion types do. Given that view, it should not be surprising that we can take the feature structures out of their role as models, and treat them as assertion types on their own.

But there is something important and special to consider here. What differentiates models from assertion types is not that one bears propositional content and the other does not. The difference is in terms of the complexity or simplicity of the logical consequence relation between members of the set (assertion types or models). The set of assertion types may have a logically complex relation of consequence, but the set of models must be logically simple. As we saw above, there are degrees to this logical simplicity of models, but the key point is that if a model-theoretic account of logical consequence for a set of assertion types is to be explanatory, the relation of logical consequence for the models must be simpler than the relation of logical consequence for the assertion types.

In our discussion of Section 3.6 we discussed the potential use of the Set *BE* technique to model the consequence relation for feature logics. If we are able to complete that application, we will in doing so show that the set of feature structures used as models satisfies assumption *A7* with respect to the practice which interprets it, and as such is logically simple, in that the subsumption ordering implies, and is implied by, the relation of logical consequence between feature structures.

A language with a logically simple consequence relation is useful as a set of models, but as a language of assertion it leaves something to be desired. This is the key point: when feature structures play the role of assertion types rather than models, their relation of logical consequence is no longer assumed to be simple. There will be consequences which are not subsumptions. Logical complexity (and the inferential benefits which attend it) is made possible by the presence of nontrivial patterns of inconsistency.

Consider the example feature structures from Section 3.6. As a model for a feature logic, a feature structure which had *s* (a *Male*) as the value of *mother* for some other element would be consistent (taken to represent a relevant possibility). But if those feature structures were being used as assertion types to represent our world in a common-sense way (with elements corresponding to people, *Male* and *Female* corresponding to gender, *mother* corresponding to motherhood, etc.), that feature structure would be inconsistent. The fact that no feature structure with a male mother is consistent together with the fact that every consistent feature structure is extensible (via subsumption) to a structure in which every element is either *Male* or *Female* (but not both!), enables the inference from a feature structure in which *x* is the *mother* of *y*, to a feature structure in which *x* is *Female*. Explaining exactly why such inference works... calls for an application of the representational schema. In our detailed discussion below, we will outline how we would go about applying order-consistency techniques to the task of defining the relation of logical consequence between feature structures considered as assertion types.

5.4 First simple application: the “odd and even” language

OE

5.4.1 Introduction to the example

In this section we apply an order-consistency technique to model the relation of logical consequence for a simple language. This example, like the one presented in Section 3.4 concerns a primitive language used to describe the numerical value of a roll of a single six-sided die. This language will have four assertion types: *odd*, *even*, *oddOrEven*, and *oddAndEven*. The practice of categorization by which these assertion types are interpreted

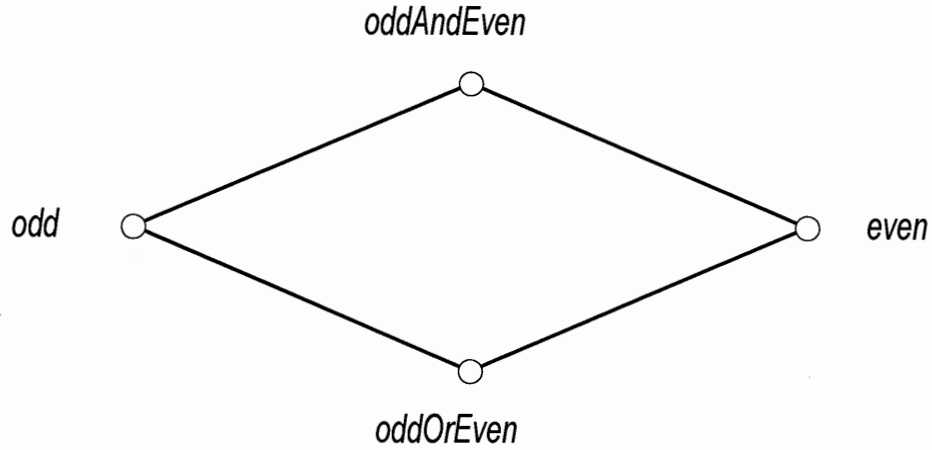


Figure 5-1: Hasse diagram of the ordered set $\langle \Sigma_{OE}, \preceq_{OE} \rangle$

coincides with the ordinary English meanings of their names, applied as predicates to the number of dots on the side of the die facing upwards after a roll. For example, if a die is rolled such that the side with five dots is facing up, the assertion types *odd* and *oddOrEven* would be assessed true by the practice, and the other types *even* and *oddAndEven* would be assessed false. Whereas, if a die is rolled such that the side with six dots is facing up, the assertion types *even* and *oddOrEven* would be assessed true, and the assertion types *odd* and *oddAndEven* would be assessed false.

5.4.2 Details of the application

Technique selection

We initially apply the Set *CL* technique.

The consistency medium *OE*

We define a consistency medium $OE = \langle \Sigma_{OE}, \preceq_{OE}, C_{OE} \rangle$.

We are given the set $\Sigma_{OE} = \{odd, even, oddOrEven, oddAndEven\}$.

We define the ordering $\preceq_{OE}$ as shown the Hasse diagram in Figure 5-1.

We define the set C_{OE} as: $C_{OE} = \Sigma_{OE} - \{oddAndEven\}$.

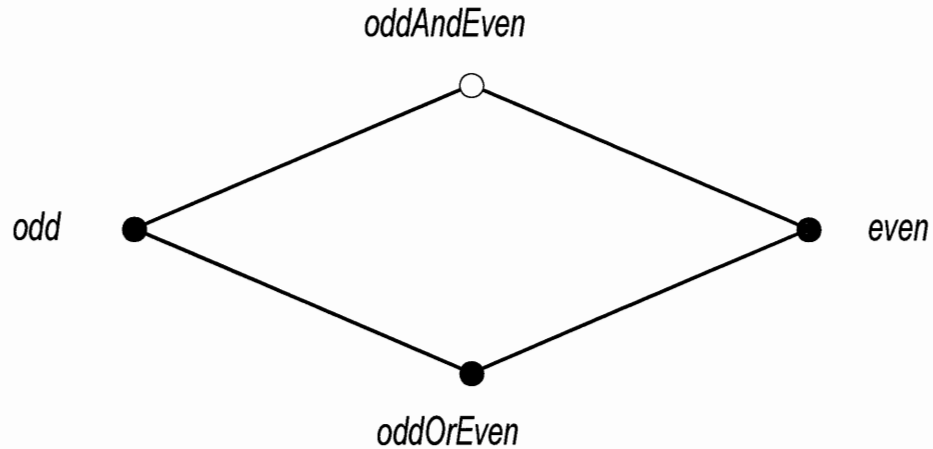


Figure 5-2: A modified Hasse diagram showing declarative medium *OE*

We indicate our assumptions of consistency on the ordering diagram by filling in the circles of the assertion types in *C*. (See Figure 5-2).

Semantic conventions for interpretations of *OEX*

Given some experience with the usage of the simple language described, we might be willing to make the following assumptions about the practice of categorization interpreting the assertion types. (In this example, there is one specific practice of categorization, not a family of acceptable practices.)

- (X1) There is some possibility in which *odd* is true.
- (X2) There is some possibility in which *even* is true.
- (X3a) Every possibility in which *odd* is true, is one in which *oddOrEven* is true.
- (X3b) Every possibility in which *even* is true, is one in which *oddOrEven* is true.
- (X3c) Every possibility in which *oddOrEven* is true, is such that either *odd* or *even* is true.
- (X4) There is no possibility in which *odd* and *even* are both true.

- (X5) $oddAndEven$ is true in just those possibilities where odd is true and $even$ is true.

Call the set of all these assumptions OEX .

We take the set U to be possible rolls of the die as in the *Roll* truth medium example (Section 3.4.2), and call the given practice of categorization interpreting OE , $\mathcal{T}_{\Sigma_{OE}}$.

We can now formalize our expression of the assumptions in the set OEX . We express those assumptions relative to the given practice of categorization $\mathcal{T}_{\Sigma_{OE}}$. These conditions partially characterize $\mathcal{T}_{\Sigma_{OE}}$.

- (X1) $\mathcal{T}_{\Sigma_{OE}}(odd) \neq \emptyset$.
- (X2) $\mathcal{T}_{\Sigma_{OE}}(even) \neq \emptyset$.
- (X3a) $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
- (X3b) $\mathcal{T}_{\Sigma_{OE}}(even) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
- (X3c) $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) \subseteq (\mathcal{T}_{\Sigma_{OE}}(odd) \cup \mathcal{T}_{\Sigma_{OE}}(even))$.
- (X4) $\mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even) = \emptyset$.
- (X5) $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even)$.

The following claims follow from the assumptions in the set OEX .

- (Y1) From X4 and X5, we have $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$.
- (Y2) From X3a, X3b, and X3c, we have $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) = \mathcal{T}_{\Sigma_{OE}}(odd) \cup \mathcal{T}_{\Sigma_{OE}}(even)$.
- (Y3a) From Y2 and X4, we have $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) - \mathcal{T}_{\Sigma_{OE}}(odd) = \mathcal{T}_{\Sigma_{OE}}(even)$.
- (Y3b) From Y2 and X4, we have $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) - \mathcal{T}_{\Sigma_{OE}}(even) = \mathcal{T}_{\Sigma_{OE}}(odd)$.
- (Y4) From X1 and X3a, we have $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) \neq \emptyset$.

Proof of the correctness of the application.

We now show that if $\mathcal{T}_{\Sigma_{OE}}$ satisfies the semantic conventions OEX , then the consistency medium OE satisfies the technique-specific assumptions for Set CL . (Remember that Set CL consists of assumptions $D1$, $D2$, $D3$, and LL .)

Proposition 67 *If the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$ satisfies the semantic conventions OEX ,*

then OE satisfies Set CL with respect to $\mathcal{T}_{\Sigma_{OE}}$.

Assume that $\mathcal{T}_{\Sigma_{OE}}$ satisfies the semantic conventions OEX .

[Show $D1$]

1. From $Y1$, we know that $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$.
2. So $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) \subseteq \mathcal{T}_{\Sigma_{OE}}(odd)$ and $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) \subseteq \mathcal{T}_{\Sigma_{OE}}(even)$.
3. From $X3a$, we know that $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
4. From $X3b$, we know that $\mathcal{T}_{\Sigma_{OE}}(even) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
5. From these facts, and the reflexivity and transitivity of $\preceq$ and $\subseteq$, we can observe that OE satisfies $D1$ with respect to $\mathcal{T}_{\Sigma_{OE}}$ by looking at the Hasse diagram in Figure5-2. ■($D1$)

[Show $D2$]

1. From $X1$ we know that $\mathcal{T}_{\Sigma_{OE}}(odd) \neq \emptyset$.
2. From $X2$ we know that $\mathcal{T}_{\Sigma_{OE}}(even) \neq \emptyset$.
3. From $Y4$, we know that $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) \neq \emptyset$.
4. From $Y1$ we know that $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$.
5. These facts enable us to verify that OE satisfies $D2$ with respect to $\mathcal{T}_{\Sigma_{OE}}$. The assertion types odd , $even$, and $oddOrEven$ are elements of C_{OE} , and the assertion type $oddAndEven$ is not.

[Show $D3a$]

This is slightly more involved. We create a table, which shows, for every combination of elements p, q of Σ_{OE} , the set R that satisfies the condition expressed by assumption $D3a$, that is, a set of assertion types such that $R \subseteq \{p, q\}_{OE}^u$ and $\bigcup \mathcal{T}_{\Sigma_{OE}}[R] = \mathcal{T}_{\Sigma_{OE}}(p) \cap \mathcal{T}_{\Sigma_{OE}}(q)$. Following each set, is a number indicating the proof of that case below.

We split the table for typographical reasons.

R		q	
		$oddAndEven$	odd
p	$oddAndEven$	$\{oddAndEven\}, 1$	-
	odd	$\{oddAndEven\}, 1$	$\{odd\}, 2$
	$even$	$\{oddAndEven\}, 1$	$\{oddAndEven\}, 3$
	$oddOrEven$	$\{oddAndEven\}, 1$	$\{odd\}, 4$

R		q	
		$even$	$oddOrEven$
p	$oddAndEven$	-	-
	odd	-	-
	$even$	$\{even\}, 2$	-
	$oddOrEven$	$\{even\}, 5$	$\{oddOrEven\}, 2$

Proofs:

First note that we only have to fill in half the table, as the entries are symmetric across the main left-right diagonal.

(1) From $Y1$, we know that $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$. So $\mathcal{T}_{\Sigma_{OE}}(p) \cap \mathcal{T}_{\Sigma_{OE}}(q) = \emptyset$ in this case. And $\bigcup \mathcal{T}_{\Sigma_{OE}}[\{oddAndEven\}] = \mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$. Also $\forall p, q \in \Sigma_{OE} (\{oddAndEven\} \subseteq \{p, q\}_{OE}^u)$.

(2) For any element p of Σ_{OE} , $\mathcal{T}_{\Sigma_{OE}}(p) \cap \mathcal{T}_{\Sigma_{OE}}(p) = \mathcal{T}_{\Sigma_{OE}}(p) = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{p\}]$. Also $\forall p \in \Sigma_{OE} (\{p\} \subseteq \{p\}_{OE}^u)$.

(3) From $X4$ we know that $\mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even) = \emptyset$, and $\bigcup \mathcal{T}_{\Sigma_{OE}}[\{oddAndEven\}] = \emptyset$. Also $\{oddAndEven\} \subseteq \{odd, even\}_{OE}^u$.

(4) From $X3a$ we know that $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$, so $\mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(oddOrEven) = \mathcal{T}_{\Sigma_{OE}}(odd) = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{odd\}]$.

Also $\{odd\} \subseteq \{odd, oddOrEven\}_{OE}^u$.

(5) Like 4, with *even* substituted for *odd*, and *X3b* for *X3a*.

Together the table shows that *OE* satisfies assumption *D3a* with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}} \blacksquare (D3a)$

[Show *D3b*]

We create a table, which shows, for every combination of elements p, q of Σ_{OE} , the set S that satisfies the condition *D3b*, that is, a set of assertion types such that $S \subseteq \uparrow_{OE}(p)$ and $\bigcup \mathcal{T}_{\Sigma_{OE}}[S] = \mathcal{T}_{\Sigma_{OE}}(p) - \mathcal{T}_{\Sigma_{OE}}(q)$. Following each set, is a number indicating the proof of that case below. (We divide the table in two for typographical reasons).

S		q	
		<i>oddAndEven</i>	<i>odd</i>
p	<i>oddAndEven</i>	$\{oddAndEven\}, 1$	$\{oddAndEven\}, 1$
	<i>odd</i>	$\{odd\}, 2$	$\{oddAndEven\}, 3$
	<i>even</i>	$\{even\}, 2$	$\{even\}, 5$
	<i>oddOrEven</i>	$\{oddOrEven\}, 2$	$\{even\}, 6$
S		q	
		<i>even</i>	<i>oddOrEven</i>
p	<i>oddAndEven</i>	$\{oddAndEven\}, 1$	$\{oddAndEven\}, 1$
	<i>odd</i>	$\{odd\}, 4$	$\{oddAndEven\}, 8$
	<i>even</i>	$\{oddAndEven\}, 3$	$\{oddAndEven\}, 9$
	<i>oddOrEven</i>	$\{odd\}, 7$	$\{oddAndEven\}, 3$

Proofs:

(1) From *Y1*, we know that $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$. So $\mathcal{T}_{\Sigma_{OE}}(p) - \mathcal{T}_{\Sigma_{OE}}(q) = \emptyset$ in this case. And $\bigcup \mathcal{T}_{\Sigma_{OE}}[\{oddAndEven\}] = \emptyset$. Also note $\{oddAndEven\} \subseteq \uparrow_{OE}(oddAndEven)$.

(2) From *Y1*, we know that $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$. So $\mathcal{T}_{\Sigma_{OE}}(p) - \mathcal{T}_{\Sigma_{OE}}(q) = \mathcal{T}_{\Sigma_{OE}}(p)$ in this case. And $\bigcup \mathcal{T}_{\Sigma_{OE}}[\{p\}] = \mathcal{T}_{\Sigma_{OE}}(p)$. Also note: $\forall p \in \Sigma_{OE} (\{p\} \subseteq \uparrow_{OE}(p))$.

(3) For any element p of Σ_{OE} , $\mathcal{T}_{\Sigma_{OE}}(p) - \mathcal{T}_{\Sigma_{OE}}(p) = \emptyset = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{oddAndEven\}]$. Also note $oddAndEven \succeq_{OE} p$ for any element p of Σ_{OE} , so $\forall p \in \Sigma_{OE} (\{oddAndEven\} \subseteq \uparrow_{OE}(p))$.

(4) From *X4*, we know that $\mathcal{T}_{\Sigma_{OE}}(odd) - \mathcal{T}_{\Sigma_{OE}}(even) = \mathcal{T}_{\Sigma_{OE}}(odd) = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{odd\}]$. Also note: $\{odd\} \subseteq \uparrow_{OE}(odd)$.

(5) Like (4) but with *odd* and *even* exchanged.

(6) From Y3a, we know that $\mathcal{T}_{\Sigma_{OE}}(oddOrEven) - \mathcal{T}_{\Sigma_{OE}}(odd) = \mathcal{T}_{\Sigma_{OE}}(even) = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{even\}]$.

Also note: $\{even\} \subseteq \uparrow_{OE}(oddOrEven)$.

(7) Like (6), but with *odd* and *even* exchanged, and Y3b for Y3a.

(8) From X3a, we know that $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$, so $\mathcal{T}_{\Sigma_{OE}}(odd) - \mathcal{T}_{\Sigma_{OE}}(oddOrEven) = \emptyset = \bigcup \mathcal{T}_{\Sigma_{OE}}[\{oddAndEven\}]$. Also note $oddAndEven \succeq_{OE} p$ for any element p of Σ_{OE} , so $\forall p \in \Sigma_{OE} (\{oddAndEven\} \subseteq \uparrow_{OE}(p))$.

(9) Like (8) but with *even* for *odd*, and X3b for X3a.

Together the table shows that *OE* satisfies assumption D3b with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$. ■(D3b)■Proof.

Note that since every entry in the tables for D3a and D3b is a singleton, we have also proved that $\langle \Sigma_{OE}, \preceq_{OE} \rangle$ is strongly extensible with respect to $\mathcal{T}_{\Sigma_{OE}}$.

[Show LL]

We see from Figure 5-2 that $Max_{OE} = \{odd, even\}$; and that

$p \in \Sigma_{OE}$	$Max_{OE}(p)$
<i>odd</i>	$\{odd\}$
<i>even</i>	$\{even\}$
<i>oddOrEven</i>	$\{odd, even\}$
<i>oddAndEven</i>	$\emptyset$

We can observe that *OE* satisfies LL.

Summary of the proof

Given that the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$ satisfies the semantic conventions *OEX*, it is the case that truth medium *OE* satisfies assumption Set *CL* with respect to $\mathcal{T}_{\Sigma_{OE}}$. As a result, the relation LLC_{OE} makes the representational relation of logical consequence $ALC_{\langle OE, \mathcal{T}_{\Sigma_{OE}} \rangle}$ explanatorily intelligible. This completes the application of the Set *CL* technique.

If truth medium *OE* satisfies assumption Set *CL* with respect to $\mathcal{T}_{\Sigma_{OE}}$, then truth medium *OE* satisfies assumption Set *CG* with respect to $\mathcal{T}_{\Sigma_{OE}}$ (since Set *CG* = $\{D1, D2, D3\}$ is included within Set *CL*).

So we have also shown that if the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$ satisfies the semantic conventions OEX , it is the case that truth medium OE satisfies assumption Set CG with respect to $\mathcal{T}_{\Sigma_{OE}}$. As a result, the medium OE is an application of the Set CG technique as well, and the relation LC_{OE} also makes the representational relation of logical consequence $ALC_{\langle OE, \mathcal{T}_{\Sigma_{OE}} \rangle}$ explanatorily intelligible.

We present both relations below.

The intelligible proxies for logical consequence

The Set CL proxy for logical consequence: LLC We gave the extension of Max_{OE} above (in the proof that OE satisfies LL).

Given that extension of Max_{OE} , we find that the extension of LLC_{OE} is as follows:

$LLC_{OE}(p, q)$		q			
		$oddOrEven$	odd	$even$	$oddAndEven$
p	$oddOrEven$	true	false	false	false
	odd	true	true	false	false
	$even$	true	false	true	false
	$oddAndEven$	true	true	true	true

Given that we accept that the practice $\mathcal{T}_{\Sigma_{OE}}$ satisfies all of the assumptions in the set OEX , then the table above makes the representational relation of logical consequence $ALC_{\langle OE, \mathcal{T}_{\Sigma_{OE}} \rangle}$ intelligible.

The Set CG proxy for logical consequence: LC Given our definition of OE , we find that the extension of CC_{OE} is as follows:

$p \in \Sigma_{OE}$	$CC_{OE}(p)$
odd	$\{odd, oddOrEven\}$
$even$	$\{even, oddOrEven\}$
$oddOrEven$	$\{odd, even, oddOrEven\}$
$oddAndEven$	$\emptyset$

Given that extension of CC_{OE} , the extension of LC_{OE} is the same as the extension of LLC_{OE} above.

5.5 Second simple application: the “powerset of odd and even” language POE

5.5.1 Introduction to the example

In this section we apply an order-consistency technique to model the relation of logical consequence for a language with slightly more structure than the previous example. This example, like the ones presented in Section 3.4, and the previous section (Section 5.4) concerns a simple language used to describe the numerical value of a roll of a single six-sided die. This example will differ in that we shall use a different method to define the set of assertion types. In those earlier examples, we simply enumerated the set of assertion types. In this case, we will define the set of assertion types structurally, using the powerset operator. The relation between the languages OE and POE , is analogous to that between the WFF - and $\mathcal{P}(WFF)$ -languages of propositional logic. Note carefully however, that our treatment of the assertion type $\emptyset$ will be different for the POE language than for the $\mathcal{P}(WFF)$ -language of propositional logic.

Our construction will begin with the set of consistent assertion types from the previous example, namely $C_{OE} = \{odd, even, oddOrEven\}$. We will assume these types are interpreted by the same practice of categorization as before, namely $\mathcal{T}_{\Sigma_{OE}}$. We will then apply the powerset operator to obtain our new set of assertion types. Let us call that new set Σ_{POE} (as we will call the consistency medium we form around it POE). So $\Sigma_{POE} = \mathcal{P}(C_{OE})$.

We intend that the assertion types in Σ_{POE} be interpreted as follows. If $P \in \Sigma_{POE}$ is nonempty, we intend that P be interpreted as the conjunction of the claims made by the elements of P as interpreted by the practice $\mathcal{T}_{\Sigma_{OE}}$. If $P \in \Sigma_{POE}$ is empty, we intend that P be true in just those cases where any element of C_{OE} would be true as interpreted by the practice $\mathcal{T}_{\Sigma_{OE}}$.

We can capture those intentions via the following definition:

($POE1$) For all $P \in \Sigma_{POE}$, $\mathcal{T}_{\Sigma_{POE}}(P) = \text{if } P = \emptyset \text{ then } \bigcup \mathcal{T}_{\Sigma_{OE}}[C_{OE}] \text{ else } \bigcap \mathcal{T}_{\Sigma_{OE}}[P]$.

Notice the subtle shift from the discussion of the semantic conventions of the $\mathcal{P}(WFF)$ -language of propositional logic in Section 3.5. Those conventions ensured that the empty set was interpreted as true in every possibility, that is, as a logical truth. Here we limit the interpretation of the empty set to those possibilities in which some assertion type from the base medium is true. Note that in the case of the $\mathcal{P}(WFF)$ -language of propositional logic, the semantic conventions (which imply the law of excluded middle), imply that the set of those possibilities in which some assertion type from the set WFF is true via the practice $\mathcal{T}_{WFF}$ just is the set of all possibilities $\mathcal{U}$.

Our task is to apply an order-consistency technique to model the representational relation of logical consequence between the assertion types in Σ_{POE} as interpreted by $\mathcal{T}_{\Sigma_{POE}}$.

5.5.2 Details of the application

Technique selection

We initially apply the Set CL technique.

The consistency medium POE

We seek to define a consistency medium $POE = \langle \Sigma_{POE}, \preceq_{POE}, C_{POE} \rangle$.

We have already defined that $\Sigma_{POE} = \mathcal{P}(C_{OE})$ above.

We define the set C_{POE} as: $C_{POE} = \Sigma_{POE} - \{\{odd, even\}, \{oddOrEven, odd, even\}\}$

We define the ordering $\preceq_{POE}$ as the inclusion order on Σ_{POE} . This is the same formulation we used in modelling the $\mathcal{P}(WFF)$ -language of propositional logic.

The complete consistency medium POE is shown the modified Hasse diagram in Figure 5-3.

Semantic conventions

We assume that $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy $POE1$, and that $\mathcal{T}_{\Sigma_{OE}}$ satisfies OEX . In the following, when we say $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, we mean $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy $POE1$, and $\mathcal{T}_{\Sigma_{OE}}$ satisfies OEX .

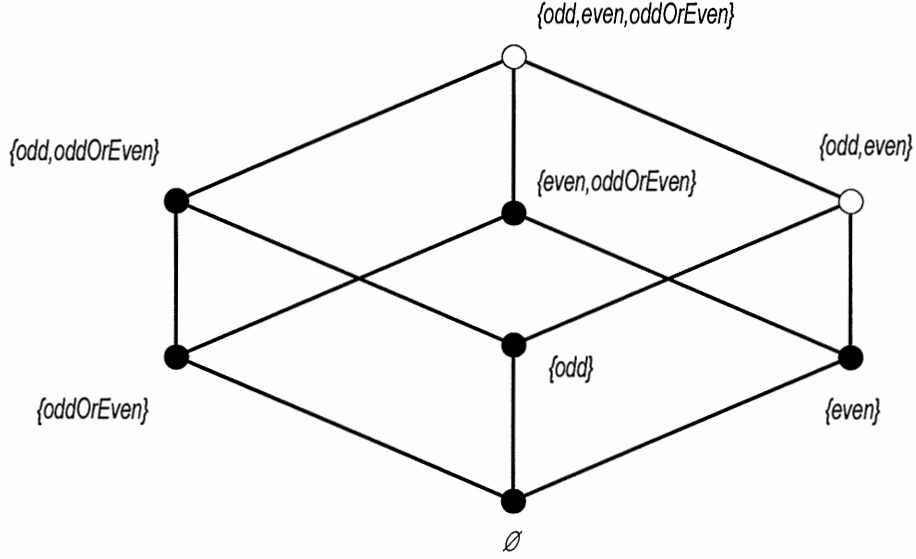


Figure 5-3: Modified Hasse diagram showing the declarative medium *POE*.

Proof of the correctness of the application

In this section, we prove that if $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, then *POE* satisfies Set *CL* with respect to $\mathcal{T}_{\Sigma_{POE}}$.

We begin by noting some equivalences. The proofs of these Lemmas can be found at the end of the chapter.

Lemma 68 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, the following equivalences hold.*

- (Z1) $\mathcal{T}_{\Sigma_{POE}}(\{\text{odd}, \text{even}, \text{oddOrEven}\}) = \mathcal{T}_{\Sigma_{POE}}(\{\text{odd}, \text{even}\}) = \mathcal{T}_{\Sigma_{OE}}(\text{oddAndEven}).$
- (Z2) $\mathcal{T}_{\Sigma_{POE}}(\{\text{odd}, \text{oddOrEven}\}) = \mathcal{T}_{\Sigma_{POE}}(\{\text{odd}\}) = \mathcal{T}_{\Sigma_{OE}}(\text{odd}).$
- (Z3) $\mathcal{T}_{\Sigma_{POE}}(\{\text{even}, \text{oddOrEven}\}) = \mathcal{T}_{\Sigma_{POE}}(\{\text{even}\}) = \mathcal{T}_{\Sigma_{OE}}(\text{even}).$
- (Z4) $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \mathcal{T}_{\Sigma_{POE}}(\{\text{oddOrEven}\}) = \mathcal{T}_{\Sigma_{OE}}(\text{oddOrEven}).$

We now identify some background lemmas before proceeding to the main proof.

Lemma 69 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that for every $P \in \Sigma_{POE}$, that $\mathcal{T}_{\Sigma_{POE}}(P) \subseteq \mathcal{T}_{\Sigma_{POE}}(\emptyset)$.*

Lemma 70 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that: For any $P, Q \in \Sigma_{POE}$, $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P \cup Q)$.*

Lemma 71 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that: For any $P, Q \in \Sigma_{POE}$, $\mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P) \cap (\mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q))$.*

Lemma 72 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that: For any $P \in \Sigma_{POE}$, there is a $Q \in \Sigma_{POE}$ such that $\mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(P)$.*

We are now ready to show the correctness of the application.

Proposition 73 *If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, then POE satisfies Set CL with respect to $\mathcal{T}_{\Sigma_{POE}}$.*

Assume that $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions.

[Show that POE satisfies D1 with respect to $\mathcal{T}_{\Sigma_{POE}}$]

1. That is, [Show that $\forall P, Q \in \Sigma_{POE}$, $P \preceq_{POE} Q$ implies $\mathcal{T}_{\Sigma_{POE}}(Q) \subseteq \mathcal{T}_{\Sigma_{POE}}(P)$]

Follows from the interaction of POE1 and $\mathcal{T}_{\Sigma_{POE}}$. Details are found at the end of the chapter.

[Show that POE satisfies D2 with respect to $\mathcal{T}_{\Sigma_{POE}}$]

That is, [Show that $\forall P \in \Sigma_{POE}$, that $P \in C_{POE}$ iff $\mathcal{T}_{\Sigma_{POE}}(P) \neq \emptyset$]

We enumerate the elements of Σ_{POE} , and state for each whether or not the set of possibilities it represents is empty, and the assumptions which support that claim.

1. $\mathcal{T}_{\Sigma_{POE}}(\emptyset) \neq \emptyset$, by Z4, and Y4.
2. $\mathcal{T}_{\Sigma_{POE}}(\{odd\}) \neq \emptyset$, by Z2 and X1.
3. $\mathcal{T}_{\Sigma_{POE}}(\{even\}) \neq \emptyset$, by Z3 and X2.

4. $\mathcal{T}_{\Sigma_{POE}}(\{oddOrEven\}) \neq \emptyset$, by Z4 and Y4.
5. $\mathcal{T}_{\Sigma_{POE}}(\{odd, oddOrEven\}) \neq \emptyset$, by Z2 and X1.
6. $\mathcal{T}_{\Sigma_{POE}}(\{even, oddOrEven\}) \neq \emptyset$, by Z3 and X2.
7. $\mathcal{T}_{\Sigma_{POE}}(\{odd, even\}) = \emptyset$, by Z1 and Y1.
8. $\mathcal{T}_{\Sigma_{POE}}(\{odd, even, oddOrEven\}) = \emptyset$, by Z1 and Y1.

Comparing these results with the definition of C_{POE} indicates that POE satisfies $D2$ with respect to $\mathcal{T}_{\Sigma_{POE}}$. ■ (D2)

[Show that POE satisfies $D3a$ with respect to $\mathcal{T}_{\Sigma_{POE}}$]

1. That is, $\left[\text{Show } \forall P, Q \in \Sigma_{POE} \left(\begin{array}{c} \exists \Gamma \subseteq \{P, Q\}_{POE}^u \text{ such that} \\ \bigcup \mathcal{T}_{\Sigma_{POE}}[\Gamma] = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) \end{array} \right) \right]$
2. Let P, Q be arbitrary elements of Σ_{POE} .
3. [Claim: $\Gamma = \{P \cup Q\}$ satisfies the existential claim for $D3a$.]
 - (a) $P \cup Q \succeq_{POE} P$, and $P \cup Q \succeq_{POE} Q$, so $\{P \cup Q\} \subseteq \{P, Q\}_{POE}^u$.
 - (b) $\bigcup \mathcal{T}_{\Sigma_{POE}}[\{P \cup Q\}] = \mathcal{T}_{\Sigma_{POE}}(P \cup Q)$.
 - (c) By Lemma 70, $\mathcal{T}_{\Sigma_{POE}}(P \cup Q) = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q)$. ■ Claim. ■ (D3a)

[Show that POE satisfies $D3b$ with respect to $\mathcal{T}_{\Sigma_{POE}}$]

1. That is, $\left[\text{Show } \forall P, Q \in \Sigma_{POE} \left(\begin{array}{c} \exists \Delta \subseteq \uparrow_{POE} P \text{ such that} \\ \bigcup \mathcal{T}_{\Sigma_{POE}}[\Delta] = \mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q) \end{array} \right) \right]$
2. Let P, Q be arbitrary elements of Σ_{POE} .
3. Let R be an element of Σ_{POE} such that $\mathcal{T}_{\Sigma_{POE}}(R) = \mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q)$. By Lemma 72 we know that some such R exists.
4. By Lemma 71, $\mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(R)$.
5. By Lemma 70, $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(R) = \mathcal{T}_{\Sigma_{POE}}(P \cup R)$.

6. [Claim: $\Delta = \{P \cup R\}$ satisfies the existential claim for *D3b*.]

(a) $P \cup R \succeq_{POE} P$, so $\{P \cup R\} \subseteq \uparrow_{POE} P$.

(b) $\bigcup \mathcal{T}_{\Sigma_{POE}}[\{P \cup R\}] = \mathcal{T}_{\Sigma_{POE}}(P \cup R) = \mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q)$. ■ Claim.

■(D3b). ■ Proposition.

Note that our proof shows that given that $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, then $\langle \Sigma_{POE}, \preceq_{POE} \rangle$ is also strongly extensible with respect to $\mathcal{T}_{\Sigma_{POE}}$. This is a consequence of the fact that for both parts a and b, the witness which was used to demonstrate the existential claim was a singleton.

[Show that *POE* satisfies *LL*]

We can see from Figure 5-3 that $Max_{POE} = \{\{odd, oddOrEven\}, \{even, oddOrEven\}\}$.

By inspecting Figure 5-3, we find that the extension of the function Max_{POE} is as follows:

$P \in \Sigma_{POE}$	$Max_{POE}(P)$
$\emptyset$ or $\{oddOrEven\}$	$\{\{odd, oddOrEven\}, \{even, oddOrEven\}\}$
$\{odd\}$ or $\{odd, oddOrEven\}$	$\{\{odd, oddOrEven\}\}$
$\{even\}$ or $\{even, oddOrEven\}$	$\{\{even, oddOrEven\}\}$
$\{odd, even\}$ or $\{odd, even, oddOrEven\}$	$\emptyset$

We see from this table that *POE* satisfies *LL*.

Summary of the proof

Given that $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that truth medium *POE* satisfies assumption Set *CL* with respect to $\mathcal{T}_{\Sigma_{POE}}$. As a result, the relation LLC_{POE} makes the representational relation of logical consequence $ALC_{\langle POE, \mathcal{T}_{\Sigma_{POE}} \rangle}$ explanatorily intelligible. This completes the application of the Set *CL* technique.

If truth medium *POE* satisfies assumption Set *CL* with respect to $\mathcal{T}_{\Sigma_{POE}}$, then truth medium *POE* satisfies assumption Set *CG* with respect to $\mathcal{T}_{\Sigma_{POE}}$ (since Set *CG* is included within Set *CL*). So we have also shown that if $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that truth medium *POE* satisfies assumption Set *CG* with respect

to $\mathcal{T}_{\Sigma_{POE}}$. As a result, the medium POE is an application of the Set CG technique as well, and the relation LC_{POE} also makes the representational relation of logical consequence $ALC_{\langle POE, \mathcal{T}_{\Sigma_{POE}} \rangle}$ explanatorily intelligible.

We present both relations below.

The intelligible proxy for logical consequence

The Set CL proxy for logical consequence: LLC We gave the extension of Max_{POE} above (in the proof that POE satisfies LL).

Given that extension of Max_{POE} , we find that the extension of LLC_{POE} is as follows:

(note that we split the table in half due to formatting restrictions).

$LLC_{POE}(P, Q)$		Q	
		$\emptyset$ or $\{oddOrEven\}$	$\{odd\}$ or $\{odd,$ $oddOrEven\}$
P	$\emptyset$ or $\{oddOrEven\}$	true	false
	$\{odd\}$ or $\{odd,$ $oddOrEven\}$	true	true
	$\{even\}$ or $\{even,$ $oddOrEven\}$	true	false
	$\{odd,$ $even\}$ or $\{odd,$ $even,$ $oddOrEven\}$	true	true

		Q	
		$\{even\}$ or $\{even, oddOrEven\}$	$\{odd, even\}$ or $\{odd, even, oddOrEven\}$
$LLC_{POE}(P, Q)$			
P	$\emptyset$ or $\{oddOrEven\}$	false	false
	$\{odd\}$ or $\{odd, oddOrEven\}$	false	false
	$\{even\}$ or $\{even, oddOrEven\}$	true	false
	$\{odd, even\}$ or $\{odd, even, oddOrEven\}$	true	true

Given that we accept that $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, then the table above makes the representational relation of logical consequence $ALC_{\langle POE, \mathcal{T}_{\Sigma_{POE}} \rangle}$ intelligible.

The Set CG proxy for logical consequence: LC Given our definition of POE , we find that the extension of CC_{POE} is as follows:

$P \in \Sigma_{POE}$	$CC_{POE}(P)$
$\emptyset$ or $\{oddOrEven\}$	$\{\emptyset,$ $\{oddOrEven\},$ $\{odd\},$ $\{odd, oddOrEven\},$ $\{even\},$ $\{even, oddOrEven\}\}$
$\{odd\}$ or $\{odd, oddOrEven\}$	$\{\emptyset,$ $\{oddOrEven\},$ $\{odd\},$ $\{odd, oddOrEven\}\}$
$\{even\}$ or $\{even, oddOrEven\}$	$\{\emptyset,$ $\{oddOrEven\},$ $\{even\},$ $\{even, oddOrEven\}\}$
$\{odd, even\}$ or $\{odd, even, oddOrEven\}$	$\emptyset$

Given the extension of CC_{POE} , we find that the extension of LC_{POE} is the same as the extension of LLC_{POE} above.

Using one consistency medium to model the consequence relation of another.

Compare the chart of LLC_{POE} with the chart of LLC_{OE} on page 203 . Notice that it is possible to define a function h from Σ_{POE} to Σ_{OE} such that $\forall P, Q \in \Sigma_{POE}$, $LLC_{POE}(P, Q)$ iff $LLC_{OE}(h(P), h(Q))$. This shows that it is possible to model the assumed consequence relation on $\Sigma_{POE} \times \Sigma_{POE}$ using a different consistency medium (in this case one with a set of assumption types with a smaller cardinality.) The concept of using one consistency medium to model the consequence relation of another will be useful in a later proof (in Chapter 9) when we discuss minimal consistency media.

5.6 Propositional logic example

We defer this example until the Section 6.11.

5.7 Languages in which feature structures are considered as assertions in their own right

5.7.1 Introduction

In this section, we are going to consider the application of order-consistency techniques to languages in which feature structures are considered as assertions in their own right. We have already discussed feature structures twice. First, we talked about their use as models in a feature logic (in Section 3.6). Remember that there, the models were considered to be feature structure/distinguished element pairs. Then, we had the chance to introduce the concept of feature structures as assertions in their own right earlier in this chapter (in Section 5.3.4). Our plan for this section is as follows.

We will begin by setting up a preliminary idea of what a language of feature structure assertion types might look like (including a concept of subsumption that we might apply to them). We will then outline the application of order-consistency techniques to languages in which feature structures are used as assertions in their own right.

5.7.2 A language of feature structure assertion types

There are many different approaches to feature structures discussed in the literature. For our purposes in this section, we will take a simple approach. When we thought of feature structures as models, it was necessary to consider them as feature structure/distinguished element pairs, since the sentences of feature logic languages are interpreted modally from the point of view of some element within a structure. It does not seem that the extra structure of a distinguished element would be necessary for a language of feature structures used as assertion types. So let us consider a language made up of the set of all feature structures of a particular signature (Call it FS).

We would like to be able to define a concept of subsumption between the elements of

FS. As it stands, our concept of subsumption is defined on feature structure/element pairs. Let us define a form of subsumption for feature structures (without a specified distinguished element) as follows.

Given two feature structures $\mathbb{A}$ and $\mathbb{B}$, say that $\mathbb{A}$ subsumes (in the feature structure form of subsumption) $\mathbb{B}$ just in case $\exists d \in D^{\mathbb{A}}$ such that $P(d) = \mathbb{A}$, and $\exists e \in D^{\mathbb{B}}$ such that $P(e) = \mathbb{B}$, such that $\langle \mathbb{A}, d \rangle$ subsumes (in the feature structure/element form of subsumption) $\langle \mathbb{B}, e \rangle$.

Since $\mathbb{A}$ and $\mathbb{B}$ are feature structures (not feature systems), we know that there exist $d \in D^{\mathbb{A}}$ and $e \in D^{\mathbb{B}}$ such that $P(d) = \mathbb{A}$, $P(e) = \mathbb{B}$. That means that from d we can reach (via a path of features) every element in $\mathbb{A}$, and from e we can reach (via a path of features) every element in $\mathbb{B}$. So if $\langle \mathbb{A}, d \rangle$ subsumes $\langle \mathbb{B}, e \rangle$ (in the feature structure/element way), then all the attributes (features and subdomains) of each element in $\mathbb{A}$ can be mapped onto the attributes (features and subdomains) of some corresponding element in $\mathbb{B}$. For example, feature structure $\mathbb{C}$ (in Figure 3-3 on page 106) subsumes (in the feature structure way) the feature structure $P(s)$ (in Figure 3-2 on page 104), since $\langle \mathbb{C}, s \rangle$ subsumes $\langle P(s), s \rangle$ (in the feature structure/element way).

5.7.3 Semantic conventions for a language of feature structure assertion types

Here we switch our usual order and discuss semantic conventions before describing the consistency medium itself. We know that since we will be using a consistency medium, the medium will have one kind of representative element: the feature structures. So the semantic conventions for a language of feature structure assertion types will constrain the practices of categorization interpreting those feature structures. Such semantic conventions would be defined to take feature structure signatures as a parameter.

There are several things our discussion so far should lead us to expect about these conventions. One is that the semantic conventions for interpretations of feature structures as assertions should be expected to imply that, for all feature structures $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{A}$ subsumes $\mathbb{B}$, then the possibilities represented by $\mathbb{B}$ are a subset of the possibilities represented by $\mathbb{A}$.

This rule is at the heart of the way in which we intend to interpret feature structures in general. (We intend subsumption to correspond to informativeness).

We should also expect that in the general case, not all consequences will be subsumptions. That is, in general, the converse of the rule just stated will not hold. The consequence relation will, in general, be more complex; and the source of this complexity will be that, with respect to certain acceptable practices of categorization interpreting the feature structures, certain feature structures will be interpreted as inconsistent. This means that the semantic conventions for interpretations of feature structures as assertion types will be different than the semantic conventions for feature structures as models.

5.7.4 The consistency medium

Now we can describe the consistency medium we could use to model logical consequence for a language of feature structure assertion types. Like any consistency medium, there will be three components: the set of assertion types, a partial order, and a subset of the assertion types taken to be consistent. In the case of a language of feature structure assertion types, the set of assertion types could be the set FS of feature structures. We could use the subsumption order as the partial order. And then we would need to specify the consistent subset.

How would we carry this out? The basic idea is that we would specify the set of consistent feature structures using a set of constraints written in the language of some feature logic. A feature structure would be deemed consistent if it had some extension (via the subsumption relation) which satisfied all the constraints at every element in the structure.

So for instance, consider the constraints on consistency expressed in the example in Section 5.3.4. No male can be a mother. And every person is either male or female (but not both). We could express these constraints using a three-valued feature logic following the standard Kleene evaluation scheme (Rounds 1997, 490; Dawar and Vijay-Shanker 1991) as follows:

mother : $\neg Male$

$(Male \vee Female) \wedge \neg(Male \wedge Female)$

Let us consider some of the attributes of this proposal.

1. Modality. The work in feature logics has shown that it is natural to specify and evaluate constraints from the perspective of elements within a structure. The semantics of feature logic sentences were designed for this very purpose.
2. Involvement of the subsumption ordering. Our specification of consistency has an element of complexity, in that we bring the subsumption relation into the picture. We don't say simply: we deem consistent those feature structures that satisfy every constraint at every element. This is because feature structures are inherently partial. Given that partiality, there may not be enough information in some structure to permit the determinate evaluation of a constraint. For example, if no gender is specified for an element in a structure from our example, then the value of "*Male* $\vee$ *Female*" for that element (on a three-valued logical scheme) would be *unknown*. Because of the subsumption ordering (and its property that if $\mathbb{A}$ subsumes $\mathbb{B}$, then the possibilities represented by $\mathbb{B}$ are a subset of the possibilities represented by $\mathbb{A}$), we know that if $\mathbb{B}$ is consistent, then any $\mathbb{A}$ that subsumes $\mathbb{B}$, that is, any $\mathbb{A}$ for which $\mathbb{B}$ is an extension on the subsumption ordering, must also be consistent. So we can express the constraints relative to those feature structures for which they would be determinate.
3. Maximality. Our approach to specifying consistency does not require maximal extensions, just extensions that are rich enough to show that all constraints are satisfied.

We will see a simple example of this approach to specifying consistency in the proof in Chapter 6, showing that the range of applicability of the Set *PP* technique is included within that of the Set *CG* technique.

A possible extension

We can consider an extension to the general approach to specifying consistency above. We could shift our focus from simple feature structures to typed feature structures (of the kind discussed in Carpenter 1992), where the elements of the structure are typed (and the types have an ordering of their own). In that case, constraints could be specified on a type-by-type

basis; and the definition of consistency could be extended as follows: A feature structure would be deemed consistent if it had some extension (via the subsumption relation) which satisfied, at every element of the structure, all the constraints defined for that type of element. This would enable the distribution of the specification of consistency across the different types of element used in the feature structure.

5.7.5 What is required to complete this application?

There are two major tasks required to complete this application. First, the semantic conventions for interpretations of feature structures as assertion types would need to be specified. As we have discussed, these conventions would share some things in common with the semantic conventions for feature structures as models (like subsumption implying consequence), but would also have some differences (in that consistency of all feature structures would not be assumed).

The second major task would be to give proofs that showing that if the semantic conventions are satisfied, then the technique-specific assumptions for Set *CG* are satisfied. This would justify the usage of the Set *CG* proxy for consequence relation *LC*. If a set of feature structures further satisfied *LL*, that is, it had a maximal extension for every consistent structure, then the usage of the Set *CL* proxy for consequence relation *LLC* would be justified as well (and its extension would be equivalent to *LC*).

5.8 Considering the applicability of order-consistency techniques

5.8.1 Can we, in general, define an appropriate ordering $\preceq$ on the set of assertion types Σ ?

We have now presented several examples of the application of order-consistency representative techniques. One thing should be very obvious from these examples and the discussion which preceded them. In order to apply these techniques, we need to identify an ordering $\preceq_\Sigma$ on the set of assertion types Σ , such that we would be willing to assume that $\langle \Sigma, \preceq_\Sigma \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_\Sigma$, and further, that $\langle \Sigma, \preceq_\Sigma \rangle$

is weakly extensible with respect to $\mathcal{T}_\Sigma$. We have seen that for some common languages of assertion, like the $\mathcal{P}(WFF)$ -language of propositional logic, such orderings can in fact be identified. This raises an important question: can we in general, define an appropriate ordering $\preceq_\Sigma$ on a set of assertion types Σ as interpreted by a practice of categorization $\mathcal{T}_\Sigma$?

The quick answer is “no, we can’t.” Consider the example from Section 3.4. Given the set of assertion types in Σ_{Roll} and the practice of categorization $\mathcal{T}_{\Sigma_{Roll}}$ it is easy to see that no ordering $\preceq_{\Sigma_{Roll}}$ on Σ_{Roll} can satisfy weak extensibility with respect to $\mathcal{T}_{\Sigma_{Roll}}$. Look at the pair of assertion types *odd* and *five*. There is no set of assertion types S , such that the possibilities collectively represented by the members of S (that is, $\bigcup \mathcal{T}_{\Sigma_{Roll}}[S]$) equals the difference between the possibilities represented by *odd* and the possibilities represented by *five* (that is, $\mathcal{T}_{\Sigma_{Roll}}(odd) - \mathcal{T}_{\Sigma_{Roll}}(five)$). Given the absence of such assertion types it is impossible to construct some order $\preceq_{\Sigma_{Roll}}$ on the existing types so that $\langle \Sigma_{Roll}, \preceq_{\Sigma_{Roll}} \rangle$ satisfies weak extensibility with respect to $\mathcal{T}_{\Sigma_{Roll}}$.

5.8.2 The possibility of completions

As we have just seen, we cannot in general define an ordering $\preceq_\Sigma$ on a set of assertion types Σ as interpreted by a practice of categorization $\mathcal{T}_\Sigma$, such that $\langle \Sigma, \preceq_\Sigma \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_\Sigma$, and $\langle \Sigma, \preceq_\Sigma \rangle$ is weakly extensible with respect to $\mathcal{T}_\Sigma$. But that is not the whole story. Completions, analogous to the Dedekind-MacNeille completion in lattice theory (Davey and Priestley 1990, 41ff), are possible.

If we are given an application of the Set *PP* technique (that is, a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$) such that A satisfies assumption Set *PP* with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we can define an operator *comp* which takes A , $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, as arguments, and constructs from them a partially ordered truth medium $comp(A)$ and practices of categorization $\mathcal{T}_{\Sigma_{comp(A)}}$ and $\mathcal{T}_{M_{comp(A)}}$, such that

1. $\langle \Sigma_{comp(A)}, \preceq_{comp(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{comp(A)}}$,
2. $\langle \Sigma_{comp(A)}, \preceq_{comp(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{comp(A)}}$,
3. $comp(A)$ satisfies assumption Set *PP* with respect to practices of categorization

$\mathcal{T}_{\Sigma_{comp(A)}}$ and $\mathcal{T}_{M_{comp(A)}}$, therefore, by the Corollary to the Intelligibility of Consequence Theorem for Truth media (Corollary 14),

$MC_{comp(A)}$ makes the assumed relation of logical consequence

$ALC_{\langle \Sigma_{comp(A)}, \mathcal{T}_{\Sigma_{comp(A)}} \rangle}$ intelligible,

4. An image of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$, the representational relation of logical consequence which the given application of the

Set PP technique makes intelligible, is embedded within $ALC_{\langle \Sigma_{comp(A)}, \mathcal{T}_{\Sigma_{comp(A)}} \rangle}$

(that is, there is a function $h : \Sigma_A \rightarrow \Sigma_{comp(A)}$ such that

$$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{comp(A)}, \mathcal{T}_{\Sigma_{comp(A)}} \rangle}(h(p), h(q)) \right).$$

As a result, $MC_{comp(A)}$ makes $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ intelligible as well.

To see this, realize that it is the case that

$$\forall p, q \in \Sigma_A \left(\begin{array}{c} ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff} \\ ALC_{\langle \Sigma_{comp(A)}, \mathcal{T}_{\Sigma_{comp(A)}} \rangle}(h(p), h(q)) \text{ iff} \\ MC_{comp(A)}(h(p), h(q)) \end{array} \right).$$

We will see the application of this concept of completions in the proof in the next chapter. In our presentation there, we split the functionality of the operator $comp$ across two operators on truth media and associated practices, namely, $negcomp$ and pow . This division will enable us to handle varying initial conditions more elegantly.

This concludes the linear text of the chapter. The remainder of the material in this chapter are the proofs of propositions mentioned earlier in the text.

5.9 Proofs of propositions from the text

Proof of Lemma 68:

(Lemma 68): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, the following equivalences hold.

- (Z1) $\mathcal{T}_{\Sigma_{POE}}(\{odd, even, oddOrEven\}) = \mathcal{T}_{\Sigma_{POE}}(\{odd, even\}) = \mathcal{T}_{\Sigma_{OE}}(oddAndEven)$.

1. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\{odd, even\}) = \mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even)$.

2. By $X4$, $\mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even) = \emptyset$.
 3. By $Y1$, $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$.
 4. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\{odd, even, oddOrEven\}) = \mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(even) \cap \mathcal{T}_{\Sigma_{OE}}(oddOrEven) = \emptyset$. ■ (Z1)
- (Z2) $\mathcal{T}_{\Sigma_{POE}}(\{odd, oddOrEven\}) = \mathcal{T}_{\Sigma_{POE}}(\{odd\}) = \mathcal{T}_{\Sigma_{OE}}(odd)$.
 1. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\{odd, oddOrEven\}) = \mathcal{T}_{\Sigma_{OE}}(odd) \cap \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 2. By $X3a$, $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 3. So $\mathcal{T}_{\Sigma_{POE}}(\{odd, oddOrEven\}) = \mathcal{T}_{\Sigma_{OE}}(odd)$.
 4. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\{odd\}) = \mathcal{T}_{\Sigma_{OE}}(odd)$. ■ (Z2)
 - (Z3) $\mathcal{T}_{\Sigma_{POE}}(\{even, oddOrEven\}) = \mathcal{T}_{\Sigma_{POE}}(\{even\}) = \mathcal{T}_{\Sigma_{OE}}(even)$.
Like (Z2) but with *even* substituted for *odd*, and $X3b$ for $X3a$. ■ (Z3)
 - (Z4) $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \mathcal{T}_{\Sigma_{POE}}(\{oddOrEven\}) = \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 1. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \bigcup \mathcal{T}_{\Sigma_{OE}}[C_{OE}] = \mathcal{T}_{\Sigma_{OE}}(odd) \cup \mathcal{T}_{\Sigma_{OE}}(even) \cup \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 2. By $X3a$, $\mathcal{T}_{\Sigma_{OE}}(odd) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 3. By $X3b$, $\mathcal{T}_{\Sigma_{OE}}(even) \subseteq \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 4. So $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
 5. By $POE1$, $\mathcal{T}_{\Sigma_{POE}}(\{oddOrEven\}) = \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$. ■ (Z4)

Proof of Lemma 69:

(Lemma 69): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that for every $P \in \Sigma_{POE}$, that $\mathcal{T}_{\Sigma_{POE}}(P) \subseteq \mathcal{T}_{\Sigma_{POE}}(\emptyset)$.

1. Assume that the set of assertion types Σ_{OE} satisfies the assumptions in OEX with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$.
2. Let P be an arbitrary element of Σ_{POE} .

3. Case 1. $P = \emptyset$.

(a) $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{POE}}(\emptyset)$, so $\mathcal{T}_{\Sigma_{POE}}(P) \subseteq \mathcal{T}_{\Sigma_{POE}}(\emptyset)$. ■(Case 1).

4. Case 2. $P \neq \emptyset$.

(a) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(P) = \bigcap \mathcal{T}_{\Sigma_{OE}}[P]$.

(b) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \bigcup \mathcal{T}_{\Sigma_{OE}}[C_{OE}]$.

(c) Since $P \in \Sigma_{POE}$, $P \subseteq C_{OE}$ by the definition of Σ_{POE} .

(d) So $\mathcal{T}_{\Sigma_{POE}}(P) \subseteq \mathcal{T}_{\Sigma_{POE}}(\emptyset)$. ■(Case 2). ■Lemma.

Proof of Lemma 70:

(Lemma 70): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that:

For any $P, Q \in \Sigma_{POE}$, $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P \cup Q)$.

1. Assume that the set of assertion types Σ_{OE} satisfies the assumptions in *OEX* with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$.

2. Let P, Q be arbitrary elements of Σ_{POE} .

3. Case 1. ($P = \emptyset, Q = \emptyset$)

(a) $P = Q$ so $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{POE}}(Q)$ so $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{POE}}(\emptyset)$.

(b) $P \cup Q = \emptyset$, so $\mathcal{T}_{\Sigma_{POE}}(P \cup Q) = \mathcal{T}_{\Sigma_{POE}}(\emptyset) = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q)$. ■(Case 1).

4. Case 2. ($P = \emptyset, Q \neq \emptyset$)

(a) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(P) = \bigcup \mathcal{T}_{\Sigma_{OE}}[C_{OE}]$.

(b) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[Q]$.

(c) Since $Q \in \Sigma_{POE}$, $Q \subseteq C_{OE}$, by the definition of Σ_{POE} .

(d) So $\mathcal{T}_{\Sigma_{POE}}(Q) \subseteq \mathcal{T}_{\Sigma_{POE}}(P)$.

(e) So $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(Q)$.

(f) $P \cup Q = Q$.

(g) So $\mathcal{T}_{\Sigma_{POE}}(P \cup Q) = \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q)$. ■ (Case 2).

5. Case 3. ($P \neq \emptyset, Q = \emptyset$)

(a) Same as Case 2, with P and Q reversed. ■ (Case 3)

6. Case 4. ($P \neq \emptyset, Q \neq \emptyset$)

(a) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(P) = \bigcap \mathcal{T}_{\Sigma_{OE}}[P]$.

(b) By *POE1*, $\mathcal{T}_{\Sigma_{POE}}(Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[Q]$.

(c) So $\mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[P \cup Q]$.

(d) In this case (Case 4), $P \cup Q \neq \emptyset$.

(e) So by *POE1*, $\mathcal{T}_{\Sigma_{POE}}(P \cup Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[P \cup Q] = \mathcal{T}_{\Sigma_{POE}}(P) \cap \mathcal{T}_{\Sigma_{POE}}(Q)$ ■ (Case 4) ■ Lemma.

Proof of Lemma 71:

(Lemma 71): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that:

For any $P, Q \in \Sigma_{POE}$, $\mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(P) \cap (\mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q))$.

1. Assume that the set of assertion types Σ_{OE} satisfies all of the assumptions in *OEX* with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$.

2. Let P, Q be arbitrary elements of Σ_{POE} .

3. ($\subseteq$)

(a) Let $u \in \mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q)$.

(b) So $u \in \mathcal{T}_{\Sigma_{POE}}(P)$ and $u \notin \mathcal{T}_{\Sigma_{POE}}(Q)$.

(c) By Lemma 69, $\mathcal{T}_{\Sigma_{POE}}(P) \subseteq \mathcal{T}_{\Sigma_{POE}}(\emptyset)$.

(d) So $u \in \mathcal{T}_{\Sigma_{POE}}(\emptyset)$, and therefore $u \in \mathcal{T}_{\Sigma_{POE}}(P) \cap (\mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q))$. ■ ($\subseteq$)

4. ($\supseteq$)

- (a) Let $u \in \mathcal{T}_{\Sigma_{POE}}(P) \cap (\mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q))$.
- (b) So $u \in \mathcal{T}_{\Sigma_{POE}}(P)$ and $u \in (\mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(Q))$.
- (c) So $u \notin \mathcal{T}_{\Sigma_{POE}}(Q)$.
- (d) So $u \in \mathcal{T}_{\Sigma_{POE}}(P) - \mathcal{T}_{\Sigma_{POE}}(Q)$. ■ ($\supseteq$) ■ Lemma.

Proof of Lemma 72:

(Lemma 72): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, it is the case that:

For any $P \in \Sigma_{POE}$, there is a $Q \in \Sigma_{POE}$ such that $\mathcal{T}_{\Sigma_{POE}}(Q) = \mathcal{T}_{\Sigma_{POE}}(\emptyset) - \mathcal{T}_{\Sigma_{POE}}(P)$.

1. Assume that the set of assertion types Σ_{OE} satisfies the assumptions in *OEX* with respect to the practice of categorization $\mathcal{T}_{\Sigma_{OE}}$.
2. By Z4, $\mathcal{T}_{\Sigma_{POE}}(\emptyset) = \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
3. Consider the following cases:
 - (a) Case 1. $P = \{odd, even\}$ or $P = \{odd, even, oddOrEven\}$
 - i. By Z1, $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{OE}}(oddAndEven)$.
 - ii. By Y1, $\mathcal{T}_{\Sigma_{OE}}(oddAndEven) = \emptyset$.
 - iii. So by POE1, $Q = \{oddOrEven\}$ satisfies the existential claim.

■(Case 1.)
 - (b) Case 2. $P = \{odd, oddOrEven\}$ or $P = \{odd\}$
 - i. By Z2, $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{OE}}(odd)$.
 - ii. By Y3a and POE1, we see that $Q = \{even\}$ satisfies the existential claim.

■(Case 2.)
 - (c) Case 3. $P = \{even, oddOrEven\}$ or $P = \{even\}$
 - i. By Z3, $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{OE}}(even)$.
 - ii. By Y3b and POE1, we see that $Q = \{odd\}$ satisfies the existential claim.

■(Case 3.)
 - (d) Case 4. $P = \emptyset$ or $P = \{oddOrEven\}$

- i. By Z4, $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{OE}}(oddOrEven)$.
- ii. By Z1, Y1, we see that $Q = \{odd, even\}$ satisfies the existential claim.
 ■(Case 4.)■Lemma.

Proof of Proposition 73:

(Proposition 73): If $\mathcal{T}_{\Sigma_{POE}}$ and $\mathcal{T}_{\Sigma_{OE}}$ satisfy the semantic conventions, then POE satisfies Set CL with respect to $\mathcal{T}_{\Sigma_{POE}}$.

(Part of Proposition 73)

proof that POE satisfies D1 with respect to $\mathcal{T}_{\Sigma_{POE}}$.

1. Let P, Q be arbitrary elements of Σ_{POE} .
2. Assume $P \preceq_{POE} Q$. So $P \subseteq Q$ by the definition of $\preceq_{POE}$.
3. Case 1. ($P = \emptyset, Q = \emptyset$)
 - (a) $\mathcal{T}_{\Sigma_{POE}}(P) = \mathcal{T}_{\Sigma_{POE}}(Q)$ so $\mathcal{T}_{\Sigma_{POE}}(Q) \subseteq \mathcal{T}_{\Sigma_{POE}}(P)$. ■(Case 1).
4. Case 2. ($P = \emptyset, Q \neq \emptyset$)
 - (a) By $POE1$, we have $\mathcal{T}_{\Sigma_{POE}}(P) = \bigcup \mathcal{T}_{\Sigma_{OE}}[C_{OE}]$.
 - (b) By $POE1$, we have $\mathcal{T}_{\Sigma_{POE}}(Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[Q]$.
 - (c) Since $Q \in \Sigma_{POE}$, $Q \subseteq C_{OE}$ so $\mathcal{T}_{\Sigma_{POE}}(Q) \subseteq \mathcal{T}_{\Sigma_{POE}}(P)$. ■(Case 2).
5. Case 3. ($P \neq \emptyset, Q = \emptyset$)
 - (a) This case is not possible since $P \subseteq Q$ ■(Case 3)
6. Case 4. ($P \neq \emptyset, Q \neq \emptyset$)
 - (a) By $POE1$, we have $\mathcal{T}_{\Sigma_{POE}}(P) = \bigcap \mathcal{T}_{\Sigma_{OE}}[P]$.
 - (b) By $POE1$, we have $\mathcal{T}_{\Sigma_{POE}}(Q) = \bigcap \mathcal{T}_{\Sigma_{OE}}[Q]$.
 - (c) Since $P \subseteq Q$, we have $\mathcal{T}_{\Sigma_{POE}}(Q) \subseteq \mathcal{T}_{\Sigma_{POE}}(P)$. ■(Case 4) ■(D1).

Part IV

Comparing the Ranges of Applicability of the Representational Techniques Presented

In this Part, we demonstrate additional results about the relative ranges of applicability of the representational techniques we have presented. We already know that the range of applicability of the Set E technique is a subset of the range of applicability of the Set PP technique, i.e. $ra(\text{Set } E) \subseteq ra(\text{Set } PP)$, and that the range of applicability of the Set CL technique is a subset of the range of applicability of the Set CG technique, i.e. $ra(\text{Set } CL) \subseteq ra(\text{Set } CG)$.

- Chapter 6. We will show that the range of applicability of the Set PP technique is a subset of the range of applicability of the Set CG technique, i.e. $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$. This proof will present a methodology for transforming an application of the Set PP technique into an application of the Set CG technique, such that the constructed application of the Set CG technique makes intelligible the relation of consequence which was made intelligible by the original application of the Set PP technique.
- Chapter 7. We will use the methodology presented in the proof of Chapter 6, to show that the range of applicability of the Set PP technique is a subset of the range of applicability of the Set CL technique, i.e. $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$. This proof will consist in showing that the applications of the Set CG technique constructed by the methodology presented in Chapter 6, not only satisfy the Set CG technique-specific assumptions, but satisfy assumption LL (the representational version of Lindenbaum's Lemma) as well.
- Chapter 8. We will show that the range of applicability of the Set CG technique is a subset of the range of applicability of the Set BE technique, i.e. $ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$. This proof will present a methodology for transforming an application of the Set CG technique into an application of the Set BE technique, such that the constructed application of the Set BE technique makes intelligible the relation of consequence which was made intelligible by the original application of the Set CG technique.

In virtue of the results described above (and presented below!), we will know:
 $ra(\text{Set } E) \subseteq ra(\text{Set } PP) \subseteq ra(\text{Set } CL) \subseteq ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$.

In this Part, we will also present the following result:

- Chapter 9. We give a canonical method for constructing a minimal equivalent for any application of the order-consistency technique *CG*. The result is minimal with respect to the cardinality of the set of assertion types used. We will make use of the concepts introduced in Chapter 8.

Chapter 6

Demonstrating that the Range of the Set PP Technique is a Subset of the Range of the Set CG Technique

6.1 Introduction

In this chapter, we will demonstrate that any interpreted set of assertion types in the range of applicability of the Set PP technique is in the range of applicability of the Set CG technique. Intuitively, this result means that any interpreted set of assertion types for which we can make the representational concept of logical consequence explanatorily intelligible by an application of the Set PP technique, is one for which we can do the same with the Set CG technique. This result is one element of the more global picture we have developed organizing all the techniques under consideration into a linear order on the basis of their relative ranges of applicability.

Here is the structure of the chapter. In the initial section, we will examine in detail just what the claim to be proven means, and what constitutes the desired result. We will then describe the structure of a proof realizing that result. The central concept in the proof will

be a methodology for taking the elements of an application of the Set *PP* technique and constructing from them an application of the Set *CG* technique, such that the constructed application of the Set *CG* technique makes intelligible the relation of consequence which was made intelligible by the given application of the Set *PP* technique. This methodology will take the form of a collection of operators on interpreted truth media. These operators will serve to “move” the relation of consequence from its initial context in the application of the Set *PP* technique, through a sequence of intermediate contexts, into the context of an application of the Set *CG* technique. This movement from context to context is similar in spirit to the way in which Barwise and Seligman move relations of consequence from one local logic to another (Barwise and Seligman 1997). The structural overview will be followed an organized presentation of the elements of the proof.

We will conclude the chapter by applying the methodology of the chapter to the task of constructing an application of the Set *CG* technique for the $\mathcal{P}(WFF)$ -language of propositional logic. We have already constructed an application of the Set *E* technique (and hence an application of the Set *PP* technique) for this language, so it will serve as an instructive example to run that application through the methodology of the chapter in order to create the desired application of the Set *CG* technique.

6.2 Definition of the desired result

6.2.1 Introducing a condition sufficient to show the result

Let us consider what is required to demonstrate the claim expressed in the chapter title.

We can show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$ if we can show that:

(PP → CG) Given any application of the Set *PP* technique, we can construct an application of the Set *CG* technique such that the representational relation of logical consequence which was made intelligible by the given application of the Set *PP* technique is embedded within the representational relation of logical consequence made intelligible by the constructed application of the Set *CG* technique.

Let's unpack the elements of the condition $(PP \rightarrow CG)$.

- An application of the Set PP technique consists in a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set PP technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Such an application makes the representational relation of logical consequence $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ intelligible (via the proxy relation MC_A).
- An application of the Set CG technique consists in a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting the assertion types of D , such that D satisfies the Set CG technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$. Such an application makes the representational relation of logical consequence $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ intelligible (via the proxy relation LC_D).
- $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ is embedded within $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ just in case there is a function $k : \Sigma_A \rightarrow \Sigma_D$ such that $\forall p, q \in \Sigma_A, \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \right)$ iff $\left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(k(p), k(q)) \right)$.

6.2.2 Showing that the condition is sufficient

Now we can see that $(PP \rightarrow CG)$ is sufficient to show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$. We will be given an interpreted set of assertion types $\langle \Sigma, \mathcal{T}_{\Sigma} \rangle$ in the range of applicability of the Set PP model-theoretic technique. We want to show that $\langle \Sigma, \mathcal{T}_{\Sigma} \rangle$ is in the range of applicability of the Set CG order-consistency technique.

1. Applying the general definition of “range of applicability” from Section 1.7.2 to the Set PP technique tells us that:

$\langle \Sigma, \mathcal{T}_{\Sigma} \rangle$ is in the **range of applicability of the Set PP technique** exactly when there is a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set PP technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and the representational relation of logical consequence for Σ as interpreted by $\mathcal{T}_{\Sigma}$ (i.e. $ALC_{\langle \Sigma, \mathcal{T}_{\Sigma} \rangle}$) is embedded (via some function j) in the representational relation of logical consequence for the assertion types of medium A (i.e. Σ_A) as interpreted by $\mathcal{T}_{\Sigma_A}$ (i.e. $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$).

2. Applying the same definition to the Set *CG* technique, we find that $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ is in the **range of applicability of the Set *CG* technique** exactly when there is a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting the assertion types of D , such that D satisfies the Set *CG* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$, and the representational relation of logical consequence for Σ as interpreted by $\mathcal{T}_\Sigma$ (i.e. $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$) is embedded (via some function i) in the representational relation of logical consequence for the assertion types of medium D (i.e. Σ_D) as interpreted by $\mathcal{T}_{\Sigma_D}$ (i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$).
3. So if an interpreted set of assertion types $\langle \Sigma, \mathcal{T}_\Sigma \rangle \in ra(\text{Set } PP)$, then there is an application of the Set *PP* technique (a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set *PP* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$) such that $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$ is embedded (via some function j) in $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.
4. By $(PP \rightarrow CG)$, we know that there is application of the Set *CG* technique (a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting the assertion types of D , such that D satisfies the Set *CG* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$) such that $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ is embedded (via some function k) within $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$.
5. So $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$ is embedded (via the composition $k \circ j$) within $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$.
6. So $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ is in the range of applicability of the Set *CG* technique.

6.2.3 Summary

By the argument above, we see that we can demonstrate the primary claim for the chapter (that $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$), if we can show that the condition $(PP \rightarrow CG)$ is true. To demonstrate $(PP \rightarrow CG)$, we shall describe a methodology which, given any application of the Set *PP* technique (a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set *PP* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$), constructs from it an application of the Set *CG* technique (a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting

the assertion types of D , such that D satisfies the Set *CG* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$, and a function h embedding $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ in $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$.

6.3 Structure of the proof

6.3.1 First, an extension of an existing concept

Operators on truth media and their associated practices

In order to discuss the proof, we need to extend the existing concept of operators on interpreted truth media. As we have presented them so far, operators take one or more interpreted truth media as arguments, and construct an interpreted truth medium as a result. We extend the concept of these operators so that while some operators may construct an interpreted truth medium, others may alternatively construct an interpreted consistency medium. We give the new definitions below.

A unary operator uop on interpreted truth media, is one which takes a arbitrary truth medium A and the practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ by which it is interpreted, and from them, constructs a new medium $uop(A)$, and the practice(s) by which $uop(A)$ is interpreted. Some unary operators will construct a truth medium and its associated practices. Other unary operators will construct a consistency medium and its associated practice. If the new medium $uop(A)$ is a truth medium, then uop will construct the practices $\mathcal{T}_{\Sigma_{uop(A)}}$ and $\mathcal{T}_{M_{uop(A)}}$ by which $uop(A)$ is interpreted. If the new medium $uop(A)$ is a consistency medium, then uop will construct the practice $\mathcal{T}_{\Sigma_{uop(A)}}$ by which $uop(A)$ is interpreted.

A binary operator bop on interpreted truth media, is one which takes an arbitrary truth medium A and the practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ by which it is interpreted, and an arbitrary truth medium B and the practices $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$ by which it is interpreted, and from them, constructs a new medium $bop(A, B)$, and the practice(s) by which $bop(A, B)$ is interpreted. As with unary operators, some binary operators will construct a truth medium and its associated practices, while other binary operators will construct a consistency medium and its associated practice. (NB: in this dissertation, we will not have need for binary operators which construct consistency media.) If the new medium $bop(A, B)$ is a truth medium, then bop will construct the practices $\mathcal{T}_{\Sigma_{bop(A, B)}}$ and $\mathcal{T}_{M_{bop(A, B)}}$ by which $bop(A, B)$ is interpreted.

If the new medium $bop(A, B)$ is a consistency medium, then bop will construct the practice $\mathcal{T}_{\Sigma_{bop(A, B)}}$ by which $bop(A, B)$ is interpreted.

We see that the results of applying these operators are interpreted intelligibility media themselves.

6.3.2 Top level structure of the proof

We carry out the proof in three stages.

1. In the first stage, we show how to carry out the construction for those applications of the Set *PP* technique where the truth medium used in the application satisfies monotonic extension of commitment with respect to models, and is weakly extensible with respect to models.
2. In the second stage, we show how to carry out the construction for those applications of the Set *PP* technique where the truth medium used in the application is negation complete with respect to models. This construction utilizes the results of the first stage.
3. In the third stage, we show how to carry out the construction for arbitrary applications of the Set *PP* technique. This construction utilizes the results of the first two stages.

6.3.3 Stage 1: The construction in the case where the truth medium used in the application satisfies monotonic extension of commitment with respect to models and is weakly extensible with respect to models

Reviewing the given

We are given an application of the Set *PP* technique. This consists of a truth medium $A = \langle \Sigma_A, M_A, t_A \rangle$ and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ for which it is assumed that A satisfies assumption set *PP* (*B1*, *A4*, *A5*) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. We further assume that A is partially ordered, and that A satisfies monotonic extension of commitment with respect to models and that A is weakly extensible with respect to models.

Looking ahead to the end result

Our goal is to define an operator dm which given a partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set PP ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ and A satisfies monotonic extension of commitment with respect to models and A is weakly extensible with respect to models, constructs a consistency medium $dm(A)$ and a practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$ interpreting $dm(A)$, such that

1. an image of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ is embedded within $ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}$,
that is, there is a function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ such that

$$\forall p, q \in \Sigma_A$$

$$\left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}(h(p), h(q)) \right); \text{ and}$$
2. consistency medium $dm(A)$ satisfies assumption Set $CG(D1, D2, D3)$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$.

Preparing the way

We have found such an operator dm . The key to understanding our construction of dm is to think about a property of assertion types, namely our assumed concept of consistency. Consistency of assertion types is implicitly characterized within each application of the Set PP technique. We can describe this implicit characterization simply. Given that a truth medium A satisfies the assumptions in Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, for every assertion type $p \in \Sigma_A$, p is consistent (that is, according to our assumed concept of consistency, $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$) just in case the set of models of p is non-empty (that is, $t_A(p) \neq \emptyset$).

Proposition 74 *Given that a truth medium A satisfies the assumptions in Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, for every assertion type $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$ iff $t_A(p) \neq \emptyset$.*

Let p be an arbitrary element of Σ_A .

($\rightarrow$)

1. Assume $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$. By *B1*, $\bigcup \mathcal{T}_{M_A}[t_A(p)] \neq \emptyset$. So there exists an $m \in t_A(p)$, such that $\mathcal{T}_{M_A}(m) \neq \emptyset$. ■ ($\rightarrow$)

($\leftarrow$)

1. Assume $t_A(p) \neq \emptyset$. So there exists an $m \in M_A$ such that $m \in t_A(p)$.
2. Since A satisfies assumption *A5* with respect to $\mathcal{T}_{M_A}$, we know $\mathcal{T}_{M_A}(m) \neq \emptyset$.
3. Let $u \in \mathcal{T}_{M_A}(m)$. Since $m \in t_A(p)$, $u \in \bigcup \mathcal{T}_{M_A}[t_A(p)]$.
4. By *B1*, $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$. ■ ($\leftarrow$) ■ Proposition.

In line with how we have thought about consequence, we could say the following. In addition to making the assumed relation of logical consequence intelligible, an application of the Set *PP* technique makes the assumed property of consistency intelligible as well. With respect to some truth medium A and assertion type $p \in \Sigma_A$, the Set *PP* technique provides the intelligible property $t_A(p) \neq \emptyset$ as a proxy for the unintelligible, but assumed, property of consistency $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$.

The discussion above shows how the intelligible proxy for consistency is defined in terms of a relationship between two forms of representation: the logically simple (Etchemendy would say “logically transparent”) models; and the (usually) logically complex assertion types. The relation of truth-in-a-model (captured in our formalization of the model-theoretic techniques by the function t) links the two forms of representation and is the basis for the intelligible proxy for consistency.

We have set ourselves a task. Given a relation of consequence which is made intelligible by the Set *PP* technique, we wish to construct an application of the Set *CG* technique that makes that relation intelligible. The key step in that task will be to transfer the property of consistency from the truth medium given to the consistency medium we construct. We face an immediate challenge. While the Set *PP* technique has two forms of representation at its disposal (the assertion types and the models in any truth medium), the Set *CG* technique has only one (the assertion types in any consistency medium). This way of framing the problem suggests a route to a possible solution. One way to enable the transfer of the

property of consistency from the context of the Set *PP* technique to the context of the Set *CG* technique is to bring the two forms of representation in the given truth medium together into a single form for use as the set of assertion types in the consistency medium being constructed; and then to specify consistency in terms of that new form.

An obvious way to start carrying this out would be to form the Cartesian product of the set of assertion types and the set of models from the given truth medium. In our case, given the truth medium A , this product would be the set of ordered pairs $\Sigma_A \times M_A$. Each member of this set would have the form $\langle p, m \rangle$ where p is an assertion type from the truth medium A and m is a model from A . Take this set $\Sigma_A \times M_A$ as our provisional definition of $\Sigma_{dm(A)}$. (We will see we have need to adjust it slightly in a moment). So each pair $\langle p, m \rangle \in \Sigma_A \times M_A$ would be an assertion type of $\Sigma_{dm(A)}$. As assertion types, they would need some practice of categorization of possibility by which they would be interpreted. To construct that practice, we have available to us the practices by which Σ_A and M_A were interpreted in the context of the truth medium A , namely, $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Let us choose to interpret the elements of our provisional $\Sigma_{dm(A)}$ as follows: for all $\langle p, m \rangle \in \Sigma_{dm(A)}$, $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$. That is, we consider each pair $\langle p, m \rangle$ to be true just in case both of its individual components would have been true in their original interpretations.

The definition we have proposed for $\Sigma_{dm(A)}$ does bring together assertion types and models into a single form of representation. But it has a potential limitation. Looking ahead, we are going to want to map the elements of Σ_A into elements of $\Sigma_{dm(A)}$ in such a way as to be able to embed the assumed relationship of consequence $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ within the assumed relationship of consequence $ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}$. A natural way to make this happen will be to have a function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$. In that case, we would have

$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}(h(p), h(q)) \right)$ by application of the definition of ALC .

This is where the limitation of our development of $\Sigma_{dm(A)}$ appears. As specified so far, it is not guaranteed that given some $p \in \Sigma_A$, we can identify an $m \in M_A$ such that $\langle p, m \rangle$ when interpreted as an element of $\Sigma_{dm(A)}$ via $\mathcal{T}_{\Sigma_{dm(A)}}$, will have the property that $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle)$. Given our construction so far, it is possible that for every $m \in$

M_A , the combination $\langle p, m \rangle$ carries strictly more information than p alone, that is, that $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle)$ (which we have defined as $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$) is a proper subset of $\mathcal{T}_{\Sigma_A}(p)$. This condition would block our ability to define a function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$.

We can remove this limitation. The technique we will employ will be to supplement the set M_A with a special element (call it $\perp$) not already in M_A , such that we will take $\Sigma_{dm(A)}$ to be $\Sigma_A \times (M_A \cup \{\perp\})$ and redefine $\mathcal{T}_{\Sigma_{dm(A)}}$ in such a way that $\forall p \in \Sigma_A$, $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) =$ if $m = \perp$ then $\mathcal{T}_{\Sigma_A}(p)$ else $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$. Then $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ is defined so that for all $p \in \Sigma_A$, $h(p) = \langle p, \perp \rangle$ ensures that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$. The symbol $\perp$ can be understood as representing “no information with respect to models.”

Given our preparations so far, we can now describe how we will identify those assertion types in $\Sigma_{dm(A)}$ which are consistent, that is, are members of the set $\{\langle p, m \rangle \in \Sigma_{dm(A)} \mid \mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset\}$. Note that this set will be $C_{dm(A)}$ in our construction. The assertion types in $\Sigma_{dm(A)}$ can be divided into two disjoint subsets; those for which the second component is $\perp$ and those for which it isn't (in which case the second component is a member of M_A).

First, let us consider the case where an element of $\Sigma_{dm(A)}$ has the form $\langle p, \perp \rangle$. In this case, we know that $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A}(p)$. So $\langle p, \perp \rangle$ will be consistent with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$ just in case p is consistent with respect to $\mathcal{T}_{\Sigma_A}$. And we know that in the given truth medium A interpreted according to the practice $\mathcal{T}_{\Sigma_A}$ (with respect to which it satisfies assumption Set PP), an element p of Σ_A is consistent ($\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$) just in case p has a model (i.e. $t_A(p) \neq \emptyset$). Further, we know that every combination of p and a model m of M_A exists in $\Sigma_{dm(A)}$. So we can define the consistency condition for elements of the form $\langle p, \perp \rangle$ as follows. For all $\langle p, m \rangle \in \Sigma_{dm(A)}$, if $m = \perp$, then $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$ just in case there exists a $\langle p, m' \rangle \in \Sigma_{dm(A)}$ such that $m' \in t_A(p)$.

Now we can consider the case where an element of $\Sigma_{dm(A)}$ has the form $\langle p, m \rangle$ such that $m \neq \perp$. In this case, $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$. So $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$ just in case $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$. We show below that given the assumption that A satisfies Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know that $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$ iff $m \in t_A(p)$. So we can define the consistency condition for elements of the form $\langle p, m \rangle$ such that $m \neq \perp$ as

follows. For all $\langle p, m \rangle \in \Sigma_{dm(A)}$, if $m \neq \perp$, then $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$ just in case $m \in t_A(p)$.

Proposition 75 *For arbitrary truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for all $p \in \Sigma_A$ and $m \in M_A$, $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$ iff $m \in t_A(p)$.*

The proof can be found at the end of the chapter.

The structure of the plan

We have now laid the foundation for our construction of $dm(A)$ and the proofs of its adequacy. We will take $\Sigma_{dm(A)}$ to be $\Sigma_A \times (M_A \cup \{\perp\})$ ¹, and define $\mathcal{T}_{\Sigma_{dm(A)}}$ such that for all $\langle p, m \rangle \in \Sigma_{dm(A)}$, $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) =$ if $m = \perp$ then $\mathcal{T}_{\Sigma_A}(p)$ else $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$. Those definitions will permit the definition of an ordering $\preceq_{dm(A)}$ such that $\langle \Sigma_{dm(A)}, \preceq_{dm(A)} \rangle$ will satisfy monotonic extension of commitment and weak extensibility with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$ (thus $dm(A)$ will satisfy *D1* and *D3* with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$). We will define $C_{dm(A)}$ as described in the previous section, which will assure that $dm(A)$ satisfies *D2* with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$. Our construction of $\Sigma_{dm(A)}$ and $\mathcal{T}_{\Sigma_{dm(A)}}$ ensure that there is a function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ such that $\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}(h(p), h(q)) \right)$.

We could try and define the operator dm and give the proofs of its adequacy in one go. However, that would be more complex than we would like. Instead, we will factor the construction, and the proofs, using a number of operators on interpreted truth media. We will use the following three operators:

1. *models*; a unary operator on interpreted truth media which yields an interpreted partially ordered truth medium as a result (definition in Section 6.6.1);
2. $+$; a binary operator on interpreted partially ordered truth media which yields a interpreted partially ordered truth medium as a result (definition in Section 6.7.1);

¹In actuality, we will take $\Sigma_{dm(A)}$ to be $\Sigma_A \times (\bigcup t_A[\Sigma_A] \cup \{\perp\})$. That is, instead of using the set M_A of all models, we use the set $\bigcup t_A[\Sigma_A]$ of those models which are models of some assertion type. We exclude those members of M_A which are not models of any assertion type. As shown in the proofs to follow, this does not affect the sufficiency of our definition of $C_{dm(A)}$.

3. dm ; a unary operator on interpreted partially ordered truth media which yields an interpreted consistency medium as a result (definition in Section 6.5.1).

The detailed definitions of these operators and proofs of their properties are given later in this chapter. The overall plan of construction and proof can be described as follows.

1. The construction begins with the partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set PP ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ and A satisfies monotonic extension of commitment with respect to models and A is weakly extensible with respect to models.
2. Given the preceding, and the definitions of $models$, and $+$ as found in Sections 6.6.1, and 6.7.1, respectively, the partially ordered truth medium $A + models(A)$ and associated practices of categorization $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$ have the following properties:

- (a) $\Sigma_{A+models(A)} = \Sigma_A \times (\bigcup t_A[\Sigma_A] \cup \{\perp\})$.
- (b) $A + models(A)$ satisfies assumption set PP with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$ (Proposition 90).
- (c) $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$. (Proposition 91).
- (d) $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$ (Proposition 92).
- (e) The function $h : \Sigma_A \rightarrow \Sigma_{A+models(A)}$, defined for $p \in \Sigma_A$ as $h(p) = \langle p, \perp \rangle$, is such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{A+models(A)}}(h(p))$ (Proposition 93). Applying the definition of ALC , this implies: for all $p, q \in \Sigma_A$, $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q)$ iff $ALC_{\langle \Sigma_{A+models(A)}, \mathcal{T}_{\Sigma_{A+models(A)}} \rangle}(h(p), h(q))$.

3. Given the preceding, the definition of dm as found in Section 6.5.1 implies that the consistency medium $dm(A)$ and associated practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$ are constructed as follows:

- (a) $\Sigma_{dm(A)}$ is taken to be $\Sigma_{A+models(A)}$.
- (b) $\preceq_{dm(A)}$ is taken to be $\preceq_{A+models(A)}$.
- (c) $\mathcal{T}_{\Sigma_{dm(A)}}$ is taken to be $\mathcal{T}_{\Sigma_{A+models(A)}}$.
- (d) $C_{dm(A)}$ is defined as described earlier in this section.

4. Given the preceding, and the definition of dm as found in Section 6.5.1, the consistency medium $dm(A)$ and associated practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$ have the following properties:

- (a) $\langle \Sigma_{dm(A)}, \preceq_{dm(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$. This is equivalent to saying that $dm(A)$ satisfies $D1$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$ (Proposition 80).
- (b) $dm(A)$ satisfies $D2$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$ (Proposition 81).
- (c) $\langle \Sigma_{dm(A)}, \preceq_{dm(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$.
This is equivalent to saying that $dm(A)$ satisfies $D3$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$ (Proposition 82).
- (d) The function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ defined for $p \in \Sigma_A$ as $h(p) = \langle p, \perp \rangle$, is such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$ (Proposition 83).

Applying the definition of ALC , this implies:

$$\text{for all } p, q \in \Sigma_A, ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}(h(p), h(q)).$$

Proving the above claims will demonstrate that dm as defined, takes an application of the Set PP technique (for which the truth medium in that application is assumed to satisfy monotonic extension of commitment with respect to models and to be weakly extensible with respect to models) and constructs an application of the Set CG technique such that the relation of logical consequence made intelligible by the original application of the Set PP technique is made intelligible by the constructed application of the Set CG technique.

In effect, what we see in the above is the “movement” of the assumed relation of consequence $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ from the context of the initial truth medium A and its practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, through a sequence of intermediate contexts, into the context of the consistency medium $dm(A)$ and its practice $\mathcal{T}_{\Sigma_{dm(A)}}$.

6.3.4 Stage 2: The construction in the case where the truth medium used in the application is negation complete with respect to models

Reviewing the given

We are given an application of the Set *PP* technique. This consists of a truth medium $A = \langle \Sigma_A, M_A, t_A \rangle$ and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ for which it is assumed that A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and that A is negation complete with respect to models.

Reviewing the goal

Our goal is to construct a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , such that

1. an image of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ is embedded within $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
that is, there is a function $h : \Sigma_A \rightarrow \Sigma_D$ such that
 $\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right)$; and
2. consistency medium D satisfies assumption Set *CG* ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$.

The approach

Given the approach we have worked out above, we can now consider the case where the truth medium used in the application of the Set *PP* technique is known to be negation complete with respect to models (but may not be known to satisfy monotonic extension of commitment with respect to models, nor be known to be weakly extensible with respect to models). Our approach uses the unary operator *pow* on interpreted truth media which yields an interpreted partially ordered truth medium as a result. We introduced this operator in Section 4.8.4 in the context of applying the Set *E* technique to the $\mathcal{P}(WFF)$ -language of propositional logic.

The overall plan of construction can be described as follows.

1. The construction begins with the truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$

and $\mathcal{T}_{M_A}$ and A is negation complete with respect to models.

2. Given the preceding, and the definition of pow as found in Section 4.8.4, the partially ordered truth medium $pow(A)$ and associated practices of categorization $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$ have the following properties:
 - (a) $pow(A)$ is a partially ordered truth medium satisfying assumption set PP with respect to $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$ (Proposition 94).
 - (b) $pow(A)$ satisfies monotonic extension of commitment with respect to models (Proposition 64).
 - (c) $pow(A)$ is weakly extensible with respect to models (Proposition 65).
 - (d) The function $f : \Sigma_A \rightarrow \Sigma_{pow(A)}$ defined for $p \in \Sigma_A$ as $f(p) = \{p\}$, is such that $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{pow(A)}}(f(p))$ (Proposition 95). Therefore, we know that $\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{pow(A)}, \mathcal{T}_{\Sigma_{pow(A)}} \rangle}(f(p), f(q)) \right)$ by application of the definition of ALC .
3. Given the above, $pow(A)$ and its practices $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$ can then be submitted to the procedure outlined in Section 6.3.3 (Stage 1). The result will be a consistency medium D , practice of categorization $\mathcal{T}_{\Sigma_D}$, and function $g : \Sigma_{pow(A)} \rightarrow \Sigma_D$ such that $\forall p, q \in \Sigma_{pow(A)} \left(ALC_{\langle \Sigma_{pow(A)}, \mathcal{T}_{\Sigma_{pow(A)}} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(g(p), g(q)) \right)$; and D satisfies $D1$, $D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.
4. We can form $h = g \circ f$, and we have $h : \Sigma_A \rightarrow \Sigma_D$ such that $\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right)$.

6.3.5 Stage 3: The construction in the case where an arbitrary truth medium is used in the application

Reviewing the given

We are given an application of the Set PP technique. This consists of a truth medium $A = \langle \Sigma_A, M_A, t_A \rangle$ and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ for which it is assumed that A satisfies assumption set $PP(B1, A4, A5)$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Reviewing the goal

Our goal is to construct a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , such that

1. an image of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ is embedded within $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
that is, there is a function $h : \Sigma_A \rightarrow \Sigma_D$ such that
 $\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right)$; and
2. consistency medium D satisfies assumption Set CG ($D1$, $D2$, $D3$) with respect to $\mathcal{T}_{\Sigma_D}$.

The approach

Given the approaches we have worked out above, we can now consider the case where the truth medium used in the application of the Set PP technique is not known to be negation complete with respect to models. Our approach uses a unary operator *negcomp* on interpreted truth media which yields an interpreted truth medium.

1. The construction begins with the truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies assumption set PP ($B1$, $A4$, $A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. Given the definition of the operator *negcomp* (see Section 6.10.1), the truth medium $negcomp(A)$ and associated practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$ have the following properties:
 - (a) $negcomp(A)$ is negation complete with respect to models (Proposition 96).
 - (b) $negcomp(A)$ satisfies assumption set PP with respect to $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$ (Proposition 97).
 - (c) The function $f : \Sigma_A \rightarrow \Sigma_{negcomp(A)}$, defined for $p \in \Sigma_A$ as $f(p) = \langle 1, p \rangle$, is such that
 $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{negcomp(A)}}(f(p))$. (Proposition 98). Therefore, applying the

definition of ALC , we have

$$\forall p, q \in \Sigma_A, ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{negcomp(A)}, \mathcal{T}_{\Sigma_{negcomp(A)}} \rangle}(f(p), f(q))$$

3. Given the above, $negcomp(A)$ and its practices $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$ can then be submitted to the procedure outlined in Section 6.3.4 (Stage 2). The result will be a consistency medium D , practice of categorization $\mathcal{T}_{\Sigma_D}$, and function

$g : \Sigma_{negcomp(A)} \rightarrow \Sigma_D$ such that

$$\forall p, q \in \Sigma_{negcomp(A)} \left(\begin{array}{c} ALC_{\langle \Sigma_{negcomp(A)}, \mathcal{T}_{\Sigma_{negcomp(A)}} \rangle}(p, q) \text{ iff} \\ ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(g(p), g(q)) \end{array} \right);$$

and D satisfies $D1$, $D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.

4. We can form $h = g \circ f$, and we have $h : \Sigma_A \rightarrow \Sigma_D$ such that

$$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right).$$

6.3.6 Presentation of the proof

Our presentation of the proofs needed to fill out the structure above, appear in the following order.

1. The top level proofs that yield the theorems for this chapter (Section 6.4);
2. The definition of the operator dm and the proofs of its properties (Section 6.5);
3. The definition of the operator $models$ and the proofs of its properties (Section 6.6);
4. The definition of the operator $+$ and the proofs of its properties (Section 6.7);
5. Properties of $A + models(A)$ with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$ given that A satisfies assumption set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ and A satisfies monotonic extension of commitment with respect to models and is weakly extensible with respect to models (Section 6.8);
6. Proofs of additional properties of the operator pow (Section 6.9), note that the definition of the operator pow and proofs of certain properties have already been presented in Sections 4.8.4 and 4.8.8);
7. The definition of the operator $negcomp$ and the proofs of its properties (Section 6.10);

6.4 The canonical construction of an application of the Set *CG* technique given an application of the Set *PP* technique

6.4.1 The canonical construction of an application of the Set *CG* technique given an application of the Set *PP* technique constructed using a partially ordered truth medium which satisfies monotonic extension of commitment with respect to models and is weakly extensible with respect to models (Stage 1)

Theorem 76 *Given a partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models and A is weakly extensible with respect to models, and A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

then there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from $A, \mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and there is a function h from Σ_A to Σ_D such that

$$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right)$$

We will show that the consistency medium $dm(A)$, its associated practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$, and the function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$ defined for $p \in \Sigma_A$ as $h(p) = \langle p, \perp \rangle$ satisfy the above claim.

1. Assume we have an arbitrary partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ such that A satisfies monotonic extension of commitment with respect to models and A is weakly extensible with respect to models, and A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. Applying Proposition 79, we know that the consistency medium $dm(A)$ satisfies assumptions $D1, D2, D3$ with respect to derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.
3. By Proposition 83, we know the function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$, defined for $p \in \Sigma_A$ as $h(p) = \langle p, \perp \rangle$, is such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$. Apply-

ing the definition of ALC , this implies: for all $p, q \in \Sigma_A$, $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q)$ iff $ALC_{\langle \Sigma_{dm(A)}, \mathcal{T}_{\Sigma_{dm(A)}} \rangle}(h(p), h(q))$. ■ Theorem.

6.4.2 The canonical construction of an application of the Set CG technique given an application of the Set PP technique constructed using a truth medium which is negation complete with respect to models (Stage 2)

Theorem 77 *Given a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set PP ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and A is negation complete with respect to models, then there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from A , $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and there is a function h from Σ_A to Σ_D such that*

$$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q)) \right).$$

1. Let A be a truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization such that A satisfies assumption set PP ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and A is negation complete with respect to models.
2. Apply the operation pow to A , $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ forming the truth medium $pow(A)$ and the practices of categorization $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$.
3. Applying Proposition 94, we know that $pow(A)$ is a partially ordered truth medium satisfying assumption set PP ($B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$.
4. By Proposition 64, we know that $pow(A)$ satisfies monotonic extension of commitment with respect to models.
5. By Proposition 65, we know that $pow(A)$ is weakly extensible with respect to models.
6. If we apply Theorem 76 to $pow(A)$, $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$ then there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from $pow(A)$, $\mathcal{T}_{\Sigma_{pow(A)}}$ and $\mathcal{T}_{M_{pow(A)}}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and

there is a function g from $\Sigma_{pow(A)}$ to Σ_D such that

$$\forall p, q \in \Sigma_{pow(A)} \left(ALC_{\langle \Sigma_{pow(A)}, \mathcal{T}_{\Sigma_{pow(A)}} \rangle} (p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} (g(p), g(q)) \right)$$

7. Consider the function $f : \Sigma_A \rightarrow \Sigma_{pow(A)}$ defined so that for all $p \in \Sigma_A$, $f(p) = \{p\}$.

8. By Proposition 95, we have

$$\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{pow(A)}}(f(p)). \text{ Applying the definition of } ALC, \text{ this implies}$$

$$\forall p, q \in \Sigma_A, ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle} (p, q) \text{ iff } ALC_{\langle \Sigma_{pow(A)}, \mathcal{T}_{\Sigma_{pow(A)}} \rangle} (f(p), f(q)).$$

9. Let $h = g \circ f$.

10. Then h is a function from Σ_A to Σ_D such that

$$\forall p, q \in \Sigma_A, ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle} (p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} (h(p), h(q)) \blacksquare \text{Theorem.}$$

6.4.3 The canonical construction of an application of the Set *CG* technique given an application of the Set *PP* technique constructed using an arbitrary truth medium (Stage 3)

Theorem 78 *Given a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

then there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from $A, \mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and there is a function h from Σ_A to Σ_D such that

$$\forall p, q \in \Sigma_A \left(ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle} (p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} (h(p), h(q)) \right)$$

1. Let A be a truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization such that A satisfies assumption set *PP* ($B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

2. Apply the operation *negcomp* to $A, \mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ forming the truth medium *negcomp*(A) and the practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$.

3. Applying Proposition 97, we know that *negcomp*(A) satisfies assumption set *PP* ($B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$.

4. By Proposition 96, we know that $negcomp(A)$ is negation complete with respect to models.
5. If we apply Theorem 77 to $negcomp(A)$, $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$, then there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from $negcomp(A)$, $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and there is a function g from $\Sigma_{negcomp(A)}$ to Σ_D such that
$$\forall p, q \in \Sigma_{negcomp(A)} \left(ALC_{\langle \Sigma_{negcomp(A)}, \mathcal{T}_{\Sigma_{negcomp(A)}} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(g(p), g(q)) \right)$$
6. Consider the function $f : \Sigma_A \rightarrow \Sigma_{negcomp(A)}$ defined so that for all $p \in \Sigma_A$, $f(p) = \langle 1, p \rangle$.
7. By Proposition 98, we have $\forall p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{negcomp(A)}}(f(p))$.
Applying the definition of ALC , this implies
$$\forall p, q \in \Sigma_A, ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{negcomp(A)}, \mathcal{T}_{\Sigma_{negcomp(A)}} \rangle}(f(p), f(q)).$$
8. Let $h = g \circ f$.
9. Then $\forall p, q \in \Sigma_A$, $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(h(p), h(q))$ ■ Theorem.

6.5 The operator dm

6.5.1 dm defined

Given an arbitrary partially ordered truth medium A used with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,

the operator $dm()$ constructs a consistency medium $dm(A)$, and derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$ defined as follows:

Consistency medium $dm(A)$:

- $\Sigma_{dm(A)} = \Sigma_{A+models(A)}$
- $\preceq_{dm(A)} = \preceq_{A+models(A)}$

$$\bullet C_{dm(A)} = \left\{ \begin{array}{l} \langle p, m \rangle \in \Sigma_{dm(A)} \mid \\ \left(\begin{array}{l} m = \perp \text{ and there exists a } \langle p, m' \rangle \in \Sigma_{dm(A)} \\ \text{such that } m' \in t_A(p) \end{array} \right) \text{ or } \\ (m \neq \perp \text{ and } m \in t_A(p)) \end{array} \right\}$$

Derived practice $\mathcal{T}_{\Sigma_{dm(A)}}$:

- $\mathcal{T}_{\Sigma_{dm(A)}} : \Sigma_{dm(A)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:

$$\mathcal{T}_{\Sigma_{dm(A)}} = \mathcal{T}_{\Sigma_{A+models(A)}}$$

6.5.2 The dm Set CG assumptions proposition

Proposition 79 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,

then the consistency medium $dm(A)$ satisfies assumptions $D1, D2$, and $D3$ with respect to derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.

The proof is presented in three sections below, one for each of the assumptions $D1, D2, D3$ (Propositions 80, 81, 82 respectively).

6.5.3 Monotonic extension of commitment ($D1$) portion of the dm Set CG assumptions proposition

Proposition 80 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then the consistency medium $dm(A)$ satisfies $D1$ with respect to derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.

1. Let A be an arbitrary partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. From Proposition 91 we know that $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$.
3. Since $\Sigma_{A+models(A)} = \Sigma_{dm(A)}$, $\preceq_{A+models(A)} = \preceq_{dm(A)}$, and $\mathcal{T}_{\Sigma_{A+models(A)}} = \mathcal{T}_{\Sigma_{dm(A)}}$ by the definition of dm , we know that $\langle \Sigma_{dm(A)}, \preceq_{dm(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$. This is equivalent to saying that $dm(A)$ satisfies $D1$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$. ■ Proposition.

6.5.4 Consistency ($D2$) portion of the dm Set CG assumptions proposition

Proposition 81 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then the consistency medium $dm(A)$ satisfies $D2$ (consistency) with respect to the derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.

Let A be a partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

1. $\left[\text{Show } \forall \langle p, m \rangle \in \Sigma_{dm(A)} \left(\langle p, m \rangle \in C_{dm(A)} \text{ iff } \mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset \right) \right]$
2. By Proposition 80, we know that the consistency medium $dm(A)$ satisfies assumption $D1$ (monotonic extension of commitment) with respect to derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.
3. Let $\langle p, m \rangle$ be an arbitrary element of $\Sigma_{dm(A)}$.

4. Then $p \in \Sigma_A$, and $m \in \bigcup t_A[\Sigma_A] \cup \{\perp\}$, by the definition of dm .

5. We split into cases on the value of m .

6. Case 1: ($m = \perp$)

(a) [Show $\langle p, \perp \rangle \in C_{dm(A)}$ iff $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) \neq \emptyset$]

(b) ($\rightarrow$)

i. Assume $\langle p, \perp \rangle \in C_{dm(A)}$.

ii. Then there exists a $\langle p, m' \rangle \in \Sigma_{dm(A)}$ such that $m' \in t_A(p)$.

iii. $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m' \rangle) = \mathcal{T}_{\Sigma_A + models(A)}(\langle p, m' \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_{models(A)}}(m')$.

iv. Since $m' \in t_A(p)$, we know $m' \neq \perp$.

v. So $\mathcal{T}_{\Sigma_{models(A)}}(m') = \mathcal{T}_{M_A}(m')$.

vi. So $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m' \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m')$.

vii. By Proposition 75, we know that $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m' \rangle) \neq \emptyset$.

viii. By the definition of dm , we know that $\langle p, \perp \rangle \preceq_{dm(A)} \langle p, m' \rangle$.

ix. Since $dm(A)$ satisfies *D1* (monotonic extension of commitment) with respect

to $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) \preceq_{dm(A)} \langle p, m' \rangle$ implies that $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m' \rangle) \subseteq \mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle)$.

x. So $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) \neq \emptyset$. ■ ($\rightarrow$)

(c) ($\leftarrow$)

i. Assume $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) \neq \emptyset$.

ii. [Show that there exists a $\langle p, m' \rangle \in \Sigma_{dm(A)}$ such that $m' \in t_A(p)$.]

iii. $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A + models(A)}(\langle p, \perp \rangle)$.

iv. By Proposition 93, we know that $\mathcal{T}_{\Sigma_A + models(A)}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A}(p)$.

v. So $\mathcal{T}_{\Sigma_A}(p) \neq \emptyset$.

vi. Let $u \in \mathcal{T}_{\Sigma_A}(p)$.

vii. Since A satisfies Assumption *B1* with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$.

viii. So there exists an $m' \in t_A(p)$ such that $u \in \mathcal{T}_{M_A}(m')$.

ix. Since $m' \in t_A(p)$, $m' \in \bigcup t_A[\Sigma_A]$.

- x. So $m' \in \Sigma_{models(A)}$.
- xi. So $\langle p, m' \rangle \in \Sigma_{dm(A)}$.
- xii. So $\langle p, \perp \rangle \in C_{dm(A)}$. ■ ($\leftarrow$) ■ Case 1.

7. Case 2: ($m \neq \perp$)

- (a) [Show $\langle p, m \rangle \in C_{dm(A)}$ iff $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$]
- (b) ($\rightarrow$)
 - i. Assume $\langle p, m \rangle \in C_{dm(A)}$.
 - ii. So $m \in t_A(p)$.
 - iii. By Proposition 75, $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$.
 - iv. By the definition of dm , $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$.
 - v. So $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$. ■ ($\rightarrow$)
- (c) ($\leftarrow$)
 - i. Assume $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) \neq \emptyset$.
 - ii. By the definition of dm , $\mathcal{T}_{\Sigma_{dm(A)}}(\langle p, m \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$.
 - iii. By Proposition 75, $m \in t_A(p)$.
 - iv. So $m \in \bigcup t_A[\Sigma_A]$.
 - v. So $m \in \Sigma_{models(A)}$.
 - vi. So $\langle p, m \rangle \in \Sigma_{dm(A)}$.
 - vii. So $\langle p, m \rangle \in C_{dm(A)}$. ■ ($\leftarrow$) ■ Case 2. ■ Proposition.

6.5.5 Weak extensibility (D3) portion of the dm Set CG assumptions proposition

Proposition 82 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,*

if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then the consistency medium $dm(A)$ satisfies assumption $D3$ (weak extensibility) with respect to derived practice of categorization $\mathcal{T}_{\Sigma_{dm(A)}}$.

1. Let A be an arbitrary partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
 2. By Proposition 92, $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$.
 3. Since $\Sigma_{A+models(A)} = \Sigma_{dm(A)}$, $\preceq_{A+models(A)} = \preceq_{dm(A)}$, and $\mathcal{T}_{\Sigma_{A+models(A)}} = \mathcal{T}_{\Sigma_{dm(A)}}$, by the definition of dm , we know that $\langle \Sigma_{dm(A)}, \preceq_{dm(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$.
- This is equivalent to saying that $dm(A)$ satisfies $D3$ with respect to $\mathcal{T}_{\Sigma_{dm(A)}}$. ■ Proposition.

This completes the proof of the dm Set CG assumptions proposition (Proposition 79).

6.5.6 Proposition demonstrating the embedding of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$

Proposition 83 *Given arbitrary partially ordered truth medium A , and practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then the function $h : \Sigma_A \rightarrow \Sigma_{dm(A)}$, defined for $p \in \Sigma_A$ as $h(p) = \langle p, \perp \rangle$, is such that for all $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{dm(A)}}(h(p))$.*

This is immediate via Proposition 93, since $\Sigma_{dm(A)} = \Sigma_{A+models(A)}$, and $\mathcal{T}_{\Sigma_{dm(A)}} = \mathcal{T}_{\Sigma_{A+models(A)}}$ by the definition of dm . ■ Proposition.

6.6 The operator *models*

6.6.1 *models* defined

Given an arbitrary truth medium A used with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, the operator *models*() constructs a partially ordered truth medium $models(A)$, and derived practices of categorization $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$ defined as follows:

Truth medium $models(A)$:

- $\Sigma_{models(A)} = \bigcup t_A [\Sigma_A] \cup \{\perp\}$, such that $\perp \notin M_A$
(note: $\perp$ will be used for “no information with respect to models”)
- $\preceq_{models(A)}$ is defined as follows: for all $p, q \in \Sigma_{models(A)}$, $p \preceq_{models(A)} q$ iff $p = q$ or $p = \perp$
- $M_{models(A)} = M_A$
- $t_{models(A)} : \Sigma_{models(A)} \rightarrow \mathcal{P}(M_{models(A)})$ is defined as follows:
for any $\sigma \in \Sigma_{models(A)}$, $t_{models(A)}(\sigma) =$ if $\sigma = \perp$ then $\bigcup t_A [\Sigma_A]$ else $\{\sigma\}$

Derived practices $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$:

- $\mathcal{T}_{\Sigma_{models(A)}} : \Sigma_{models(A)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:
for any $\sigma \in \Sigma_{models(A)}$, $\mathcal{T}_{\Sigma_{models(A)}}(\sigma) = \bigcup \mathcal{T}_{M_A} [t_{models(A)}(\sigma)] =$
if $\sigma = \perp$ then $\bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]]$ else $\mathcal{T}_{M_A}(\sigma)$
- $\mathcal{T}_{M_{models(A)}} : M_{models(A)} \rightarrow \mathcal{P}(\mathcal{U}) = \mathcal{T}_{M_A}$

6.6.2 *models* Set PP assumption preservation proposition

Proposition 84 *For all truth media A ,*

if A satisfies assumption Set PP (i.e. B1, A4, A5)

with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,

then $models(A)$ is a partially ordered truth medium satisfying assumption set PP

with respect to practices of categorization $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$.

Proof is at the end of the chapter.

6.6.3 Monotonic extension of commitment of *models* proposition

Proposition 85 *For any truth medium A , $models(A)$ satisfies monotonic extension of commitment with respect to models.*

Proof is at the end of the chapter.

6.6.4 Weak extensibility of *models* proposition

Proposition 86 *For arbitrary truth medium A , $\text{models}(A)$ is weakly extensible with respect to models .*

Proof is at the end of the chapter.

6.7 The operator $+$

6.7.1 $+$ defined

Given a partially ordered truth medium A , interpreted by practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and a partially ordered truth medium B , interpreted by practices of categorization $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$, such that A and B have identical sets of models (i.e. $M_A = M_B$) interpreted identically (i.e. $\mathcal{T}_{M_A} = \mathcal{T}_{M_B}$), the operator $+$ constructs a partially ordered truth medium $A + B$, and derived practices of categorization $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$ defined as follows:

Truth medium $A + B$:

- $\Sigma_{A+B} = \Sigma_A \times \Sigma_B$
- $\preceq_{A+B}$ is defined as follows:
for all $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in \Sigma_{A+B}$, $\langle \alpha_1, \beta_1 \rangle \preceq_{A+B} \langle \alpha_2, \beta_2 \rangle$ iff $\alpha_1 \preceq_A \alpha_2$ and $\beta_1 \preceq_B \beta_2$
- $M_{A+B} = M_A = M_B$
- $t_{A+B} : \Sigma_{A+B} \rightarrow \mathcal{P}(M_{A+B})$ is defined as follows:
for any $\langle \alpha, \beta \rangle \in \Sigma_{A+B}$, $t_{A+B}(\langle \alpha, \beta \rangle) = t_A(\alpha) \cap t_B(\beta)$

Derived practices $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$:

- $\mathcal{T}_{\Sigma_{A+B}} : \Sigma_{A+B} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:
for any $\langle \alpha, \beta \rangle \in \Sigma_{A+B}$, $\mathcal{T}_{\Sigma_{A+B}}(\langle \alpha, \beta \rangle) = \mathcal{T}_{\Sigma_A}(\alpha) \cap \mathcal{T}_{\Sigma_B}(\beta)$
- $\mathcal{T}_{M_{A+B}} : M_{A+B} \rightarrow \mathcal{P}(\mathcal{U}) = \mathcal{T}_{M_A} = \mathcal{T}_{M_B}$

6.7.2 + Set PP assumption preservation proposition

Proposition 87 *Given a partially ordered truth medium A , interpreted by practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and a partially ordered truth medium B , interpreted by practices of categorization $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$, such that A and B have identical sets of models (i.e. $M_A = M_B$) interpreted identically (i.e. $\mathcal{T}_{M_A} = \mathcal{T}_{M_B}$), if A satisfies assumption Set PP (i.e. B1, A4, A5) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and B satisfies assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$, then $A+B$ is a partially ordered truth medium satisfying assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$.*

Proof is at the end of the chapter.

6.7.3 + Monotonic extension of commitment preservation proposition

Proposition 88 *For all partially ordered truth media A , B , with identical sets of models (i.e. $M_A = M_B$), if A satisfies monotonic extension of commitment with respect to models and B satisfies monotonic extension of commitment with respect to models, then $A+B$ satisfies monotonic extension of commitment with respect to models.*

Proof is at the end of the chapter.

6.7.4 + Weak extensibility preservation proposition

Proposition 89 *For all partially ordered truth media A , B , with identical sets of models (i.e. $M_A = M_B$), if A is weakly extensible with respect to models, and B is weakly extensible with respect to models, then $A+B$ is weakly extensible with respect to models.*

Proof is at the end of the chapter.

6.8 Properties of $A+\text{models}(A)$

6.8.1 $A + \text{models}(A)$ Set PP assumption preservation proposition

Proposition 90 *For any partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption Set PP (i.e. B1, A4, A5) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $A + \text{models}(A)$ is a partially ordered truth medium satisfying assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+\text{models}(A)}}$ and $\mathcal{T}_{M_{A+\text{models}(A)}}$.*

Proof is at the end of the chapter.

6.8.2 Monotonic extension of commitment

Proposition 91 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. A4, A5, B1) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\langle \Sigma_{A+\text{models}(A)}, \preceq_{A+\text{models}(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{A+\text{models}(A)}}$.*

Proof is at the end of the chapter.

6.8.3 Weak extensibility

Proposition 92 *Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. A4, A5, B1) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\langle \Sigma_{A+\text{models}(A)}, \preceq_{A+\text{models}(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{A+\text{models}(A)}}$.*

Proof is at the end of the chapter.

6.8.4 Proposition demonstrating the embedding of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$

Proposition 93 *Given a partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption B1 with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{A+\text{models}(A)}}(\langle p, \perp \rangle)$.*

Proof is at the end of the chapter.

6.9 The operator *pow*

6.9.1 *pow* defined

See Section 4.8.4.

6.9.2 *pow* Set PP assumption preservation proposition

Proposition 94 *For all truth media A ,
if A satisfies assumption set PP (i.e. B1, A4, A5)
with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,
then $\text{pow}(A)$ is a partially ordered truth medium satisfying assumption set PP
with respect to practices of categorization $\mathcal{T}_{\Sigma_{\text{pow}(A)}}$ and $\mathcal{T}_{M_{\text{pow}(A)}}$.*

Proof is at the end of the chapter.

6.9.3 Proposition demonstrating the embedding of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$

Proposition 95 *Given a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,
the function $h : \Sigma_A \rightarrow \Sigma_{\text{pow}(A)}$, defined for all $p \in \Sigma_A$ as $h(p) = \{p\}$, is such that
 $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{\text{pow}(A)}}(h(p))$.*

1. Let p be an arbitrary element of Σ_A .
2. $h(p) = \{p\}$ is an element of $\Sigma_{\text{pow}(A)}$.
3. By the definition of *pow*, $\mathcal{T}_{\Sigma_{\text{pow}(A)}}(\{p\}) = \bigcap \mathcal{T}_{\Sigma_A}[\{p\}] = \mathcal{T}_{\Sigma_A}(p)$. ■Proposition.

6.10 The operator *negcomp*

The goal is to define an operator which is going to take a truth medium A and produce a truth medium $\text{negcomp}(A)$ such that.

1. $\text{negcomp}(A)$ is negation complete with respect to models;

2. $negcomp(A)$ preserves assumption set PP , that is, if A satisfies assumption set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $negcomp(A)$ satisfies assumption set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$;
3. $negcomp(A)$ preserves the assumed consequence relation. That is, there is some function $h : \Sigma_A \rightarrow \Sigma_{negcomp(A)}$, such that
$$ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{negcomp(A)}, \mathcal{T}_{\Sigma_{negcomp(A)}} \rangle}(h(p), h(q)).$$

6.10.1 $negcomp$ defined

Given arbitrary truth medium A , used with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we define the truth medium $negcomp(A)$ and derived practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$ as follows:

Truth medium $negcomp(A)$:

- $\Sigma_{negcomp(A)} = \{1, -1\} \times \Sigma_A$
- $M_{negcomp(A)} = M_A$
- $t_{negcomp(A)} : \Sigma_{negcomp(A)} \rightarrow \mathcal{P}(M_{negcomp(A)})$ is defined as follows:
for any $\langle i, \alpha \rangle \in \Sigma_{negcomp(A)}$, $t_{negcomp(A)}(\langle i, \alpha \rangle) = \text{if } i = 1 \text{ then } t_A(\alpha) \text{ else } \bigcup t_A[\Sigma_A] - t_A(\alpha)$

Derived practices $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$:

- $\mathcal{T}_{\Sigma_{negcomp(A)}} : \Sigma_{negcomp(A)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:
for any $\langle i, \alpha \rangle \in \Sigma_{negcomp(A)}$, $\mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle) = \text{if } i = 1 \text{ then } \mathcal{T}_{\Sigma_A}(\alpha) \text{ else } \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A] - \mathcal{T}_{\Sigma_A}(\alpha)$
- $\mathcal{T}_{M_{negcomp(A)}} : M_{negcomp(A)} \rightarrow \mathcal{P}(\mathcal{U}) = \mathcal{T}_{M_A}$

6.10.2 Sufficiency of $negcomp$ proposition

Proposition 96 *Given any truth medium A , the truth medium $\text{negcomp}(A)$ is negation complete with respect to models.*

Proof is at the end of the chapter.

6.10.3 negcomp Set PP assumption preservation proposition

Proposition 97 *For all truth media A , if A satisfies assumption set PP (i.e. $B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\text{negcomp}(A)$ is a truth medium satisfying assumption set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{\text{negcomp}(A)}}$ and $\mathcal{T}_{M_{\text{negcomp}(A)}}$.*

6.10.4 Proposition demonstrating the embedding of $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$

Proposition 98 *Given arbitrary truth medium A , and practices $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, the function $h : \Sigma_A \rightarrow \Sigma_{\text{negcomp}(A)}$, defined for $p \in \Sigma_A$ as $h(p) = \langle 1, p \rangle$, is such that $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{\text{negcomp}(A)}}(h(p))$.*

1. Let p be an arbitrary element of Σ_A .
2. So $h(p) = \langle 1, p \rangle$ is an element of $\Sigma_{\text{negcomp}(A)}$.
3. By the definition of negcomp , $\mathcal{T}_{\Sigma_{\text{negcomp}(A)}}(h(p)) = \mathcal{T}_{\Sigma_A}(p)$. ■ Proposition.

6.11 Applying the methodology described in this chapter: Applying the Set CG technique to the $P(WFF)$ -language of propositional logic

6.11.1 Statement of the goal

Our task in this section is to construct an application of the Set CG technique for the $P(WFF)$ -language of propositional logic. We have already shown how to apply the Set E technique to the $P(WFF)$ -language of propositional logic. Since every application of the Set E technique is an application of the Set PP technique, we have also already shown

how to apply the Set *PP* technique to the $P(WFF)$ -language of propositional logic. We will take that application, and run it through the methodology of this chapter, in order to create the desired application of the Set *CG* technique.

6.11.2 The application proper

We will be given a practice of categorization $\mathcal{T}_{\mathcal{P}(WFF)}$ assumed to be an conventional interpretation of the set of assertion types $\mathcal{P}(WFF)$; that is, $\mathcal{T}_{\mathcal{P}(WFF)}$ is assumed to satisfy the semantic conventions for interpretations of $\mathcal{P}(WFF)$. We have already seen (in Section 4.8.9 and the proof of Proposition 66) how, given a conventional interpretation $\mathcal{T}_{\mathcal{P}(WFF)}$ of $\mathcal{P}(WFF)$, we can form a partially ordered truth medium *PTO* and practice of interpretation for models $\mathcal{T}_{TTA}$, such that

1. *PTO* satisfies monotonic extension of commitment with respect to models;
2. *PTO* is weakly extensible with respect to models;
3. *PTO* satisfies assumption Set *PP* with respect to $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$;

Applying Theorem 76, we know that there is a consistency medium D and an associated practice of categorization $\mathcal{T}_{\Sigma_D}$ derived from *PTO*, $\mathcal{T}_{\mathcal{P}(WFF)}$ and $\mathcal{T}_{TTA}$, such that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, and there is a function h from $\mathcal{P}(WFF)$ to Σ_D such that $\forall p, q \in \mathcal{P}(WFF) \left(ALC_{\langle \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)} \rangle} (p, q) \text{ iff } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} (h(p), h(q)) \right)$

Thus, an image of the representational relation of logical consequence for the $\mathcal{P}(WFF)$ -language of propositional logic as interpreted by $\mathcal{T}_{\mathcal{P}(WFF)}$ $\left(ALC_{\langle \mathcal{P}(WFF), \mathcal{T}_{\mathcal{P}(WFF)} \rangle} \right)$ is embedded within the representational relation of logical consequence $\left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} \right)$ made intelligible by an application of the Set *CG* technique.

6.11.3 Details of the application

It will be interesting for us to look at the particular consistency medium, practice of categorization, and function constructed by Theorem 76 in the process of showing the existential claim.

Consistency medium

The consistency medium constructed is $dm(PTO)$. Let us consider its components one at a time.

The set of assertion types: $\Sigma_{dm(PTO)}$

Unpacking the definitions involved, $\Sigma_{dm(PTO)} = \Sigma_{PTO+models(PTO)} = \mathcal{P}(WFF) \times (\bigcup t_{PT}[\mathcal{P}(WFF)] \cup \{\perp\})$.

It is easy to show that $\bigcup t_{PT}[\mathcal{P}(WFF)] = TTA$, since for any $p \in WFF$, the models of $\{p\}$ and $\{\mathcal{E}_-(p)\}$ exhaust TTA .

So $\Sigma_{dm(PTO)} = \mathcal{P}(WFF) \times (TTA \cup \{\perp\})$.

The ordering relation: $\preceq_{dm(pow(ST))}$

Unpacking the definitions involved, $\preceq_{dm(PTO)} = \preceq_{PTO+models(PTO)}$ which is defined as follows:

For all $\langle S_1, m_1 \rangle, \langle S_2, m_2 \rangle \in \Sigma_{PTO+models(PTO)}$,

$\langle S_1, m_1 \rangle \preceq_{PTO+models(PTO)} \langle S_2, m_2 \rangle$ iff $S_1 \preceq_{PTO} S_2$ and $m_1 \preceq_{models(PTO)} m_2$;

which is equivalent to:

For all $\langle S_1, m_1 \rangle, \langle S_2, m_2 \rangle \in \Sigma_{dm(PTO)}$,

$\langle S_1, m_1 \rangle \preceq_{dm(PTO)} \langle S_2, m_2 \rangle$ iff $S_1 \subseteq S_2$ and $(m_1 = m_2 \text{ or } m_1 = \perp)$.

The set of consistent assertion types: $C_{dm(PTO)}$ Unpacking the definitions involved,

$$C_{dm(PTO)} = \left\{ \begin{array}{l} \langle S, m \rangle \in \Sigma_{dm(PTO)} \mid \\ \left(\begin{array}{l} m = \perp \text{ and there exists a } \langle S, m' \rangle \in \Sigma_{dm(PTO)} \\ \text{such that } m' \in t_{PT}(S) \end{array} \right) \text{ or } \\ (m \neq \perp \text{ and } m \in t_{PT}(S)) \end{array} \right\}$$

Given that $S \subseteq WFF$, and $m \in (TTA \cup \{\perp\})$, when is it the case that m is an element of $t_{PT}(S)$? There are two cases.

Case 1: $S = \emptyset$. In this case, $t_{PT}(S) = TTA$. So $m \in t_{PT}(S)$ iff $m \in TTA$.

Case 2: $S \neq \emptyset$. In this case, $t_{PT}(S) = \bigcap t_{ST}[S]$.

So $m \in t_{PT}(S)$ iff

$m \in \bigcap t_{ST}[S]$ iff

for every $s \in S$, $m \in t_{ST}(s)$ iff

$m \in TTA$ and for every $s \in S$, $m \models s$.

We can combine the two cases and say: $m \in t_{PT}(S)$ just in case $m \in TTA$ and for every $s \in S$, $m \models s$.

So the description of $C_{dm(PTO)}$ above can be rewritten as:

$$C_{dm(PTO)} = \left\{ \begin{array}{l} \langle S, m \rangle \in \Sigma_{dm(PTO)} \mid \\ \left(\begin{array}{l} m = \perp \text{ and there exists a } \langle S, m' \rangle \in \Sigma_{dm(PTO)} \\ \text{such that } m' \in TTA \text{ and } \forall s \in S, m' \models s \end{array} \right) \text{ or } \\ (m \neq \perp \text{ and } \forall s \in S, m \models s) \end{array} \right\}$$

Practice of categorization

The practice of categorization constructed is $\mathcal{T}_{\Sigma_{dm(PTO)}}$.

Unpacking definitions, $\mathcal{T}_{\Sigma_{dm(PTO)}} = \mathcal{T}_{\Sigma_{PTO+models(PTO)}}$, which is defined as follows:

for any $\langle S, m \rangle \in \Sigma_{PTO+models(PTO)}$, $\mathcal{T}_{\Sigma_{PTO+models(PTO)}}(\langle S, m \rangle) = \mathcal{T}_{\Sigma_{PTO}}(S) \cap \mathcal{T}_{\Sigma_{models(PTO)}}(m)$.

Now $\mathcal{T}_{\Sigma_{PTO}} = \mathcal{T}_{\mathcal{P}(WFF)}$, and $\mathcal{T}_{\Sigma_{models(PTO)}} = \mathcal{T}_{TTA}$.

There are two cases for the value of $\mathcal{T}_{\Sigma_{models(PTO)}}(m)$.

(Case 1) $m = \perp$. In this case, $\mathcal{T}_{\Sigma_{models(PTO)}}(m) = \bigcup \mathcal{T}_{TTA}[\bigcup t_{PT}[\mathcal{P}(WFF)]] = \bigcup \mathcal{T}_{TTA}[TTA] = \mathcal{U}$ (this last equivalence is implied by the semantic conventions of the $\mathcal{P}(WFF)$ -language of propositional logic).

(Case 2) $m \neq \perp$. In this case, $\mathcal{T}_{\Sigma_{models(PTO)}}(m) = \mathcal{T}_{TTA}(m)$.

So $\mathcal{T}_{\Sigma_{dm(PTO)}}(\langle S, m \rangle) =$ if $m = \perp$ then $\mathcal{T}_{\mathcal{P}(WFF)}(S)$ else $\mathcal{T}_{\mathcal{P}(WFF)}(S) \cap \mathcal{T}_{TTA}(m)$.

Function h

The function h constructed is $h : \mathcal{P}(WFF) \rightarrow \Sigma_{dm(PTO)}$ defined for $\forall P \in \mathcal{P}(WFF)$ as $h(P) = \langle P, \perp \rangle$.

6.11.4 Discussion of the application

The core of the application is the definition of the set of consistent assertion types:

$$C_{dm(PTO)} = \left\{ \begin{array}{l} \langle S, m \rangle \in \Sigma_{dm(PTO)} \mid \\ \left(\begin{array}{l} m = \perp \text{ and there exists a } \langle S, m' \rangle \in \Sigma_{dm(PTO)} \\ \text{such that } m' \in TTA \text{ and } \forall s \in S, m' \models s \end{array} \right) \text{ or } \\ (m \neq \perp \text{ and } \forall s \in S, m \models s) \end{array} \right\}$$

The relation of truth-in-a-model ($\models$) from the standard model-theoretic semantic account plays the central role in this definition. As seen here (and as we discussed in the development of the proof - see Section 6.3.3), our construction directly imports the standard model-theoretic means of making consistency intelligible (consistency is equivalent to the existence of a model).

It is interesting to compare the structure of this definition to the approach we proposed for defining consistency for feature structures as assertion types in Section 5.7.4. There we proposed that a feature structure would be deemed consistent if it had some extension (via the subsumption relation) which satisfied all the constraints at every element in the structure. If we think of truth-in-a-model as a constraint between a model and a set of sentences; this definition can be seen as an instance of that general idea. A structure $\langle S, m \rangle \in \Sigma_{dm(PTO)}$ is consistent iff there exists an extension $\langle S', m' \rangle$ of $\langle S, m \rangle$ on the ordering $\preceq_{dm(PTO)}$ such that $m' \in TTA$, and $\forall s \in S', m' \models s$.

This concludes the linear text of the chapter. The remainder of the material in this chapter are the proofs of propositions mentioned earlier in the text.

6.12 Proofs of propositions from the text

Proof of Proposition 75:

(Proposition 75): For arbitrary truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and

$\mathcal{T}_{M_A}$, if A satisfies Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then for all $p \in \Sigma_A$ and $m \in M_A$, $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$ iff $m \in t_A(p)$.

Let A be an arbitrary truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies Set PP with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Let p be an arbitrary element of Σ_A , and m an arbitrary element of M_A .

[Show that $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$ iff $m \in t_A(p)$]

($\rightarrow$)

1. Assume $\mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m) \neq \emptyset$.
2. Then there is a $u \in \mathcal{U}$ such that $u \in \mathcal{T}_{\Sigma_A}(p)$ and $u \in \mathcal{T}_{M_A}(m)$.
3. Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know that $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$.
4. So there exists an $n \in t_A(p)$ such that $u \in \mathcal{T}_{M_A}(n)$.
5. Since A satisfies assumption $A4$ with respect to $\mathcal{T}_{M_A}$, $m = n$.
6. So $m \in t_A(p)$. ■ ($\rightarrow$)

($\leftarrow$)

1. Assume $m \in t_A(p)$.
2. Since A satisfies assumption $A5$ with respect to $\mathcal{T}_{M_A}$, we know that there exists a $u \in \mathcal{U}$ such that $u \in \mathcal{T}_{M_A}(m)$.
3. Since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know that $\bigcup \mathcal{T}_{M_A}[t_A(p)] = \mathcal{T}_{\Sigma_A}(p)$.
4. So $u \in \mathcal{T}_{\Sigma_A}(p)$.
5. So $u \in \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{M_A}(m)$. ■ ($\leftarrow$) ■ Proposition.

Proof of Proposition 84:

(Proposition 84): For all truth media A ,
if A satisfies assumption set PP (i.e. $B1, A4, A5$)

with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,
then $models(A)$ is a partially ordered truth medium satisfying assumption set PP
with respect to practices of categorization $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$.

Showing that $models(A)$ is a truth medium.

$models(A)$ has the structure of a truth medium. We need to show that $\Sigma_{models(A)}$ and $M_{models(A)}$ are non-empty. $\Sigma_{models(A)}$ is non-empty because $\perp \in \Sigma_{models(A)}$. $M_{models(A)}$ is non-empty because $M_{models(A)} = M_A$, and M_A is non-empty since A is a truth medium. $t_{models(A)}$ is a function $\Sigma_{models(A)} \rightarrow \mathcal{P}(M_{models(A)})$ by definition.

Showing that $models(A)$ is a partially ordered truth medium.

$\preceq_{models(A)}$ is a partial order, since $\langle \Sigma_{models(A)}, \preceq_{models(A)} \rangle$ is $\left(\bigcup t_A [\Sigma_A] \right)_{\perp}$, that is, $\bigcup t_A [\Sigma_A]$ made into an antichain and then “lifted”. See Davey and Priestley, p. 16.

Showing that $models(A)$ satisfies assumptions A4, A5 with respect to practice of categorization $\mathcal{T}_{M_{models(A)}}$.

This is immediate since $M_{models(A)} = M_A$ and $\mathcal{T}_{M_{models(A)}} = \mathcal{T}_{M_A}$ by the definition of $models$. Substituting these equivalences into statements of A4 and A5 expressed relatively to $models(A)$ and $\mathcal{T}_{M_{models(A)}}$, yields A4 and A5 expressed relatively to A and $\mathcal{T}_{M_A}$, which was assumed.

Showing that $models(A)$ satisfies assumption B1 with respect to practices of categorization $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$.

Formally, $\left[\text{Show } \forall \sigma \in \Sigma_{models(A)} \left(\bigcup \mathcal{T}_{M_{models(A)}} [t_{models(A)}(\sigma)] = \mathcal{T}_{\Sigma_{models(A)}}(\sigma) \right) \right]$

From the definition of $models$, we know $\mathcal{T}_{M_{models(A)}} = \mathcal{T}_{M_A}$, and

$\forall \sigma \in \Sigma_{models(A)}, \mathcal{T}_{\Sigma_{models(A)}}(\sigma) = \bigcup \mathcal{T}_{M_A} [t_{models(A)}(\sigma)]$.

So the above becomes

$\left[\text{Show } \forall \sigma \in \Sigma_{models(A)} \left(\bigcup \mathcal{T}_{M_A} [t_{models(A)}(\sigma)] = \bigcup \mathcal{T}_{M_A} [t_{models(A)}(\sigma)] \right) \right]$,

which is immediate.

Proof of Proposition 85:

(Proposition 85): For any truth medium A , $models(A)$ satisfies monotonic extension of commitment with respect to models.

Let A be an arbitrary truth medium. Let p, q be arbitrary elements of $\Sigma_{models(A)}$, such that $p \preceq_{models(A)} q$.

[Show $t_{models(A)}(q) \subseteq t_{models(A)}(p)$]

1. By the definition of $models$, either $p = q$ or $p = \perp$.

2. (Case 1) $p = q$

(a) $t_{models(A)}(p) = t_{models(A)}(q)$. ■ Case 1.

3. (Case 2) $p = \perp$, and $p \neq q$

(a) $t_{models(A)}(p) = \bigcup t_A [\Sigma_A]$.

(b) Since $p \neq q$, $q \neq \perp$; so $t_{models(A)}(q) = \{q\}$.

(c) Since $q \in \bigcup t_A [\Sigma_A]$, $t_{models(A)}(q) \subseteq t_{models(A)}(p)$. ■ Case 2. ■ Proposition.

Proof of Proposition 86:

(Proposition 86): For arbitrary truth medium A , $models(A)$ is weakly extensible with respect to models.

In this proof, we will use the notation $x \vee_P y$ (' x join y ') to indicate the least upper bound of x and y in the ordered set P . (See Davey and Priestley, p. 28)

Let A be an arbitrary truth medium.

(Part a.)

[Show: For every $p, q \in \Sigma_{models(A)}$,
 $\exists R \subseteq \{p, q\}_{models(A)}^u$ such that $\bigcup t_{models(A)} [R] = t_{models(A)}(p) \cap t_{models(A)}(q)$
 Let p, q be arbitrary elements of $\Sigma_{models(A)}$.]

[Claim: $R = \{p \vee_{models(A)} q\}$ if $p \vee_{models(A)} q$ exists, then $\{p \vee_{models(A)} q\}$ else $\emptyset$]

Note: if $p \vee_{models(A)} q$ exists, then $p \vee_{models(A)} q \in \{p, q\}_{models(A)}^u$.

We consider four cases: 1) $p = \perp, q = \perp$, 2) $p = \perp, q \neq \perp$, 3) $p \neq \perp, q = \perp$, 4) $p \neq \perp, q \neq \perp$

(Case 1) $p = \perp, q = \perp$

1. $t_{models(A)}(p) = \bigcup t_A [\Sigma_A]$.

2. $t_{models(A)}(q) = \bigcup t_A [\Sigma_A]$.

3. $t_{models(A)}(p) \cap t_{models(A)}(q) = \bigcup t_A [\Sigma_A]$.

4. $p \vee_{models(A)} q = \perp$.
5. Let $R = \{\perp\}$.
6. $\bigcup t_{models(A)}[R] = t_{models(A)}(\perp) = \bigcup t_A[\Sigma_A]$ ■ (Case 1)
- (Case 2) $p = \perp, q \neq \perp$
1. $t_{models(A)}(p) = \bigcup t_A[\Sigma_A]$.
2. $t_{models(A)}(q) = \{q\}$.
3. $q \in \bigcup t_A[\Sigma_A]$, since $q \in \Sigma_{models(A)}$ and $q \neq \perp$.
4. So $t_{models(A)}(p) \cap t_{models(A)}(q) = \{q\}$.
5. $p \vee_{models(A)} q = q$.
6. Let $R = \{q\}$.
7. $\bigcup t_{models(A)}[R] = t_{models(A)}(q) = \{q\}$. ■ (Case 2)
- (Case 3) $p \neq \perp, q = \perp$
- Same as Case 2, but with p and q exchanged. ■ (Case 3)
- (Case 4) $p \neq \perp, q \neq \perp$
1. $t_{models(A)}(p) = \{p\}$.
2. $t_{models(A)}(q) = \{q\}$.
3. (Case 4a) $p = q$
 - (a) $t_{models(A)}(p) \cap t_{models(A)}(q) = \{p\}$.
 - (b) $p \vee_{models(A)} q = p$.
 - (c) Let $R = \{p\}$.
 - (d) $\bigcup t_{models(A)}[R] = t_{models(A)}(p) = \{p\}$. ■ (Case 4a)
4. (Case 4b) $p \neq q$

- (a) $t_{models(A)}(p) \cap t_{models(A)}(q) = \emptyset$.
- (b) $p \vee_{models(A)} q$ does not exist.
- (c) Let $R = \emptyset$.
- (d) $\bigcup t_{models(A)}[R] = \emptyset$. ■(Case 4b) ■(Case 4) ■(Part a.)

(Part b.)

Show: For every $p, q \in \Sigma_{models(A)}$,

$$\left[\exists S \subseteq \uparrow_{models(A)} p \text{ such that } \bigcup t_{models(A)}[S] = t_{models(A)}(p) - t_{models(A)}(q) \right]$$

Let p, q be arbitrary elements of $\Sigma_{models(A)}$.

[Claim: $S = t_{models(A)}(p) - t_{models(A)}(q)$]

We consider four cases: 1) $p = \perp, q = \perp$, 2) $p = \perp, q \neq \perp$, 3) $p \neq \perp, q = \perp$, 4) $p \neq \perp, q \neq \perp$

(Case 1) $p = \perp, q = \perp$

1. $t_{models(A)}(p) = \bigcup t_A[\Sigma_A]$.
2. $t_{models(A)}(q) = \bigcup t_A[\Sigma_A]$.
3. $t_{models(A)}(p) - t_{models(A)}(q) = \emptyset$.
4. Let $S = \emptyset$.
5. $S \subseteq \uparrow_{models(A)} p$ trivially.
6. $\bigcup t_{models(A)}[S] = \emptyset$. ■(Case 1)

(Case 2) $p = \perp, q \neq \perp$

1. $t_{models(A)}(p) = \bigcup t_A[\Sigma_A]$.
2. $t_{models(A)}(q) = \{q\}$.
3. $t_{models(A)}(p) - t_{models(A)}(q) = \bigcup t_A[\Sigma_A] - \{q\}$.
4. Let $S = \bigcup t_A[\Sigma_A] - \{q\}$.
5. Since $p = \perp$, every $s \in S$ is such that $s \succeq_{models(A)} p$. So $S \subseteq \uparrow_{models(A)} p$.
6. For each member $s \in S$, $s \neq \perp$, so $t_{models(A)}(s) = \{s\}$.

7. So $\bigcup t_{models(A)}[S] = S$.■(Case 2)

(Case 3) $p \neq \perp, q = \perp$

1. $t_{models(A)}(p) = \{p\}$.

2. $t_{models(A)}(q) = \bigcup t_A[\Sigma_A]$.

3. Since $p \in \bigcup t_A[\Sigma_A]$, $t_{models(A)}(p) - t_{models(A)}(q) = \emptyset$.

4. Let $S = \emptyset$.

5. $S \subseteq \uparrow_{models(A)} p$ trivially.

6. $\bigcup t_{models(A)}[S] = \emptyset$.■(Case 3)

(Case 4) $p \neq \perp, q \neq \perp$

1. $t_{models(A)}(p) = \{p\}$.

2. $t_{models(A)}(q) = \{q\}$.

3. (Case 4a) $p = q$

(a) $t_{models(A)}(p) - t_{models(A)}(q) = \emptyset$.

(b) Let $S = \emptyset$.

(c) $S \subseteq \uparrow_{models(A)} p$ trivially.

(d) $\bigcup t_{models(A)}[S] = \emptyset$.■(Case 4a)

4. (Case 4b) $p \neq q$

(a) $t_{models(A)}(p) - t_{models(A)}(q) = \{p\}$.

(b) Let $S = \{p\}$.

(c) $S \subseteq \uparrow_{models(A)} p$ since $\preceq_{models(A)}$ is reflexive.

(d) $\bigcup t_{models(A)}[S] = t_{models(A)}(p) = \{p\}$.■(Case 4b)

■(Case 4)■(Part b.)■Proposition.

Proof of Proposition 87:

(Proposition 87): Given a partially ordered truth medium A , interpreted by practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and a partially ordered truth medium B , interpreted by practices of categorization $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$, such that A and B have identical sets of models (i.e. $M_A = M_B$) interpreted identically (i.e. $\mathcal{T}_{M_A} = \mathcal{T}_{M_B}$), if A satisfies assumption Set PP (i.e. $B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and B satisfies assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_B}$ and $\mathcal{T}_{M_B}$, then $A + B$ is a partially ordered truth medium satisfying assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$.

Showing that $A + B$ is a truth medium.

$A + B$ has the structure of a truth medium. We need to show that Σ_{A+B} and M_{A+B} are non-empty. We know that Σ_A and Σ_B are non-empty since A and B are truth media. So we know that Σ_{A+B} is non-empty since $\Sigma_{A+B} = \Sigma_A \times \Sigma_B$. We know that M_A is non-empty since A is a truth medium. So we know that M_{A+B} is non-empty because $M_{A+B} = M_A = M_B$. t_{A+B} is a function from Σ_{A+B} to $\mathcal{P}(M_{A+B})$ by definition.

Showing that $A + B$ is a partially ordered truth medium.

The ordered set $\langle \Sigma_{A+B}, \preceq_{A+B} \rangle$ is the product of Σ_A and Σ_B with the coordinate-wise order imposed. Given that $\langle \Sigma_A, \preceq_A \rangle$ and $\langle \Sigma_B, \preceq_B \rangle$ were partially ordered sets, so is $\langle \Sigma_{A+B}, \preceq_{A+B} \rangle$. See Davey and Priestley, p. 18.

Showing that $A + B$ satisfies assumptions $A4, A5$ with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$.

This is immediate since $M_{A+B} = M_A$ and $\mathcal{T}_{M_{A+B}} = \mathcal{T}_{M_A}$ by the definition of $A + B$. Substituting these equivalences into statements of $A4$ and $A5$ expressed relatively to $A + B$ and $\mathcal{T}_{M_{A+B}}$, yields $A4$ and $A5$ expressed relatively to A and $\mathcal{T}_{M_A}$, which was assumed.

Showing that $A + B$ satisfies assumption $B1$ with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+B}}$ and $\mathcal{T}_{M_{A+B}}$.

Formally, [Show $\forall \langle \alpha, \beta \rangle \in \Sigma_{A+B} (\bigcup \mathcal{T}_{M_{A+B}} [t_{A+B}(\langle \alpha, \beta \rangle)] = \mathcal{T}_{\Sigma_{A+B}}(\langle \alpha, \beta \rangle)]$

1. Let $\langle \alpha, \beta \rangle$ be an arbitrary element of Σ_{A+B} .
2. Apply the definitions of $A + B$, i.e. $\mathcal{T}_{M_{A+B}} = \mathcal{T}_{M_A}$, $\mathcal{T}_{\Sigma_{A+B}}(\langle \alpha, \beta \rangle) = \mathcal{T}_{\Sigma_A}(\alpha) \cap \mathcal{T}_{\Sigma_B}(\beta)$,

and $t_{A+B}(\langle \alpha, \beta \rangle) = t_A(\alpha) \cap t_B(\beta)$.

3. [Show $\bigcup \mathcal{T}_{M_A} [t_A(\alpha) \cap t_B(\beta)] = \mathcal{T}_{\Sigma_A}(\alpha) \cap \mathcal{T}_{\Sigma_B}(\beta)$]
4. Since A satisfies assumption B1 with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we have $\mathcal{T}_{\Sigma_A}(\alpha) = \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$. Similarly, $\mathcal{T}_{\Sigma_B}(\beta) = \bigcup \mathcal{T}_{M_B} [t_B(\beta)]$.
5. So [Show $\bigcup \mathcal{T}_{M_A} [t_A(\alpha) \cap t_B(\beta)] = \bigcup \mathcal{T}_{M_A} [t_A(\alpha)] \cap \bigcup \mathcal{T}_{M_B} [t_B(\beta)]$]
6. Apply the assumed equivalence that $\mathcal{T}_{M_A} = \mathcal{T}_{M_B}$.
7. So [Show $\bigcup \mathcal{T}_{M_A} [t_A(\alpha) \cap t_B(\beta)] = \bigcup \mathcal{T}_{M_A} [t_A(\alpha)] \cap \bigcup \mathcal{T}_{M_A} [t_B(\beta)]$]
8. ($\subseteq$)
 - (a) Let $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha) \cap t_B(\beta)]$.
 - (b) Then there is some $m \in t_A(\alpha) \cap t_B(\beta)$ such that $u \in \mathcal{T}_{M_A}(m)$.
 - (c) So $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$ and $u \in \bigcup \mathcal{T}_{M_A} [t_B(\beta)]$. ■ ($\subseteq$)
9. ($\supseteq$)
 - (a) Let $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha)] \cap \bigcup \mathcal{T}_{M_A} [t_B(\beta)]$.
 - (b) So $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$ and $u \in \bigcup \mathcal{T}_{M_A} [t_B(\beta)]$.
 - (c) So there exists an $m \in t_A(\alpha)$ such that $u \in \mathcal{T}_{M_A}(m)$, and an $n \in t_B(\beta)$, such that $u \in \mathcal{T}_{M_A}(n)$. Since $t_B : \Sigma_B \rightarrow \mathcal{P}(M_B)$, and $M_A = M_B$, $n \in M_A$. Since $u \in \mathcal{T}_{M_A}(m)$ and $u \in \mathcal{T}_{M_A}(n)$, and $m, n \in M_A$, and A satisfies A4 with respect to $\mathcal{T}_{M_A}$, we know that $m = n$.
 - (d) So $m \in t_A(\alpha)$ and an $m \in t_B(\beta)$.
 - (e) So $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha) \cap t_B(\beta)]$ ■ ($\supseteq$) ■

Proof of Proposition 88:

(Proposition 88): For all partially ordered truth media A, B , with identical sets of models (i.e. $M_A = M_B$), if A satisfies monotonic extension of commitment with respect to models and B satisfies monotonic extension of commitment with respect to models, then $A + B$ satisfies monotonic extension of commitment with respect to models.

Let A, B be arbitrary partially ordered truth media with identical sets of models (i.e. $M_A = M_B$), such that A satisfies monotonic extension of commitment with respect to models and B satisfies monotonic extension of commitment with respect to models.

1. Let $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle$ be arbitrary elements of Σ_{A+B} , such that $\langle \alpha_1, \beta_1 \rangle \preceq_{A+B} \langle \alpha_2, \beta_2 \rangle$
2. So $\alpha_1 \preceq_A \alpha_2$ and $\beta_1 \preceq_B \beta_2$ by the definition of $+$.
3. Since A satisfies monotonic extension of commitment with respect to models, $\forall \alpha_1, \alpha_2 \in \Sigma_A$, if $\alpha_1 \preceq_A \alpha_2$ then $t_A(\alpha_2) \subseteq t_A(\alpha_1)$.
4. Since B satisfies monotonic extension of commitment with respect to models, $\forall \beta_1, \beta_2 \in \Sigma_B$, if $\beta_1 \preceq_B \beta_2$ then $t_B(\beta_2) \subseteq t_B(\beta_1)$.
5. So $t_A(\alpha_2) \subseteq t_A(\alpha_1)$ and $t_B(\beta_2) \subseteq t_B(\beta_1)$.
6. So $t_A(\alpha_2) \cap t_B(\beta_2) \subseteq t_A(\alpha_1) \cap t_B(\beta_1)$
7. So $t_{A+B}(\langle \alpha_2, \beta_2 \rangle) \subseteq t_{A+B}(\langle \alpha_1, \beta_1 \rangle)$ by the definition of t_{A+B} . ■ Proposition.

Proof of Proposition 89:

(Proposition 89): For all partially ordered truth media A, B , with identical sets of models (i.e. $M_A = M_B$), if A is weakly extensible with respect to models, and B is weakly extensible with respect to models, then $A + B$ is weakly extensible with respect to models.

Let A, B be partially ordered truth media with identical sets of models (i.e. $M_A = M_B$) such that A is weakly extensible with respect to models, and B is weakly extensible with respect to models.

[Show that $A + B$ is weakly extensible with respect to models.]

(Part a.)

$$\left[\begin{array}{l} \text{For every } \langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in \Sigma_{A+B}, \\ \text{Show } \exists R \subseteq \{\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle\}_{A+B}^u \\ \text{such that } \bigcup t_{A+B}[R] = t_{A+B}(\langle \alpha_1, \beta_1 \rangle) \cap t_{A+B}(\langle \alpha_2, \beta_2 \rangle) \end{array} \right]$$

1. Let $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle$ be arbitrary elements of Σ_{A+B} .
2. So $t_{A+B}(\langle \alpha_1, \beta_1 \rangle) = t_A(\alpha_1) \cap t_B(\beta_1)$, and $t_{A+B}(\langle \alpha_2, \beta_2 \rangle) = t_A(\alpha_2) \cap t_B(\beta_2)$.

3. Since A is weakly extensible with respect to models, and $\alpha_1, \alpha_2 \in \Sigma_A$, we know that $\exists R_A \subseteq \{\alpha_1, \alpha_2\}_A^u$ such that $\bigcup t_A[R_A] = t_A(\alpha_1) \cap t_A(\alpha_2)$.
4. Similarly, since B is weakly extensible with respect to models, and $\beta_1, \beta_2 \in \Sigma_B$, we know that $\exists R_B \subseteq \{\beta_1, \beta_2\}_B^u$ such that $\bigcup t_B[R_B] = t_B(\beta_1) \cap t_B(\beta_2)$.
5. Consider $R = R_A \times R_B$.
6. (Claim a1) $[R_A \times R_B \subseteq \{\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle\}_{A+B}^u]$
 - (a) Let $\langle \alpha_3, \beta_3 \rangle \in R_A \times R_B$.
 - (b) So $\alpha_3 \in R_A$, and $\beta_3 \in R_B$.
 - (c) So $\alpha_3 \succeq_A \alpha_1$, and $\alpha_3 \succeq_A \alpha_2$ since $R_A \subseteq \{\alpha_1, \alpha_2\}_A^u$.
 - (d) And $\beta_3 \succeq_B \beta_1$, and $\beta_3 \succeq_B \beta_2$ since $R_B \subseteq \{\beta_1, \beta_2\}_B^u$.
 - (e) So $\langle \alpha_3, \beta_3 \rangle \succeq_{A+B} \langle \alpha_1, \beta_1 \rangle$ and $\langle \alpha_3, \beta_3 \rangle \succeq_{A+B} \langle \alpha_2, \beta_2 \rangle$ by definition +.
 - (f) So $\langle \alpha_3, \beta_3 \rangle \in \{\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle\}_{A+B}^u$ ■ (Claim a1.)
7. (Claim a2) $[\bigcup t_{A+B}[R] = t_{A+B}(\langle \alpha_1, \beta_1 \rangle) \cap t_{A+B}(\langle \alpha_2, \beta_2 \rangle)]$
 - (a) $\bigcup t_{A+B}[R_A \times R_B] = \bigcup_{\langle \alpha, \beta \rangle \in R_A \times R_B} t_{A+B}(\langle \alpha, \beta \rangle) = \bigcup_{\langle \alpha, \beta \rangle \in R_A \times R_B} t_A(\alpha) \cap t_B(\beta)$.
 - (b) $t_{A+B}(\langle \alpha_1, \beta_1 \rangle) \cap t_{A+B}(\langle \alpha_2, \beta_2 \rangle) = t_A(\alpha_1) \cap t_B(\beta_1) \cap t_A(\alpha_2) \cap t_B(\beta_2)$.
 - (c) $(\subseteq)$
 - i. Let $m \in \bigcup_{\langle \alpha, \beta \rangle \in R_A \times R_B} t_A(\alpha) \cap t_B(\beta)$.
 - ii. So there is an $\alpha \in R_A$ and a $\beta \in R_B$ such that $m \in t_A(\alpha)$ and $m \in t_B(\beta)$.
 - iii. Since $\alpha \in R_A$ and $m \in t_A(\alpha)$ we know $m \in \bigcup t_A[R_A]$, and hence that $m \in t_A(\alpha_1) \cap t_A(\alpha_2)$.
 - iv. Since $\beta \in R_B$ and $m \in t_B(\beta)$ we know $m \in \bigcup t_B[R_B]$, and hence that $m \in t_B(\beta_1) \cap t_B(\beta_2)$. ■ $(\subseteq)$
 - (d) $(\supseteq)$
 - i. Let $m \in t_A(\alpha_1) \cap t_B(\beta_1) \cap t_A(\alpha_2) \cap t_B(\beta_2)$.
 - ii. So $m \in \bigcup t_A[R_A]$ and $m \in \bigcup t_B[R_B]$.

- iii. So there is an $\alpha \in R_A$ such that $m \in t_A(\alpha)$, and a $\beta \in R_B$ such that $m \in t_B(\beta)$.
- iv. So there is an $\langle \alpha, \beta \rangle \in R_A \times R_B$ such that $m \in t_A(\alpha)$ and $m \in t_B(\beta)$. ■ ($\supseteq$)
 ■(Claim a2). ■(Part a)

(Part b.)

$$\left[\begin{array}{l} \text{Show: For every } \langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in \Sigma_{A+B}, \\ \text{Show} \quad \exists S \subseteq \uparrow_{A+B} \langle \alpha_1, \beta_1 \rangle \\ \text{such that } \bigcup t_{A+B}[S] = t_{A+B}(\langle \alpha_1, \beta_1 \rangle) - t_{A+B}(\langle \alpha_2, \beta_2 \rangle) \end{array} \right]$$

1. Let $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle$ be arbitrary elements of Σ_{A+B} .
2. So $t_{A+B}(\langle \alpha_1, \beta_1 \rangle) = t_A(\alpha_1) \cap t_B(\beta_1)$, and $t_{A+B}(\langle \alpha_2, \beta_2 \rangle) = t_A(\alpha_2) \cap t_B(\beta_2)$.
3. Since A is weakly extensible with respect to models, and $\alpha_1, \alpha_2 \in \Sigma_A$, we know that $\exists S_A \subseteq (\uparrow_A \alpha_1)$ such that $\bigcup t_A[S_A] = t_A(\alpha_1) - t_A(\alpha_2)$.
4. Similarly, since B is weakly extensible with respect to models, and $\beta_1, \beta_2 \in \Sigma_B$, we know that $\exists S_B \subseteq (\uparrow_B \beta_1)$ such that $\bigcup t_B[S_B] = t_B(\beta_1) - t_B(\beta_2)$.
5. Let $S = \{ \langle \alpha, \beta \rangle \mid (\alpha \in S_A \text{ and } \beta = \beta_1) \text{ or } (\alpha = \alpha_1 \text{ and } \beta \in S_B) \}$
6. (Claim b1)[$S \subseteq (\uparrow_{A+B} \langle \alpha_1, \beta_1 \rangle)$]
 - (a) Let $\langle \alpha, \beta \rangle \in S$.
 - (b) (Case 1) ($\alpha \in S_A$ and $\beta = \beta_1$)
 - i. Since $\alpha \in S_A$, $\alpha \succeq_A \alpha_1$.
 - ii. So $\langle \alpha, \beta \rangle \succeq_{A+B} \langle \alpha_1, \beta_1 \rangle$ ■(Case 1)
 - (c) (Case 2) ($\alpha = \alpha_1$ and $\beta \in S_B$)
 - i. Since $\beta \in S_B$, $\beta \succeq_B \beta_1$.
 - ii. So $\langle \alpha, \beta \rangle \succeq_{A+B} \langle \alpha_1, \beta_1 \rangle$ ■(Case 2) ■(Claim b1)
7. (Claim b2) [$\bigcup t_{A+B}[S] = t_{A+B}(\langle \alpha_1, \beta_1 \rangle) - t_{A+B}(\langle \alpha_2, \beta_2 \rangle)$]

(a) That claim is equivalent to:

$$\left[\text{Claim: } \bigcup_{\langle \alpha, \beta \rangle \in S} t_A(\alpha) \cap t_B(\beta) = (t_A(\alpha_1) \cap t_B(\beta_1)) - (t_A(\alpha_2) \cap t_B(\beta_2)) \right]$$

(b) ($\subseteq$)

- i. Let $m \in \bigcup_{\langle \alpha, \beta \rangle \in S} t_A(\alpha) \cap t_B(\beta)$.
- ii. So there is some $\langle \alpha, \beta \rangle \in S$ such that $m \in t_A(\alpha)$ and $m \in t_B(\beta)$.
- iii. (Case 1) ($\alpha \in S_A$ and $\beta = \beta_1$)
 - A. Since $\alpha \in S_A$, $m \in \bigcup t_A[S_A]$, so $m \in t_A(\alpha_1)$ and $m \notin t_A(\alpha_2)$.
 - B. So $m \in (t_A(\alpha_1) \cap t_B(\beta_1))$ and $m \notin (t_A(\alpha_2) \cap t_B(\beta_2))$ ■(Case 1)
- iv. (Case 2) ($\alpha = \alpha_1$ and $\beta \in S_B$)
 - A. Since $\beta \in S_B$, $m \in \bigcup t_B[S_B]$, so $m \in t_B(\beta_1)$ and $m \notin t_B(\beta_2)$.
 - B. So $m \in (t_A(\alpha_1) \cap t_B(\beta_1))$ and $m \notin (t_A(\alpha_2) \cap t_B(\beta_2))$ ■(Case 2) ■($\subseteq$)

(c) ($\supseteq$)

- i. Let $m \in (t_A(\alpha_1) \cap t_B(\beta_1)) - (t_A(\alpha_2) \cap t_B(\beta_2))$
- ii. So $m \in t_A(\alpha_1)$, $m \in t_B(\beta_1)$, and $m \notin (t_A(\alpha_2) \cap t_B(\beta_2))$
- iii. (Case 1) $m \notin t_A(\alpha_2)$
 - A. Since $m \in t_A(\alpha_1)$ and $m \notin t_A(\alpha_2)$, $m \in (t_A(\alpha_1) - t_A(\alpha_2))$, so $m \in \bigcup t_A[S_A]$.
 - B. So there is an $\alpha \in S_A$ such that $m \in t_A(\alpha)$.
 - C. Form the pair $\langle \alpha, \beta_1 \rangle$.
 - D. $\langle \alpha, \beta_1 \rangle$ is an element of S such that $m \in t_A(\alpha)$ and $m \in t_B(\beta_1)$ ■(Case 1)
- iv. (Case 2) $m \notin t_B(\beta_2)$
 - A. Since $m \in t_B(\beta_1)$ and $m \notin t_B(\beta_2)$, $m \in t_B(\beta_1) - t_B(\beta_2)$, so $m \in \bigcup t_B[S_B]$.
 - B. So there is a $\beta \in S_B$ such that $m \in t_B(\beta)$.
 - C. Form the pair $\langle \alpha_1, \beta \rangle$.
 - D. $\langle \alpha_1, \beta \rangle$ is an element of S such that $m \in t_A(\alpha_1)$ and $m \in t_B(\beta)$. ■(Case 2) ■($\supseteq$) ■(Claim b2) ■(Part b.) ■Proposition.

Proof of Proposition 90:

(Proposition 90): For any partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption Set PP (i.e. $B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $A + models(A)$ is a partially ordered truth medium satisfying assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$.

1. Let A be an arbitrary partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ arbitrary practices of categorization for A such that A satisfies assumption Set PP (i.e. $B1, A4, A5$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. By the *models* Set PP assumption preservation proposition (Proposition 84), $models(A)$ is a partially ordered truth medium satisfying Set PP with respect to $\mathcal{T}_{\Sigma_{models(A)}}$ and $\mathcal{T}_{M_{models(A)}}$.
3. We have $M_{models(A)} = M_A$ and $\mathcal{T}_{M_{models(A)}} = \mathcal{T}_{M_A}$ by the definition of *models*.
4. So by the $+$ Set PP assumption preservation proposition (Proposition 87), $A + models(A)$ is a partially ordered truth medium satisfying assumption Set PP with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$.

Proof of Proposition 91:

(Proposition 91): Given arbitrary partially ordered truth medium A , and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$.

1. Let A be an arbitrary partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that A satisfies monotonic extension of commitment with respect to models, and A is weakly extensible with respect to models, and A satisfies assumption set PP (i.e. $A4, A5, B1$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

2. $\left[\text{Show } \forall p, q \in \Sigma_{A+models(A)} \left(\begin{array}{c} p \preceq_{A+models(A)} q \text{ implies} \\ \mathcal{T}_{\Sigma_{A+models(A)}}(q) \subseteq \mathcal{T}_{\Sigma_{A+models(A)}}(p) \end{array} \right) \right]$
3. By the monotonic extension of commitment for *models* proposition (Proposition 85) we know that *models*(*A*) satisfies monotonic extension of commitment with respect to models.
4. By the + monotonic extension of commitment preservation proposition (Proposition 88) we know $A+models(A)$ satisfies monotonic extension of commitment with respect to models.
5. By the $A+models(A)$ Set PP assumption preservation proposition (Proposition 90) we know that $A+models(A)$ is a partially ordered truth medium satisfying assumption set *PP* with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$.
6. By Proposition 60, we have
 $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ satisfies monotonic extension of commitment with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$. ■ Proposition.

Proof of Proposition 92:

(Proposition 92): Given arbitrary partially ordered truth medium *A*, and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if *A* satisfies monotonic extension of commitment with respect to models,

and *A* is weakly extensible with respect to models, and *A* satisfies assumption set *PP* (i.e. *A4*, *A5*, *B1*) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$.

1. Let *A* be an arbitrary partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization, such that *A* satisfies monotonic extension of commitment with respect to models, and *A* is weakly extensible with respect to models, and *A* satisfies assumption set *PP* (i.e. *A4*, *A5*, *B1*) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. By the $A+models(A)$ Set PP assumption preservation proposition (Proposition 90), we know that $A+models(A)$ satisfies assumption set *PP* with respect to practices of categorization $\mathcal{T}_{\Sigma_{A+models(A)}}$ and $\mathcal{T}_{M_{A+models(A)}}$.

3. By the Weak extensibility of *models* proposition (Proposition 86), we know that $models(A)$ is weakly extensible with respect to models.
4. By the $+$ weak extensibility preservation proposition (Proposition 89), we know that $A + models(A)$ is weakly extensible with respect to models.
5. By Proposition 63, we know the partially ordered set $\langle \Sigma_{A+models(A)}, \preceq_{A+models(A)} \rangle$ is weakly extensible with respect to $\mathcal{T}_{\Sigma_{A+models(A)}}$. ■ Proposition.

Proof of Proposition 93:

(Proposition 93): Given a partially ordered truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, if A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, then $\forall p \in \Sigma_A, \mathcal{T}_{\Sigma_A}(p) = \mathcal{T}_{\Sigma_{A+models(A)}}(\langle p, \perp \rangle)$.

1. Let A be a partially ordered truth medium, and $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ practices of categorization such that A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
2. Let p be an arbitrary element of Σ_A .
3. Then $\langle p, \perp \rangle$ is an element of $\Sigma_{A+models(A)}$, since $p \in \Sigma_A$, and $\perp \in \Sigma_{models(A)}$.
4. Note that by definition of the $+$ operator,
$$\mathcal{T}_{\Sigma_{A+models(A)}}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \mathcal{T}_{\Sigma_{models(A)}}(\perp) = \mathcal{T}_{\Sigma_A}(p) \cap \bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A]].$$
5. By Proposition 6, we know that since A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, that A satisfies assumption $B4$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.
6. So $\bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A]] = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$ (Assumption $B4$).
7. So $\mathcal{T}_{\Sigma_{A+models(A)}}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A}(p) \cap \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$.
8. Since $p \in \Sigma_A$, $\mathcal{T}_{\Sigma_A}(p) \subseteq \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$.
9. So $\mathcal{T}_{\Sigma_{A+models(A)}}(\langle p, \perp \rangle) = \mathcal{T}_{\Sigma_A}(p)$. ■ Proposition.

Proof of Proposition 94:

Proposition 94: For all truth media A ,
if A satisfies assumption set PP (i.e. $B1, A4, A5$)

with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,

then $\text{pow}(A)$ is a partially ordered truth medium satisfying assumption set PP

with respect to practices of categorization $\mathcal{T}_{\Sigma_{\text{pow}(A)}}$ and $\mathcal{T}_{M_{\text{pow}(A)}}$.

Showing that $\text{pow}(A)$ is a truth medium.

$\text{pow}(A)$ has the structure of a truth medium. We need to show that $\Sigma_{\text{pow}(A)}$ and $M_{\text{pow}(A)}$ are non-empty. $\Sigma_{\text{pow}(A)}$ is non-empty because $\emptyset \in \mathcal{P}(\Sigma_A)$. We know that M_A is non-empty since A is a truth medium. So $M_{\text{pow}(A)}$ is non-empty since $M_{\text{pow}(A)} = M_A$. $t_{\text{pow}(A)}$ is a function $\Sigma_{\text{pow}(A)} \rightarrow \mathcal{P}(M_{\text{pow}(A)})$ by definition.

Showing that $\text{pow}(A)$ is a partially ordered truth medium.

$\preceq_{\text{pow}(A)}$ is a partial ordering since $\subseteq$ is.

Showing that $\text{pow}(A)$ satisfies assumptions $A4, A5$ with respect to practices of categorization $\mathcal{T}_{\Sigma_{\text{pow}(A)}}$ and $\mathcal{T}_{M_{\text{pow}(A)}}$.

This is immediate since $M_{\text{pow}(A)} = M_A$ and $\mathcal{T}_{M_{\text{pow}(A)}} = \mathcal{T}_{M_A}$ by the definition of pow . Substituting these equivalences into statements of $A4$ and $A5$ expressed relatively to $\text{pow}(A)$ and $\mathcal{T}_{M_{\text{pow}(A)}}$, yields $A4$ and $A5$ expressed relatively to A and $\mathcal{T}_{M_A}$, which was assumed.

Showing that $\text{pow}(A)$ satisfies assumption $B1$ with respect to practices of categorization $\mathcal{T}_{\Sigma_{\text{pow}(A)}}$ and $\mathcal{T}_{M_{\text{pow}(A)}}$.

Lemma 99 *Let A be an arbitrary truth medium satisfying assumption $A4$ with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Then $\forall X \subseteq \Sigma_A, X \neq \emptyset$ implies $\bigcup \mathcal{T}_{M_A} [\bigcap t_A [X]] = \bigcap_{x \in X} (\bigcup \mathcal{T}_{M_A} [t_A(x)])$.*

Let X be an arbitrary nonempty subset of Σ_A .

($\subseteq$)

1. Let a be an element of $\bigcup \mathcal{T}_{M_A} [\bigcap t_A [X]]$.
2. So $a \in \mathcal{T}_{M_A}(m)$ for some m , such that $m \in t_A(x)$ for every $x \in X$.
3. So $a \in \bigcup \mathcal{T}_{M_A} [t_A(x)]$ for every $x \in X$.
4. So $a \in \bigcap_{x \in X} (\bigcup \mathcal{T}_{M_A} [t_A(x)])$ ■ ($\subseteq$)

($\supseteq$)

1. Let a be an element of $\bigcap_{x \in X} (\bigcup \mathcal{T}_{M_A} [t_A(x)])$.
2. $\left[\text{Claim: there is some } m \in M_A \text{ such that } \left(\begin{array}{l} a \in \mathcal{T}_{M_A}(m) \text{ and} \\ \text{for every } x \in X, m \in t_A(x) \end{array} \right) \right]$
 - (a) Pick arbitrary $p \in X$. We know one exists since X is nonempty.
 - (b) By case ($\supseteq$) assumption, there is some $m \in t_A(p)$ such that $a \in \mathcal{T}_{M_A}(m)$.
 - (c) Now consider any $x \in X$. [Show $m \in t_A(x)$.]
 - (d) We know, by case ($\supseteq$) assumption, there is some $n \in t_A(x)$ such that $a \in \mathcal{T}_{M_A}(n)$.
 - (e) Since A satisfies $A4$ with respect to $\mathcal{T}_{M_A}$, we know that $m = n$.
 - (f) So $m \in t_A(x)$. Since x was an arbitrary element of X we have shown: for every $x \in X, m \in t_A(x)$. ■ Claim.
3. Given some $m \in M_A$ satisfying the claim, $m \in \bigcap t_A[X]$, and $a \in \mathcal{T}_{M_A}(m)$, so $a \in \bigcup \mathcal{T}_{M_A} [\bigcap t_A[X]]$ ■ ($\supseteq$) ■ Lemma.

($B1$)

Formally, $\left[\text{Show } \forall S \in \Sigma_{pow(A)} \left(\bigcup \mathcal{T}_{M_{pow(A)}} [t_{pow(A)}(S)] = \mathcal{T}_{\Sigma_{pow(A)}}(S) \right) \right]$

1. Let S be an arbitrary element of $\Sigma_{pow(A)}$.
2. $\left[\text{Show } \bigcup \mathcal{T}_{M_{pow(A)}} [t_{pow(A)}(S)] = \mathcal{T}_{\Sigma_{pow(A)}}(S) \right]$
3. Case 1: $S = \emptyset$, so $t_{pow(A)}(S) = \bigcup t_A[\Sigma_A]$ (by definition of pow)
 - (a) $\left[\text{Show } \bigcup \mathcal{T}_{M_{pow(A)}} [\bigcup t_A[\Sigma_A]] = \mathcal{T}_{\Sigma_{pow(A)}}(S) \right]$
 - (b) From the definition of pow , we know that $\mathcal{T}_{\Sigma_{pow(A)}}(S) = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$, and $\mathcal{T}_{M_{pow(A)}} = \mathcal{T}_{M_A}$. So the above claim to be shown becomes
 $\left[\text{Show } \bigcup \mathcal{T}_{M_A} [\bigcup t_A[\Sigma_A]] = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A] \right]$
 - (c) This is just assumption $B4$ with respect to truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. Since we have assumed that A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we apply Proposition 6 ($B1$ implies $B4$). ■ (Case 1).

4. Case 2: $S \neq \emptyset$ so $t_{pow(A)}(S) = \bigcap t_A[S]$ (by definition of pow)

(a) $\left[\text{Show } \bigcup \mathcal{T}_{M_{pow(A)}} [\bigcap t_A[S]] = \mathcal{T}_{\Sigma_{pow(A)}}(S) \right]$

(b) From the definition of pow , we know that $\mathcal{T}_{\Sigma_{pow(A)}}(S) = \bigcap \mathcal{T}_{\Sigma_A}[S]$ and $\mathcal{T}_{M_{pow(A)}} = \mathcal{T}_{M_A}$. So the above claim to be shown becomes

$$[\text{Show } \bigcup \mathcal{T}_{M_A} [\bigcap t_A[S]] = \bigcap \mathcal{T}_{\Sigma_A}[S]]$$

(c) Applying Lemma 99, this becomes

$$\left[\text{Show } \bigcap_{s \in S} (\bigcup \mathcal{T}_{M_A}[t_A(s)]) = \bigcap_{s \in S} \mathcal{T}_{\Sigma_A}(s) \right]$$

(d) Since truth medium A satisfies assumption $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $S \subseteq \Sigma_A$ we know that

$$\forall s \in S (\bigcup \mathcal{T}_{M_A}[t_A(s)] = \mathcal{T}_{\Sigma_A}(s)). \text{ This gives the desired result.}$$

■(Case 2).■(B1).■Proposition.

Proof of Proposition 96:

First we prove a lemma.

Lemma 100 For any truth medium A , $\bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}] = \bigcup t_A[\Sigma_A]$

Proof.

1. Let A be an arbitrary truth medium.

2. ($\subseteq$)

(a) Let $m \in \bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}]$

(b) So there is an $\langle i, \alpha \rangle \in \Sigma_{negcomp(A)}$ such that $m \in t_{negcomp(A)}(\langle i, \alpha \rangle)$.

(c) By the definition of $\Sigma_{negcomp(A)}$, $i = 1$ or $i = -1$, and $\alpha \in \Sigma_A$.

(d) (Case 1): $i = 1$

i. So $t_{negcomp(A)}(\langle i, \alpha \rangle) = t_A(\alpha)$.

ii. So $m \in t_A(\alpha)$.

iii. So $m \in \bigcup t_A[\Sigma_A]$. ■ (Case 1)

(e) (Case 2): $i = -1$

- i. So $t_{negcomp(A)}(\langle i, \alpha \rangle) = \bigcup t_A[\Sigma_A] - t_A(\alpha)$
- ii. So $m \in \bigcup t_A[\Sigma_A] - t_A(\alpha)$
- iii. So $m \in \bigcup t_A[\Sigma_A]$. ■ (Case 2) ■ ($\subseteq$)

3. ($\supseteq$)

- (a) Let $m \in \bigcup t_A[\Sigma_A]$.
- (b) So there is a $\sigma \in \Sigma_A$ such that $m \in t_A(\sigma)$.
- (c) Form $\langle 1, \sigma \rangle$.
- (d) $\langle 1, \sigma \rangle \in \Sigma_{negcomp(A)}$.
- (e) $t_{negcomp(A)}(\langle 1, \sigma \rangle) = t_A(\sigma)$ by the definition of $t_{negcomp(A)}$.
- (f) So $m \in t_{negcomp(A)}(\langle 1, \sigma \rangle)$.
- (g) So $m \in \bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}]$. ■ ($\supseteq$) ■ (Lemma)

(Proposition 96): Given any truth medium A , the truth medium $negcomp(A)$ is negation complete with respect to models.

1. Let A be an arbitrary truth-medium.

2. [Show that $negcomp(A)$ is negation complete with respect to models.]

3. That is,
$$\left[\begin{array}{c} \text{Show that for all } \langle i, \alpha \rangle \in \Sigma_{negcomp(A)}, \\ \exists R \subseteq \Sigma_{negcomp(A)} \text{ such that} \\ \bigcup t_{negcomp(A)}[R] = \bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}] - t_{negcomp(A)}(\langle i, \alpha \rangle) \end{array} \right]$$

4. Let $\langle i, \alpha \rangle$ be an arbitrary element of $\Sigma_{negcomp(A)}$.

5. By Lemma 100, we know that $\bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}] = \bigcup t_A[\Sigma_A]$.

6. So the claim to be shown becomes

[Show $\exists R \subseteq \Sigma_{negcomp(A)}$ such that $\bigcup t_{negcomp(A)}[R] = \bigcup t_A[\Sigma_A] - t_{negcomp(A)}(\langle i, \alpha \rangle)$]

7. By the definition of $\Sigma_{negcomp(A)}$, we know that $i = 1$ or $i = -1$, and that $\alpha \in \Sigma_A$.

8. (Case 1): $i = 1$

- (a) $t_{negcomp(A)}(\langle i, \alpha \rangle) = t_{negcomp(A)}(\langle 1, \alpha \rangle) = t_A(\alpha)$.
- (b) Let $R = \{\langle -1, \alpha \rangle\}$.
- (c) $R \subseteq \Sigma_{negcomp(A)}$, since $-1 \in \{1, -1\}$ and $\alpha \in \Sigma_A$.
- (d) $\bigcup t_{negcomp(A)}[R] = t_{negcomp(A)}(\langle -1, \alpha \rangle) = \bigcup t_A[\Sigma_A] - t_A(\alpha)$.
- (e) So $\bigcup t_{negcomp(A)}[R] = \bigcup t_A[\Sigma_A] - t_{negcomp(A)}(\langle i, \alpha \rangle)$. ■ (Case 1)

9. (Case 2): $i = -1$

- (a) $t_{negcomp(A)}(\langle i, \alpha \rangle) = \bigcup t_A[\Sigma_A] - t_A(\alpha)$.
- (b) Let $R = \{\langle 1, \alpha \rangle\}$.
- (c) $R \subseteq \Sigma_{negcomp(A)}$, since $1 \in \{1, -1\}$ and $\alpha \in \Sigma_A$.
- (d) $\bigcup t_{negcomp(A)}[R] = t_{negcomp(A)}(\langle 1, \alpha \rangle) = t_A(\alpha)$.
- (e) Claim: $t_A(\alpha) = \bigcup t_A[\Sigma_A] - (\bigcup t_A[\Sigma_A] - t_A(\alpha))$.
 - i. Since $\alpha \in \Sigma_A$, we have $t_A(\alpha) \subseteq \bigcup t_A[\Sigma_A]$.
 - ii. For all sets P, Q , if $P \subseteq Q$, then $P = Q - (Q - P)$. ■ Claim.
- (f) So $\bigcup t_{negcomp(A)}[R] = \bigcup t_A[\Sigma_A] - (\bigcup t_A[\Sigma_A] - t_A(\alpha)) = \bigcup t_A[\Sigma_A] - t_{negcomp(A)}(\langle i, \alpha \rangle)$. ■ (Case 2) ■ Proposition.

Proof of Proposition 97:

(Proposition 97): For all truth media A ,
 if A satisfies assumption set PP (i.e. $B1, A4, A5$)
 with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$,
 then $negcomp(A)$ is a truth medium satisfying assumption set PP
 with respect to practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$.

Let A be an arbitrary truth medium satisfying assumption set PP (i.e. $B1, A4, A5$) with respect to practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$.

Showing that $negcomp(A)$ is a truth medium.

$negcomp(A)$ has the structure of a truth medium. We need to show that $\Sigma_{negcomp(A)}$ and $M_{negcomp(A)}$ are non-empty. $\Sigma_{negcomp(A)}$ is non-empty because Σ_A is nonempty (A is a

truth medium), and $\Sigma_{negcomp(A)} = \{1, -1\} \times \Sigma_A$. $M_{negcomp(A)}$ is non-empty because M_A is nonempty (A is a truth medium), and $M_{negcomp(A)} = M_A$.

$t_{negcomp(A)}$ is a function $\Sigma_{negcomp(A)} \rightarrow \mathcal{P}(M_{negcomp(A)})$ by definition.

Showing that $negcomp(A)$ satisfies assumptions A4, A5 with respect to practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$.

This is immediate since $M_{negcomp(A)} = M_A$ and $\mathcal{T}_{M_{negcomp(A)}} = \mathcal{T}_{M_A}$ by the definition of $negcomp(A)$. Substituting these equivalences into statements of A4 and A5 expressed relatively to $negcomp(A)$ and $\mathcal{T}_{M_{negcomp(A)}}$, yields A4 and A5 expressed relatively to A and $\mathcal{T}_{M_A}$, which was assumed.

Showing that $negcomp(A)$ satisfies assumption B1 with respect to practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$.

Lemma 101 *If A is an arbitrary truth medium satisfying assumption A4 with respect to practice of categorization $\mathcal{T}_{M_A}$, then*

for all $\alpha \in \Sigma_A$, $\bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A] - t_A(\alpha)] = \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] - \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$

Let A be an arbitrary truth medium satisfying assumption A4. Let α be an arbitrary element of Σ_A .

($\subseteq$)

1. Let u be an element of $\bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A] - t_A(\alpha)]$
2. So $u \in \mathcal{T}_{M_A}(m)$ for some $m \in \bigcup t_A [\Sigma_A] - t_A(\alpha)$.
3. Then $m \in \bigcup t_A [\Sigma_A]$ and $m \notin t_A(\alpha)$.
4. So $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]]$.
5. [Show $u \notin \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$]

(a) Assume $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$. [Show contradiction.]

(b) So, there is an $n \in t_A(\alpha)$ such that $u \in \mathcal{T}_{M_A}(n)$.

(c) Since A satisfies assumption A4 with respect to practice of categorization $\mathcal{T}_{M_A}$, we know $m = n$.

(d) So $m \in t_A(\alpha)$. Contradiction.

6. So $u \notin \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$.

7. So $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]] - \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$. ■ ($\subseteq$)

($\supseteq$)

1. Say that $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A]]$, and $u \notin \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$.

2. So there is an $m \in \bigcup t_A [\Sigma_A]$ such that $u \in \mathcal{T}_{M_A} (m)$.

3. [Show that $m \notin t_A(\alpha)$]

4. Assume $m \in t_A(\alpha)$. [Show contradiction.]

(a) If $m \in t_A(\alpha)$, then since $u \in \mathcal{T}_{M_A} (m)$, $u \in \bigcup \mathcal{T}_{M_A} [t_A(\alpha)]$. Contradiction.

5. So $m \notin t_A(\alpha)$.

6. So $m \in \bigcup t_A [\Sigma_A] - t_A(\alpha)$

7. So $u \in \bigcup \mathcal{T}_{M_A} [\bigcup t_A [\Sigma_A] - t_A(\alpha)]$ ■ ($\supseteq$) ■ Lemma.

(B1)

$$\left[\text{Show } \forall \langle i, \alpha \rangle \in \Sigma_{negcomp(A)} \left(\bigcup \mathcal{T}_{M_{negcomp(A)}} [t_{negcomp(A)}(\langle i, \alpha \rangle)] = \mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle) \right) \right]$$
 Note that $\mathcal{T}_{M_{negcomp(A)}} = \mathcal{T}_{M_A}$, by definition of *negcomp*.

1. Let $\langle i, \alpha \rangle$ be an arbitrary element of $\Sigma_{negcomp(A)}$.

By the definition of $\Sigma_{negcomp(A)}$, we know that $i = 1$ or $i = -1$, and that $\alpha \in \Sigma_A$.

2. (Case 1): $i = 1$

(a) $t_{negcomp(A)}(\langle i, \alpha \rangle) = t_A(\alpha)$.

(b) $\mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle) = \mathcal{T}_{\Sigma_A}(\alpha)$.

(c) Since $\alpha \in \Sigma_A$ and we know that A satisfies B1 with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we know that $\bigcup \mathcal{T}_{M_A} [t_A(\alpha)] = \mathcal{T}_{\Sigma_A}(\alpha)$.

- (d) Substituting we get $\bigcup \mathcal{T}_{M_{negcomp(A)}} [t_{negcomp(A)}(\langle i, \alpha \rangle)] = \mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle)$
 ■ (Case 1)

3. (Case 2): $i = -1$

- (a) $t_{negcomp(A)}(\langle i, \alpha \rangle) = \bigcup t_A[\Sigma_A] - t_A(\alpha)$.
 (b) $\mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle) = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A] - \mathcal{T}_{\Sigma_A}(\alpha)$.
 (c) Since $\alpha \in \Sigma_A$ and we know that A satisfies $B1$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, we have $\bigcup \mathcal{T}_{M_A}[t_A(\alpha)] = \mathcal{T}_{\Sigma_A}(\alpha)$.
 (d) By Proposition 6, we know that A satisfies $B4$ with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. So $\bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A]] = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A]$
 (e) Putting the previous two step results together, we get $\bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A]] - \bigcup \mathcal{T}_{M_A}[t_A(\alpha)] = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A] - \mathcal{T}_{\Sigma_A}(\alpha)$.
 (f) By Lemma 101, we know that $\bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A]] - \bigcup \mathcal{T}_{M_A}[t_A(\alpha)] = \bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A] - t_A(\alpha)]$.
 (g) So $\bigcup \mathcal{T}_{M_A}[\bigcup t_A[\Sigma_A] - t_A(\alpha)] = \bigcup \mathcal{T}_{\Sigma_A}[\Sigma_A] - \mathcal{T}_{\Sigma_A}(\alpha)$.
 (h) So $\bigcup \mathcal{T}_{M_A}[t_{negcomp(A)}(\langle i, \alpha \rangle)] = \mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle)$.
 (i) Since $\mathcal{T}_{M_{negcomp(A)}} = \mathcal{T}_{M_A}$,
 we have $\bigcup \mathcal{T}_{M_{negcomp(A)}}[t_{negcomp(A)}(\langle i, \alpha \rangle)] = \mathcal{T}_{\Sigma_{negcomp(A)}}(\langle i, \alpha \rangle)$.
 ■ (Case 2) ■ (B1) ■ Proposition.

Chapter 7

Demonstrating that the Range of the Set *PP* Technique is a Subset of the Range of the Set *CL* Technique

7.1 Introduction

In this chapter, we will use the methodology presented in the proof of Chapter 6 to show that the range of applicability of the Set *PP* technique is a subset of the range of applicability of the Set *CL* technique, i.e. $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$. This proof consists in showing that the applications of the Set *CG* technique constructed by the methodology presented in Chapter 6 not only satisfy the Set *CG* technique-specific assumptions (*D1*, *D2*, *D3*), but satisfy assumption *LL* (the representational version of Lindenbaum's Lemma) as well.

7.2 Definition of desired results

Let us consider what is required to demonstrate the claim expressed in the chapter title. We can show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$ if we can show that:

$(PP \rightarrow CL)$ Given any application of the Set PP technique, we can construct an application of the Set CL technique such that the representational relation of logical consequence which was made intelligible by the given application of the Set PP technique is embedded within the representational relation of logical consequence made intelligible by the constructed application of the Set CL technique.

The argument that condition $(PP \rightarrow CL)$ is adequate to show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$ is parallel to that used in Chapter 6 to prove that condition $(PP \rightarrow CG)$ is adequate to show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$.

7.3 Overview of the proof

7.3.1 A simplified goal

We already know that $(PP \rightarrow CG)$ is true from Chapter 6. There, we described a methodology (the Stage 3 methodology of Section 6.3.5) which,

1. given any application of the Set PP technique (a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set PP technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$),
2. constructs from it
 - (a) an application of the Set CG technique (a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting the assertion types of D , such that D satisfies the Set CG technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$), and
 - (b) a function h embedding $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$ in $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$.

In this chapter, we will prove that $(PP \rightarrow CL)$ is true by showing that the application of the Set CG technique constructed by the Stage 3 methodology of Chapter 6 is in fact also an application of the Set CL technique. To show that result, it is sufficient to show that

the consistency medium D constructed by the Stage 3 methodology of Chapter 6 satisfies condition LL (the representational version of Lindenbaum's Lemma).

7.3.2 Demonstrating the goal

We want to show that the consistency medium D constructed by the Stage 3 methodology of Chapter 6 satisfies condition LL . The proof will take the following steps:

1. We will be given an arbitrary application of the Set PP technique.
2. To that application, we will apply the Stage 3 methodology constructing a consistency medium D . Note that applying the Stage 3 methodology will involve nested applications of the Stage 2 and Stage 1 methodologies. (Section 7.4)
3. We will then define a set $PM \subseteq \Sigma_D$. We will prove that PM is the set of maximal extensions of D (that is, $PM = Max_D$). (Section 7.5)
4. We will then show that every consistent assertion type of D has a maximal extension (that is, for every $p \in C_D$, there is an $x \in PM$ such that $x \succeq_D p$). (Section 7.6)

The above demonstrates that D satisfies LL .

7.4 Applying the Stage 3 methodology

We are given an application of the Set PP technique (a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion types and models of A , such that A satisfies the Set PP technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$).

The Stage 3 methodology constructs $negcomp(A)$, and associated practices of categorization $\mathcal{T}_{\Sigma_{negcomp(A)}}$ and $\mathcal{T}_{M_{negcomp(A)}}$; and then submits them to the Stage 2 methodology. The Stage 2 methodology constructs $pow(negcomp(A))$, and associated practices of categorization $\mathcal{T}_{\Sigma_{pow(negcomp(A))}}$ and $\mathcal{T}_{M_{pow(negcomp(A))}}$; and then submits them to the Stage 1 methodology. The Stage 1 methodology constructs $dm(pow(negcomp(A)))$ and $\mathcal{T}_{\Sigma_{dm(pow(negcomp(A)))}}$. These results are passed back to the Stage 2 methodology, which passes them back to the Stage 3 methodology. The consistency medium $dm(pow(negcomp(A)))$ and the practice of

categorization $\mathcal{T}_{\Sigma_{dm(pow(negcomp(A)))}}$ interpreting its assertion types constitute the application of the Set *CG* technology constructed by the Stage 3 methodology from A , $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$. In the following, we will refer to $dm(pow(negcomp(A)))$ as simply D . We will not have need to refer to $\mathcal{T}_{\Sigma_{dm(pow(negcomp(A)))}}$ since LL is defined in terms of the consistency medium alone and is independent of any particular interpretation of that medium.

Computing D is a straightforward application of the definitions of the operators dm , $+$, pow , and $negcomp$. There are two simplifying propositions.

Proposition 102 *For any truth medium A ,*

$$\bigcup t_{negcomp(A)} [\Sigma_{negcomp(A)}] = \bigcup t_A [\Sigma_A].$$

1. This follows from the definition of $negcomp$.

Proposition 103 *For any truth medium A ,*

$$\bigcup t_{pow(negcomp(A))} [\Sigma_{pow(negcomp(A))}] = \bigcup t_A [\Sigma_A].$$

1. This follows from the definition of pow and Proposition 102.

After simplifications, we can express D as follows (details are at the end of the chapter).

- $\Sigma_D = (\mathcal{P}(\Sigma_{negcomp(A)})) \times (\bigcup t_A [\Sigma_A] \cup \{\perp\})$, where $\perp \notin M_A$.

- $\preceq_D$ is defined as follows:

For all $\langle S_1, m_1 \rangle, \langle S_2, m_2 \rangle \in \Sigma_D$,

$\langle S_1, m_1 \rangle \preceq_D \langle S_2, m_2 \rangle$ iff $(S_1 \subseteq S_2 \text{ and } (m_1 = m_2 \text{ or } m_1 = \perp))$

$$\bullet C_D = \left\{ \begin{array}{l} \langle S, m \rangle \in \Sigma_D \mid \\ \left(\begin{array}{l} m = \perp \text{ and } \exists \langle S, m' \rangle \in \Sigma_D \\ \text{such that } m' \in t_{pow(negcomp(A))}(S) \end{array} \right) \text{ or } \\ (m \neq \perp \text{ and } m \in t_{pow(negcomp(A))}(S)) \end{array} \right\}$$

7.5 The set of maximal extensions

In this section, we define a set $PM \subseteq \Sigma_D$, and prove that PM is the set of maximal extensions of D (that is, $PM = Max_D$). We will define PM in terms of a function PS on

M_A , the set of models for the truth medium in the given application of the Set PP technique. For any model $m \in M_A$, $PS(m)$ is the set of all assertion types $\langle i, \alpha \rangle$ in $\Sigma_{negcomp(A)}$ for which m is a model of $\langle i, \alpha \rangle$.

$PS : M_A \rightarrow \mathcal{P}(\Sigma_{negcomp(A)})$ is defined as follows:

For $m \in M_A$, $PS(m) = \{\langle i, \alpha \rangle \in \Sigma_{negcomp(A)} \mid m \in t_{negcomp(A)}(\langle i, \alpha \rangle)\}$

Now we are ready to define PM , our proposed set of maximal elements for D . Recall that the assertion types of D are pairs, whose first element is a set of assertion types from $\Sigma_{negcomp(A)}$ and whose second element is either a model from the set $\bigcup t_A[\Sigma_A]$ or the distinguished element $\perp$. PM will contain exactly one element for every $m \in M_A$ for which m is a model of some assertion type in $\Sigma_{negcomp(A)}$. (That is, PM will contain one element for every $m \in \bigcup t_{negcomp(A)}[\Sigma_{negcomp(A)}]$ which we know by Proposition 102 to be equal to $\bigcup t_A[\Sigma_A]$). That element of PM will be a pair. Given that the second element is a model $m \in \bigcup t_A[\Sigma_A]$, the first element will be the set of all assertion types $\langle i, \alpha \rangle$ in $\Sigma_{negcomp(A)}$ for which m is a model of $\langle i, \alpha \rangle$, that is, the set $PS(m)$.

Now we define PM as follows:

$$PM = \{\langle PS(m), m \rangle \mid m \in \bigcup t_A[\Sigma_A]\}$$

Proposition 104 *The set PM as defined above is equal to Max_D .*

[Show $PM = Max_D$]

Recall that

$$Max_D = \left\{ \begin{array}{l} \langle S, m \rangle \in C_D \mid \\ \forall \langle S', m' \rangle \in \Sigma_D (\langle S', m' \rangle \succ_D \langle S, m \rangle \rightarrow \langle S', m' \rangle \notin C_D) \end{array} \right\}$$

($\subseteq$)

1. Let $\langle PS(m), m \rangle \in PM$. So $m \in \bigcup t_A[\Sigma_A]$.
2. [Show that $\langle PS(m), m \rangle$ is consistent, that is, $\langle PS(m), m \rangle \in C_D$]
 - (a) Since $PS(m) \subseteq \Sigma_{negcomp(A)}$ and $m \in \bigcup t_A[\Sigma_A]$; $\langle PS(m), m \rangle \in \Sigma_D$.
 - (b) $\left[\begin{array}{l} \text{Show that } m \in t_{pow(negcomp(A))}(PS(m)); \\ \text{this shows that } \langle PS(m), m \rangle \in C_D \text{ since } m \neq \perp \end{array} \right]$
 - i. (Case 1) $PS(m) = \emptyset$.

- A. This is not possible. Since $m \in \bigcup t_A[\Sigma_A]$, there is some $\alpha \in \Sigma_A$ such that $m \in t_A(\alpha)$. So $\langle 1, \alpha \rangle \in PS(m)$. ■ (Case 1).
- ii. (Case 2) $PS(m) \neq \emptyset$.
- A. $t_{pow(negcomp(A))}(PS(m)) = \bigcap t_{negcomp(A)}[PS(m)]$.
- B. For any $\langle i, \alpha \rangle \in PS(m)$, $m \in t_{negcomp(A)}(\langle i, \alpha \rangle)$ by the definition of PS .
- C. So $m \in t_{pow(negcomp(A))}(PS(m))$ ■ (Case 2).
3. $\left[\begin{array}{l} \text{Show that anything strictly greater than } \langle PS(m), m \rangle \text{ is inconsistent, that is,} \\ \forall \langle S', m' \rangle \in \Sigma_D (\langle S', m' \rangle \succ_D \langle PS(m), m \rangle \rightarrow \langle S', m' \rangle \notin C_D) \end{array} \right]$
- (a) Let $\langle S', m' \rangle \in \Sigma_D$ such that $\langle S', m' \rangle \succ_D \langle PS(m), m \rangle$.
- (b) Since $m \neq \perp$, we have $PS(m) \subset S'$ and $m = m'$.
- (c) [Show $\langle S', m \rangle \notin C_D$]
- i. $PS(m) \subset S'$, so there exists an $\langle i, \alpha \rangle \in S'$ such that $\langle i, \alpha \rangle \notin PS(m)$.
- ii. Then $m \notin t_{negcomp(A)}(\langle i, \alpha \rangle)$.
- iii. Since $S' \neq \emptyset$, $t_{pow(negcomp(A))}(S') = \bigcap t_{negcomp(A)}[S']$.
- iv. So $m \notin t_{pow(negcomp(A))}(S')$.
- v. Since $m \neq \perp$, $\langle S', m \rangle \notin C_D$.
4. So $\langle PS(m), m \rangle$ is maximal for D , that is, $\langle PS(m), m \rangle \in Max_D$. ■ ($\subseteq$)
- ($\supseteq$)
1. Let $\langle S, m \rangle$ be a maximal assertion type of D . So $\langle S, m \rangle$ is consistent, and anything strictly greater than $\langle S, m \rangle$ is inconsistent, that is,
- $\langle S, m \rangle \in C_D$ such that $\forall \langle S', m' \rangle \in \Sigma_D (\langle S', m' \rangle \succ_D \langle S, m \rangle \rightarrow \langle S', m' \rangle \notin C_D)$.
2. [Show $\langle S, m \rangle \in PM$, that is, show that $m \in \bigcup t_A[\Sigma_A]$ and $S = PS(m)$]
- (a) [Show that $m \in \bigcup t_A[\Sigma_A]$]
- i. We know that $m \in \bigcup t_A[\Sigma_A] \cup \{\perp\}$. Assume that $m = \perp$.
- [Show a contradiction.]

- ii. $\langle S, \perp \rangle \in C_D$ implies $\exists \langle S, m' \rangle \in \Sigma_D$ such that $m' \in t_{pow(negcomp(A))}(S)$.
 - iii. Since $t_{pow(negcomp(A))}(S) \subseteq \bigcup t_A [\Sigma_A]$
(applying Proposition 103), $m' \in \bigcup t_A [\Sigma_A]$.
 - iv. Then $\langle S, m' \rangle \in C_D$, since $m' \neq \perp$, and $m' \in t_{pow(negcomp(A))}(S)$.
 - v. But $\langle S, m' \rangle \succ_D \langle S, m \rangle$. This contradicts the claim that $\langle S, m \rangle \in Max_D$. So $m \in \bigcup t_A [\Sigma_A]$.
- (b) [Show that $S = PS(m)$]
- i. ($\subseteq$)
 - A. Let $\langle i, \alpha \rangle \in S$.
 - B. Since $\langle S, m \rangle \in C_D$ and $m \neq \perp$, $m \in t_{pow(negcomp(A))}(S)$.
 - C. Since $S \neq \emptyset$, $t_{pow(negcomp(A))}(S) = \bigcap t_{negcomp(A)} [S]$.
 - D. So for every $\langle i, \alpha \rangle \in S$, $m \in t_{negcomp(A)}(\langle i, \alpha \rangle)$.
 - E. So $\langle i, \alpha \rangle \in PS(m)$. ■ ($\subseteq$)
 - ii. ($\supseteq$)
 - A. Let $\langle i, \alpha \rangle \in PS(m)$.
 - B. Assume $\langle i, \alpha \rangle \notin S$. [Show contradiction.]
 - C. Since $\langle S, m \rangle \in C_D$ and $m \neq \perp$, $m \in t_{pow(negcomp(A))}(S)$.
 - D. [Show $\langle S, m \rangle \prec_D \langle PS(m), m \rangle$]
 - E. (Case 1) $S = \emptyset$.
 $\langle S, m \rangle \prec_D \langle PS(m), m \rangle$. ■ (Case 1).
 - F. (Case 2) $S \neq \emptyset$.
 $t_{pow(negcomp(A))}(S) = \bigcap t_{negcomp(A)} [S]$.
 So $\forall \langle j, \beta \rangle \in S$, $m \in t_{negcomp(A)}(\langle j, \beta \rangle)$.
 So $S \subseteq PS(m)$.
 So $\langle S, m \rangle \prec_D \langle PS(m), m \rangle$. ■ (Case 2).
 - G. Since $\langle PS(m), m \rangle \in C_D$ (by the same argument as that in main level $\subseteq$ step 2), $\langle S, m \rangle$ is not maximal. Contradiction.
 - H. So $\langle i, \alpha \rangle \in S$. ■ ($\supseteq$)

3. So $\langle S, m \rangle \in PM.$ ■ ($\supseteq$) ■ Proposition.

7.6 Demonstrating that D satisfies LL

Now we show that D satisfies LL . To show this, we need to show that every consistent assertion type of D has a maximal extension.

Proposition 105 *Every consistent assertion type of D has a maximal extension. That is, for every $\langle S, m \rangle \in C_D$, there is a $\langle S', m' \rangle \in PM$ such that $\langle S', m' \rangle \succeq_D \langle S, m \rangle$.*

1. Let $\langle S, m \rangle \in C_D$.

2. (Case 1) $m \neq \perp$

(a) [Show $\langle PS(m), m \rangle$ is a maximal extension of $\langle S, m \rangle$]

(b) $m \in t_{pow(negcomp(A))}(S)$.

(c) [Show $S \subseteq PS(m)$]

(d) (Case 1a) $S = \emptyset$

i. $S \subseteq PS(m)$ trivially. ■ (Case 1a)

(e) (Case 1b) $S \neq \emptyset$

i. $t_{pow(negcomp(A))}(S) = \bigcap t_{negcomp(A)}[S]$.

ii. So $\forall \langle i, \alpha \rangle \in S, m \in t_{negcomp(A)}(\langle i, \alpha \rangle)$.

iii. So $S \subseteq PS(m)$ by the definition of PS . ■ (Case 1b)

(f) So $\langle S, m \rangle \preceq_D \langle PS(m), m \rangle$ and $\langle PS(m), m \rangle \in PM.$ ■ (Case 1)

3. (Case 2) $m = \perp$

(a) Then $\exists \langle S, m' \rangle \in \Sigma_D$ such that $m' \in t_{pow(negcomp(A))}(S)$.

(b) Applying Proposition 103, $m' \in \bigcup t_A[\Sigma_A]$.

(c) So $\langle S, m' \rangle \in C_D$, and $\langle S, m \rangle \preceq_D \langle S, m' \rangle$.

- (d) By Case 1, we know that given an $\langle S, m' \rangle \in C_D$, such that $m' \neq \perp$, $\langle PS(m'), m' \rangle$ is a maximal extension of $\langle S, m' \rangle$. By transitivity, $\langle PS(m'), m' \rangle$ is a maximal extension of $\langle S, m \rangle$ as well.

■(Case 2) ■Proposition.

7.7 Conclusion

We have shown that the consistency medium D constructed by the Stage 3 methodology of Chapter 6 satisfies condition LL . This shows that the application of the Set CG technique constructed by the Stage 3 methodology of Chapter 6 is in fact also an application of the Set CL technique (since D satisfies LL as well as satisfying $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$). In doing so, we have shown that $(PP \rightarrow CL)$ is true, and therefore, that the range of applicability of the Set PP technique is a subset of the range of applicability of the Set CL technique, i.e. $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$.

This concludes the linear text of the chapter. The remainder of the material in this chapter are the details of the definition of D .

7.8 Details of the definition of D

We begin with $A = \langle \Sigma_A, M_A, t_A \rangle$

In Stage 3, we apply the operator $negcomp$, yielding $negcomp(A)$. (We don't need to consider the practices of categorization since LL is defined solely in terms of the consistency medium).

- $\Sigma_{negcomp(A)} = \{1, -1\} \times \Sigma_A$
- $M_{negcomp(A)} = M_A$
- $t_{negcomp(A)}$ is defined for any $\langle i, \alpha \rangle \in \Sigma_{negcomp(A)}$ as follows:
if $i = 1$ then $t_A(\alpha)$ else $\bigcup t_A[\Sigma_A] - t_A(\alpha)$

Stage 3 submits $negcomp(A)$ to Stage 2.

Stage 2 applies the operator pow to $negcomp(A)$, yielding $pow(negcomp(A))$.

- $\Sigma_{pow(negcomp(A))} = \mathcal{P}(\{1, -1\} \times \Sigma_A)$
- $\preceq_{pow(negcomp(A))}$ is defined as the inclusion order on $\Sigma_{pow(negcomp(A))}$, that is,
for all $P, Q \in \Sigma_{pow(negcomp(A))}$, $P \preceq_{pow(negcomp(A))} Q$ just in case $P \subseteq Q$
- $M_{pow(negcomp(A))} = M_A$
- $t_{pow(negcomp(A))}$ is defined for any $S \in \Sigma_{pow(negcomp(A))}$ as follows:
if $S = \emptyset$ then $\bigcup t_A[\Sigma_A]$ else $\bigcap t_{negcomp(A)}[S]$ (this applies Proposition 102)

Stage 2 submits $pow(negcomp(A))$ to Stage 1.

Stage 1 applies the operator dm to $pow(negcomp(A))$, constructing the consistency medium $dm(pow(negcomp(A)))$. The construction involves applying the operators $models$ and $+$, constructing the truth medium $pow(negcomp(A)) + models(pow(negcomp(A)))$.

First we consider $models(pow(negcomp(A)))$.

- $\Sigma_{models(pow(negcomp(A)))} = \bigcup t_A[\Sigma_A] \cup \{\perp\}$, where $\perp \notin M_A$ (Apply Proposition 103).
- $\preceq_{models(pow(negcomp(A)))}$ is defined as follows,
for all $p, q \in \Sigma_{models(pow(negcomp(A)))}$, $p \preceq_{models(pow(negcomp(A)))} q$ just in case $p = q$ or $p = \perp$
- $M_{models(pow(negcomp(A)))} = M_A$
- $t_{models(pow(negcomp(A)))}$ is defined for any $p \in \Sigma_{models(pow(negcomp(A)))}$ as follows:
if $p = \perp$ then $\bigcup t_A[\Sigma_A]$ else $\{p\}$ (this applies Proposition 103)

Now we apply the operator $+$ to construct
 $pow(negcomp(A)) + models(pow(negcomp(A)))$,
which we abbreviate as “ $p + m$ ”.

- $\Sigma_{p+m} = \mathcal{P}(\{1, -1\} \times \Sigma_A) \times (\bigcup t_A[\Sigma_A] \cup \{\perp\})$, where $\perp \notin M_A$
- $\preceq_{p+m}$ is defined as follows:
for all $\langle S_1, m_1 \rangle, \langle S_2, m_2 \rangle \in \Sigma_{p+m}$,
 $\langle S_1, m_1 \rangle \preceq_{p+m} \langle S_2, m_2 \rangle$ iff $(S_1 \subseteq S_2 \text{ and } (m_1 = m_2 \text{ or } m_1 = \perp))$

- $M_{p+m} = M_A$
- t_{p+m} is defined for any $\langle S, m \rangle \in \Sigma_{p+m}$ as follows:
 (if $S = \emptyset$ then $\bigcup t_A [\Sigma_A]$ else $\bigcap t_{negcomp(A)} [S]$) $\cap$ (if $m = \perp$ then $\bigcup t_A [\Sigma_A]$ else $\{m\}$)

Now we apply the operator dm to $pow(negcomp(A))$, constructing the consistency medium $dm(pow(negcomp(A)))$, which we will refer to as D .

- $\Sigma_D = \Sigma_{p+m}$

- $\preceq_D = \preceq_{p+m}$

- $C_D = \left\{ \begin{array}{l} \langle S, m \rangle \in \Sigma_D \mid \\ \left(\begin{array}{l} m = \perp \text{ and } \exists \langle S, m' \rangle \in \Sigma_D \\ \text{such that } m' \in t_{pow(negcomp(A))}(S) \end{array} \right) \text{ or } \\ \left(m \neq \perp \text{ and } m \in t_{pow(negcomp(A))}(S) \right) \end{array} \right\}$

Chapter 8

Demonstrating that the Range of the Set *CG* Technique is a Subset of the Range of the Set *BE* Technique

8.1 Introduction

In this chapter, we will demonstrate that any interpreted set of assertion types in the range of applicability of the Set *CG* technique is in the range of applicability of the Set *BE* technique. Intuitively, this result means that any interpreted set of assertion types for which we can make the representational concept of logical consequence explanatorily intelligible by an application of the Set *CG* technique, is one for which we can do the same with the Set *BE* technique. This result is one element of the more global picture we have developed organizing all the techniques under consideration into a linear order on the basis of their relative ranges of applicability.

Here is the structure of the chapter. We begin by discussing in detail just what the claim to be proven means, and what constitutes the desired result. The central concept in the proof will be a methodology for taking the elements of an application of the Set

CG technique and constructing from them an application of the Set *BE* technique, such that the relation of consequence which was made intelligible by the given application of the Set *CG* technique is also made intelligible by the constructed application of the Set *BE* technique. This methodology will take the form of an operator on interpreted consistency media which will have as its results interpreted truth media.

Since in this case, we are “moving” a relation of consequence from a consistency medium to a truth medium, the methodology will need to construct a set of models. The concept of “consequence sets” will enable us to carry out the construction. We will introduce the idea of consequence sets, explore some of its properties, and then show how we can use that concept to define the desired operator. We will conclude the chapter by giving proofs showing that the defined operator achieves the result we have set out for it.

8.2 Definition of desired results

Let us consider what is required to demonstrate the claim expressed in the chapter title.

We can show that $ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$ if we can show that:

$(CG \rightarrow BE)$ Given any application of the Set *CG* technique, we can construct an application of the Set *BE* technique such that the representational relation of logical consequence which was made intelligible by the given application of the Set *CG* technique is embedded within the representational relation of logical consequence made intelligible by the constructed application of the Set *BE* technique.

The argument that condition $(CG \rightarrow BE)$ is adequate to show that $ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$ is parallel to that used in Chapter 6 to prove that condition $(PP \rightarrow CG)$ is adequate to show that $ra(\text{Set } PP) \subseteq ra(\text{Set } CG)$.

To demonstrate $(CG \rightarrow BE)$, we shall describe a methodology which, given any application of the Set *CG* technique (a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting the assertion types of D , such that D satisfies the Set *CG* technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_D}$), constructs from it an application of the Set *BE* technique (a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$ interpreting the assertion

types and models of A , such that A satisfies the Set BE technique-specific assumptions with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$), and a function h embedding $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ in $ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}$.

8.3 Structure of the proof

8.3.1 Reviewing the given

We are given an application of the Set CG technique. This consists of a consistency medium $D = \langle \Sigma_D, \preceq_D, C_D \rangle$ and a practice of categorization $\mathcal{T}_{\Sigma_D}$ for which it is assumed that D satisfies Set CG ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$.

8.3.2 Looking ahead to the end result

Our goal will be to define an operator tm which given a consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$ such that D satisfies Set CG ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$, constructs a truth medium $tm(D)$ and practices of categorization $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$ interpreting the assertion types and models of $tm(D)$ such that

1. an image of $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ is embedded within $ALC_{\langle \Sigma_{tm(D)}, \mathcal{T}_{\Sigma_{tm(D)}} \rangle}$
(that is, there is a function $h : \Sigma_D \rightarrow \Sigma_{tm(D)}$ such that
 $\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{tm(D)}, \mathcal{T}_{\Sigma_{tm(D)}} \rangle}(h(p), h(q)) \right)$); and
2. truth medium $tm(D)$ satisfies assumption Set BE ($B1, B2, A7$) with respect to $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$.

8.3.3 Preparing the way

Our presentation of the proof will take the following course. First, we will introduce a new concept: the concept of “**consequence-sets.**” We will then develop some of the properties of consequence sets. Following that, we will use the concept of consequence sets to define the operator tm , and then prove that the operator tm meets the conditions characterized above.

8.4 Consequence sets

8.4.1 The function $Cons$

The function $Cons$ is defined in relation to some consistency medium D . A subscript is used to indicate the name of the appropriate medium, e.g. $Cons_D$. For any assertion type $p \in \Sigma_D$, $Cons_D(p)$ is the set of LC -consequences of p relative to D . Given a consistency medium D , we define $Cons_D : \Sigma_D \rightarrow \mathcal{P}(\Sigma_D)$ as follows:

for any $p \in \Sigma_D$, $Cons_D(p) = \{q \in \Sigma_D \mid LC_D(p, q)\}$

Recall that $LC_D(p, q)$ is equivalent to $CC_D(p) \subseteq CC_D(q)$. For more details, see Section 4.4.3.

For any assertion type $p \in \Sigma_D$, we call $Cons_D(p)$ the **consequence set** of p with respect to D .

8.4.2 Truth-equivalence and $Cons$

Given some set of assertion types Σ and a practice of categorization $\mathcal{T}_\Sigma$ interpreting Σ , two assertion types $p, q \in \Sigma$ are **truth-equivalent** with respect to $\mathcal{T}_\Sigma$, if they represent exactly the same set of possibilities, that is, if $\mathcal{T}_\Sigma(p) = \mathcal{T}_\Sigma(q)$. This proposition claims that if a consistency medium satisfies Set CG with respect to the practice of categorization interpreting its assertion types, then two assertion types are truth equivalent just in case they share the same consequence sets.

Proposition 106 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies Set CG ($D1, D2$, and $D3$) with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p, q \in \Sigma_D$, $\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_D}(q)$ iff $Cons_D(p) = Cons_D(q)$.*

Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$ a practice of categorization, such that D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.

Let p, q be arbitrary elements of Σ_D .

($\rightarrow$)

1. Assume $\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_D}(q)$. [Show $Cons_D(p) = Cons_D(q)$]

2. So $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$ and $\mathcal{T}_{\Sigma_D}(q) \subseteq \mathcal{T}_{\Sigma_D}(p)$.
3. Since D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$ we can apply the Intelligibility of Consequence Theorem for Consistency Media (Theorem 55) twice, yielding $LC_D(p, q)$ and $LC_D(q, p)$.
4. By Proposition 49, LC is transitive, so $LC_D(p, q)$ implies $Cons_D(q) \subseteq Cons_D(p)$; and $LC_D(q, p)$ implies $Cons_D(p) \subseteq Cons_D(q)$.
5. So $Cons_D(p) = Cons_D(q)$. ■ ($\rightarrow$)

($\leftarrow$)

1. Assume $Cons_D(p) = Cons_D(q)$ [Show $\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_D}(q)$]
2. ($\subseteq$)
 - (a) By Proposition 48, we know that LC is reflexive. So $LC_D(q, q)$.
 - (b) So $q \in Cons_D(q)$, by the definition of $Cons$.
 - (c) So $q \in Cons_D(p)$, since $Cons_D(p) = Cons_D(q)$.
 - (d) So $LC(p, q)$, by the definition of $Cons$.
 - (e) Given that D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, we can apply the Intelligibility of Consequence Theorem for Consistency Media (Theorem 55), yielding $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$. ■ ($\subseteq$)
3. ($\supseteq$) Same as ($\subseteq$) but with p and q exchanged. ■ ($\supseteq$) ■ ($\leftarrow$) ■ Proposition.

8.4.3 The set of unique consequence sets

Given a consistency medium D interpreted via a practice of categorization $\mathcal{T}_{\Sigma_D}$, each element σ of Σ_D picks out some set of possibilities $\mathcal{T}_{\Sigma_D}(\sigma)$. Some elements of Σ_D will pick out the same set of possibilities (i.e. be truth-equivalent). From the proof above, we have seen that, providing that D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, this is exactly when the assertion types involved have the same consequence sets with respect to D .

Let us define the set Ω_D as follows: for any consistency medium D , $\Omega_D = \text{Cons}_D[\Sigma_D]$. So Ω_D is the set of unique consequence sets for the assertion types of D . Given the results of Proposition 106, if consistency medium D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then the elements of Ω_D correspond on a one-to-one basis with the differentiable sets of possibilities represented by the assertion types of D (those sets of possibilities being the elements of $\mathcal{T}_{\Sigma_D}[\Sigma_D]$). We will use the elements of Ω_D as the models in our construction of $tm(D)$ from D .

8.4.4 A simple result about consequence sets

Recognizing this fact is important in our definition of tm .

Proposition 107 *The null set ($\emptyset$) cannot be a member of $\text{Cons}_D[\Sigma_D]$.*

Proof: If $\emptyset \in \text{Cons}_D[\Sigma_D]$, then there exists a $p \in \Sigma_D$ such that $\text{Cons}_D(p) = \emptyset$. But since LC is reflexive (Proposition 48), $\forall p \in \Sigma_D, p \in \text{Cons}_D(p)$. So $\forall p \in \Sigma_D, \text{Cons}_D(p) \neq \emptyset$.

8.5 The operator tm

We are now ready to define an operator tm which satisfies the conditions described in Section 8.3.2.

Given an arbitrary consistency medium D , with practice of categorization $\mathcal{T}_{\Sigma_D}$, we define the truth medium $tm(D)$ and derived practices of categorization $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$ as follows:

8.5.1 Truth medium $tm(D)$:

- $\Sigma_{tm(D)} = \Sigma_D$

The assertion types for the constructed truth medium are identical to the assertion types from the given consistency medium.

- $M_{tm(D)} = \Omega_D$

The models for the constructed truth medium are the consequence sets for the as-

sertion types from the given consistency medium (the set $Cons_D[\Sigma_D]$). $Cons_D$ is the closure of the relation LC , the proxy for consequence relation for the Set CG technique.

- $t_{tm(D)} : \Sigma_{tm(D)} \rightarrow \mathcal{P}(M_{tm(D)})$ is defined as follows:

for any $\sigma \in \Sigma_{tm(D)}$, $t_{tm(D)}(\sigma) = \{S \in M_{tm(D)} \mid \sigma \in S\}$

The relation of truth-in-a-model for the constructed truth medium is simply set membership. An assertion type σ is true-in-a-model S just in case σ is a member of S . Remember that S is the consequence set of some assertion type in the given consistency medium.

8.5.2 Derived practices $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$:

- $\mathcal{T}_{\Sigma_{tm(D)}} = \mathcal{T}_{\Sigma_D}$

The assertion types for the constructed truth medium (which are identical to the assertion types from the given consistency medium) are interpreted identically as well.

- $\mathcal{T}_{M_{tm(D)}} : M_{tm(D)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:

for any $S \in M_{tm(D)}$, $\mathcal{T}_{M_{tm(D)}}(S) = \bigcap \mathcal{T}_{\Sigma_D}[S]$

The set of possibilities represented by a model S in the constructed truth medium (which is the consequence set of some assertion type in the given consistency medium) is the intersection of the sets of possibilities represented by the members of S , as they were interpreted in the given consistency medium. Note that we don't have to consider the case where S is null, since no consequence set can be null (Proposition 107).

8.6 Proof of properties of tm

8.6.1 Some supporting propositions

Given a consistency medium satisfying the Set CG assumptions with respect to a practice of categorization by which it is interpreted, it is the case that, for any assertion type p , the possibilities represented by the consequence set of p are equal to the possibilities represented

by p .

Proposition 108 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p \in \Sigma_D, \mathcal{T}_{M_{tm}(D)}(Cons_D(p)) = \mathcal{T}_{\Sigma_D}(p)$.*

1. Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$ a practice of categorization, such that D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.
2. Let p be an arbitrary element of Σ_D .
3. [Show $\mathcal{T}_{M_{tm}(D)}(Cons_D(p)) = \mathcal{T}_{\Sigma_D}(p)$]
4. Note: $\mathcal{T}_{M_{tm}(D)}(Cons_D(p)) = \bigcap \mathcal{T}_{\Sigma_D}[Cons_D(p)]$
5. ($\subseteq$)
 - (a) Since $p \in Cons_D(p)$ (by reflexivity of LC , Proposition 48), $\bigcap \mathcal{T}_{\Sigma_D}[Cons_D(p)] \subseteq \mathcal{T}_{\Sigma_D}(p)$ ■ ($\subseteq$)
6. ($\supseteq$)
 - (a) Let $q \in Cons_D(p)$.
 - (b) By the definition of $Cons$, $LC_D(p, q)$.
 - (c) By the Intelligibility of Consequence Theorem for Consistency Media (Theorem 55), $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$.
 - (d) Since q was arbitrary, we have $\forall q \in Cons_D(p), \mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$.
 - (e) So $\mathcal{T}_{\Sigma_D}(p) \subseteq \bigcap \mathcal{T}_{\Sigma_D}[Cons_D(p)]$. ■ ($\supseteq$) ■ Proposition.

Given a consistency medium satisfying the Set CG assumptions with respect to a practice of categorization by which it is interpreted, it is the case that, for any assertion type and any consequence set, the assertion type is a member of the consequence set if and only if the possibilities represented by the consequence set are a subset of the possibilities represented by the assertion type.

Proposition 109 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p \in \Sigma_D, \forall S \in \Omega_D, p \in S$ iff $\mathcal{T}_{M_{tm}(D)}(S) \subseteq \mathcal{T}_{\Sigma_D}(p)$.*

1. Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$ a practice of categorization, such that D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.
2. Let p be an arbitrary element of Σ_D .
3. Let $S \in \Omega_D$, that is there exists a $\theta \in \Sigma_D$ such that $S = Cons_D(\theta)$.
4. [Show $p \in S$ iff $\mathcal{T}_{M_{tm}(D)}(S) \subseteq \mathcal{T}_{\Sigma_D}(p)$.]
5. $(\rightarrow)$
 - (a) Assume $p \in S$.
 - (b) Since $\mathcal{T}_{M_{tm}(D)}(S) = \bigcap \mathcal{T}_{\Sigma_D}[S]$, $\mathcal{T}_{M_{tm}(D)}(S) \subseteq \mathcal{T}_{\Sigma_D}(p)$. ■ $(\rightarrow)$
6. $(\leftarrow)$
 - (a) Assume $\mathcal{T}_{M_{tm}(D)}(S) \subseteq \mathcal{T}_{\Sigma_D}(p)$.
 - (b) [Show $p \in S$]
 - (c) By Proposition 108 we know that $\mathcal{T}_{M_{tm}(D)}(Cons_D(\theta)) = \mathcal{T}_{\Sigma_D}(\theta)$, so $\mathcal{T}_{M_{tm}(D)}(S) = \mathcal{T}_{\Sigma_D}(\theta)$.
 - (d) So $\mathcal{T}_{\Sigma_D}(\theta) \subseteq \mathcal{T}_{\Sigma_D}(p)$.
 - (e) Since D satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_D}$, we have $LC_D(\theta, p)$, by the Intelligibility of Consequence Theorem for Consistency Media (Theorem 55).
 - (f) So $p \in S$. ■ $(\leftarrow)$ ■ Proposition.

8.6.2 The tm Set BE assumptions proposition

Here we show that given an application of the Set CG technique, the operator tm constructs an application of the Set BE technique.

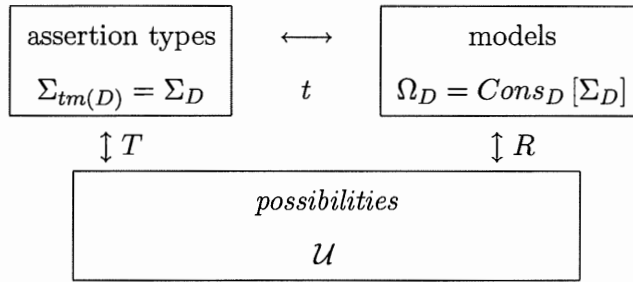
Proposition 110 *Given that consistency medium D satisfies Set CG ($D1, D2$, and $D3$) with respect to practice of categorization $\mathcal{T}_{\Sigma_D}$, truth medium $tm(D)$ satisfies Set BE ($B1, B2, A7$) with respect to practices of categorization $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$.*

Proof:

Let D be an arbitrary consistency medium, and $\mathcal{T}_{\Sigma_D}$ a practice of categorization, such that D satisfies $D1, D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.

(B1)

We have seen previously (in Section 2.3.6) that assumption $B1$ is equivalent to a principle of commutation for relations. Let's redraw the picture presented there as it relates to our current proof.



1. a relation T on $\mathcal{U} \times \Sigma_D$ such that $u T p$ iff $u \in \mathcal{T}_{\Sigma_D}(p)$;
2. a relation R on $\mathcal{U} \times M$ such that $u R m$ iff $u \in \mathcal{T}_{M_{tm(D)}}(m)$;
3. a relation t on $M \times \Sigma$ such that $m t p$ iff $m \in t_{tm(D)}(p)$.

Then we can express $B1$ in the form of a constraint:

T = the relational composition of t and R in that order: i.e.

$u T p$ iff there exists an m such that $u R m$ and $m t p$

We will show that this constraint is satisfied as follows:

($\leftarrow$)

By definition of tm , $m t p$ iff $p \in m$.

By Proposition 109, $p \in m$ iff $(u R m \text{ implies } u T p)$ ■ ($\leftarrow$)

($\rightarrow$)

By Proposition 108, $u T p$ iff $u R Cons_D(p)$. And since $p \in Cons_D(p)$ (LC is reflexive), $Cons_D(p) t p$. ■ ($\rightarrow$) ■ Constraint equivalent to $B1$.

We now give a different form of the proof of $B1$, followed by the proofs of $B2$ and $A7$.

1. $\left[\text{Show } \forall p \in \Sigma_{tm(D)} \left(\bigcup \mathcal{T}_{M_{tm(D)}} [t_{tm(D)}(p)] = \mathcal{T}_{\Sigma_{tm(D)}}(p) \right) \right]$

2. Since $\Sigma_{tm(D)} = \Sigma_D$, $\mathcal{T}_{\Sigma_{tm(D)}} = \mathcal{T}_{\Sigma_D}$, this is equivalent to

$\left[\text{Show } \forall p \in \Sigma_D \left(\bigcup \mathcal{T}_{M_{tm(D)}} [t_{tm(D)}(p)] = \mathcal{T}_{\Sigma_D}(p) \right) \right]$

3. Let p be an arbitrary element of Σ_D .

4. By definition tm , $t_{tm(D)}(p) = \{S \in \Omega_D \mid p \in S\}$

5. ($\subseteq$)

(a) Let $S \in t_{tm(D)}(p)$.

(b) So $S \in \Omega_D$ such that $p \in S$.

(c) By Proposition 109, $\mathcal{T}_{M_{tm(D)}}(S) \subseteq \mathcal{T}_{\Sigma_D}(p)$.

(d) So $\bigcup \mathcal{T}_{M_{tm(D)}} [t_{tm(D)}(p)] \subseteq \mathcal{T}_{\Sigma_D}(p)$. ■ ($\subseteq$)

6. ($\supseteq$)

(a) We know that $p \in Cons_D(p)$, since LC -consequence is reflexive (Proposition 48).

(b) So $Cons_D(p) \in t_{tm(D)}(p)$.

(c) By Proposition 108, we know that $\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{M_{tm(D)}}(Cons_D(p))$.

(d) So $\mathcal{T}_{\Sigma_D}(p) \subseteq \bigcup \mathcal{T}_{M_{tm(D)}} [t_{tm(D)}(p)]$ ■ ($\supseteq$) ■ ($B1$)

($B2$)

1. $\left[\text{Show } \forall p \in \Sigma_{tm(D)} \forall S \in M_{tm(D)} \left(\text{if } \mathcal{T}_{M_{tm(D)}}(S) \subseteq \mathcal{T}_{\Sigma_{tm(D)}}(p), \text{ then } S \in t_{tm(D)}(p) \right) \right]$

2. Since $\Sigma_{tm(D)} = \Sigma_D$, $M_{tm(D)} = \Omega_D$, $t_{tm(D)}(p) = \{S \in \Omega_D \mid p \in S\}$,

$\mathcal{T}_{\Sigma_{tm(D)}} = \mathcal{T}_{\Sigma_D}$, the above simplifies to

$\left[\text{Show } \forall p \in \Sigma_D \forall S \in \Omega_D \left(\text{if } \mathcal{T}_{M_{tm(D)}}(S) \subseteq \mathcal{T}_{\Sigma_D}(p), \text{ then } p \in S \right) \right]$

3. This is just one direction of the equivalence shown in Proposition 109. ■(B2)

(A7)

1. The elements of $M_{tm(D)}$ are partially ordered by set inclusion.

2. [Show $\forall P, Q \in M_{tm(D)}, \mathcal{T}_{M_{tm(D)}}(P) \subseteq \mathcal{T}_{M_{tm(D)}}(Q)$ iff $Q \subseteq P$]

3. Let P and Q be elements of $M_{tm(D)}$.

4. ($\rightarrow$)

(a) Assume that $\mathcal{T}_{M_{tm(D)}}(P) \subseteq \mathcal{T}_{M_{tm(D)}}(Q)$.

(b) Then $\exists p \in \Sigma_D$ such that $Cons_D(p) = P$, and $\exists q \in \Sigma_D$ such that $Cons_D(q) = Q$.

(c) So by Proposition 108, $\mathcal{T}_{M_{tm(D)}}(P) = \mathcal{T}_{\Sigma_D}(p)$ and $\mathcal{T}_{M_{tm(D)}}(Q) = \mathcal{T}_{\Sigma_D}(q)$.

(d) So $\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q)$.

(e) So since D satisfies Set CG with respect to $\mathcal{T}_{\Sigma_D}$, we have $LC_D(p, q)$, by the Intelligibility of Consequence Theorem for Consistency Media (Theorem 55).

(f) So $Cons_D(q) \subseteq Cons_D(p)$, by the definition of $Cons$ and the transitivity of LC (Proposition 49).

(g) So $Q \subseteq P$. ■ ($\rightarrow$)

5. ($\leftarrow$)

(a) Assume that $Q \subseteq P$.

(b) So $\bigcap \mathcal{T}_{\Sigma_D}[P] \subseteq \bigcap \mathcal{T}_{\Sigma_D}[Q]$.

(c) So $\mathcal{T}_{M_{tm(D)}}(P) \subseteq \mathcal{T}_{M_{tm(D)}}(Q)$ by the definition of tm . ■ ($\leftarrow$) ■ (A7) ■ Proposition.

8.7 Canonical construction of an application of the Set BE technique given an application of the Set CG technique

Theorem 111 *Given an application of the Set CG technique*

(a consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$

such that D satisfies Set CG ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$,

then there is an application of the Set BE technique

(a truth medium A and practices of categorization $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$

such that A satisfies assumption set BE with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$)

and a function $h : \Sigma_D \rightarrow \Sigma_A$ for which

$$\forall p, q \in \Sigma_D \left(\begin{array}{l} ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff} \\ ALC_{\langle \Sigma_A, \mathcal{T}_{\Sigma_A} \rangle}(h(p), h(q)) \end{array} \right).$$

We will show that the truth medium $tm(D)$, its associated practices of categorization $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$, and the identity function $h : \Sigma_D \rightarrow \Sigma_{tm(D)}$ defined for all $p \in \Sigma_D$ as $h(p) = p$, satisfy the above claim.

1. Assume we have an arbitrary consistency medium D and practice of categorization $\mathcal{T}_{\Sigma_D}$ such that D satisfies Set CG ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$.
2. Applying Proposition 110, we know that $tm(D)$ satisfies assumption set BE with respect to $\mathcal{T}_{\Sigma_{tm(D)}}$ and $\mathcal{T}_{M_{tm(D)}}$.
3. Since, $\Sigma_{tm(D)} = \Sigma_D$, and $\mathcal{T}_{\Sigma_{tm(D)}} = \mathcal{T}_{\Sigma_D}$ by the definition of tm , we know that

$$ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle} = ALC_{\langle \Sigma_{tm(D)}, \mathcal{T}_{\Sigma_{tm(D)}} \rangle}.$$

So we know that

$$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{tm(D)}, \mathcal{T}_{\Sigma_{tm(D)}} \rangle}(h(p), h(q)) \right) \blacksquare \text{Theorem.}$$

This completes the proof that $(CG \rightarrow BE)$, and therefore, we have shown that $ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$.

Chapter 9

Constructing a Minimal Equivalent for any Application of the Set *CG* Technique

9.1 The question at hand

An important question which attends the Set *CG* technique can be expressed informally as follows: Given some application of the Set *CG* technique making some relation of consequence intelligible, is there a “smaller” application which can make that same relation of consequence intelligible? By “smaller,” what we are interested in is an application in which the consistency medium involved uses a lesser number of assertion types than the medium in the given application. A refined (and more powerful) version of this question could be expressed as follows: Can we give a canonical method for constructing a minimal (i.e. “smallest”) equivalent for any application of the Set *CG* technique?¹ The answer to this latter question is “yes,” and the method we use will make use of the concepts introduced in the previous chapter.

¹Special thanks to Jon Barwise for asking this question.

9.2 Formalizing the question

An application of the Set *CG* technique consists in a consistency medium $D = \langle \Sigma_D, \preceq_D, C_D \rangle$ and a practice of categorization $\mathcal{T}_{\Sigma_D}$ for which it is assumed that D satisfies the Set *CG* assumptions ($D1, D2, D3$) with respect to $\mathcal{T}_{\Sigma_D}$. We can refer to such an application of the Set *CG* technique by the pair $\langle D, \mathcal{T}_{\Sigma_D} \rangle$.

We can formalize the question posed above as follows:

Can we define an operator *min*, which, given an application of the Set *CG* technique $\langle D, \mathcal{T}_{\Sigma_D} \rangle$ constructs an application of the Set *CG* technique $\langle \min(D), \mathcal{T}_{\Sigma_{\min(D)}} \rangle$ such that

1. The relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$,
i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
is embedded within that made intelligible by $\langle \min(D), \mathcal{T}_{\Sigma_{\min(D)}} \rangle$; and
2. There is no application of the Set *CG* technique $\langle E, \mathcal{T}_{\Sigma_E} \rangle$ such that
 - (a) the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$,
i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
is embedded within that made intelligible by $\langle E, \mathcal{T}_{\Sigma_E} \rangle$; and
 - (b) $|\Sigma_E| < |\Sigma_{\min(D)}|$

We use here the standard notation $|S|$ for the cardinality of a set S . So this condition, if satisfied, would mean that the number of assertion types for consistency medium E is less than the number of assertion types for consistency medium $\min(D)$.

9.3 Answering the question

The key to answering the question is recognizing that the least number of assertion types which we can use in an application of the Set *CG* technique and still preserve the consequence relation made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$ is given by the cardinality of $\mathcal{T}_{\Sigma_D}[\Sigma_D]$; that is, the number of differentiable sets of possibilities represented by the elements of Σ_D via the practice of categorization $\mathcal{T}_{\Sigma_D}$. We will show below that no smaller number of assertion types is capable of doing the job.

Recall from Section 8.4.3 that for any consistency medium D , the members of the set $\Omega_D (= \text{Cons}_D[\Sigma_D])$ correspond on a one-to-one basis with the differentiable sets of possibilities represented by assertion types of Σ_D via the practice of categorization $\mathcal{T}_{\Sigma_D}$. This fact is key to our answer, which begins with the definition of an operator min .

9.4 Definition of the operator min

We are now ready to define an operator min which satisfies the conditions described in Section 9.2.

Given an arbitrary consistency medium D , and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , we define the consistency medium $\text{min}(D)$ and derived practices of categorization $\mathcal{T}_{\Sigma_{\text{min}(D)}}$ as follows:

9.4.1 Consistency medium $\text{min}(D)$:

- $\Sigma_{\text{min}(D)} = \Omega_D (= \text{Cons}_D[\Sigma_D])$

The assertion types for the minimal medium are the consequence sets of the given medium (these were the models in the truth medium constructed by tm in Chapter 8).

- $\preceq_{\text{min}(D)}$ = the inclusion relation on $\Sigma_{\text{min}(D)}$,
that is, for all $P, Q \in \Sigma_{\text{min}(D)}$, $P \preceq_{\text{min}(D)} Q$ iff $P \subseteq Q$
- $C_{\text{min}(D)} = \text{Cons}_D[C_D]$

The consistent assertion types for the minimal medium are the consequence sets of the consistent assertion types in the given medium.

9.4.2 Derived practice $\mathcal{T}_{\Sigma_{\text{min}(D)}}$:

- $\mathcal{T}_{\Sigma_{\text{min}(D)}} : \Sigma_{\text{min}(D)} \rightarrow \mathcal{P}(\mathcal{U})$ is defined as follows:
for any $P \in \Sigma_{\text{min}(D)}$, $\mathcal{T}_{\Sigma_{\text{min}(D)}}(P) = \bigcap \mathcal{T}_{\Sigma_D}[P]$

We give the assertion types in the minimal medium the same interpretation they had as models in the truth medium constructed by tm in Chapter 8. Because of this

equivalence, we will be importing some of the results from that chapter. Note that we don't have to consider the case where P is null, since no consequence set can be null. (Proposition 107).

9.5 Proving the correctness of the answer

In this section, we prove that the operator min meets the requirements outlined in Section 9.2. The proof will take two parts. The first part will show goal 1; that the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$ is made intelligible by $\langle min(D), \mathcal{T}_{\Sigma_{min(D)}} \rangle$. The second part will show goal 2; that there is no “smaller” application of the Set CG technique which can do the same.

9.5.1 Showing that the original relation of consequence is made intelligible by the new application

Stating the goal formally

We are given an application of the Set CG technique $\langle D, \mathcal{T}_{\Sigma_D} \rangle$, that is, a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , such that D satisfies $D1$, $D2$, and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$. Our claim is that the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$ (i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$) is embedded within that made intelligible by $\langle min(D), \mathcal{T}_{\Sigma_{min(D)}} \rangle$. To demonstrate this claim, we need to show that: (1) an image of $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ is embedded within $ALC_{\langle \Sigma_{min(D)}, \mathcal{T}_{\Sigma_{min(D)}} \rangle}$ (that is, there is a function $h : \Sigma_D \rightarrow \Sigma_{min(D)}$ such that

$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{min(D)}, \mathcal{T}_{\Sigma_{min(D)}} \rangle}(h(p), h(q)) \right)$; and (2) that consistency medium $min(D)$ satisfies assumptions $D1$, $D2$, $D3$ with respect to $\mathcal{T}_{\Sigma_{min(D)}}$.

Identifying the embedding function h

We define the embedding function $h : \Sigma_D \rightarrow \Sigma_{min(D)}$ as follows:

for all $p \in \Sigma_D$, $h(p) = Cons_D(p)$.

To demonstrate that $\forall p, q \in \Sigma_D \left(\begin{array}{c} ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff} \\ ALC_{\langle \Sigma_{min(D)}, \mathcal{T}_{\Sigma_{min(D)}} \rangle}(h(p), h(q)) \end{array} \right)$

it suffices to show that $\forall p \in \Sigma_D \left(\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_{\min(D)}}(h(p)) \right)$ and then apply the definition of *ALC*.

We have already proven this result, although in a different guise. To see this, recognize the following equivalences:

1. $\Sigma_{\min(D)} = M_{tm(D)}$; the set of assertion types in the consistency medium $\min(D)$ equals the set of models for the truth medium constructed from D by the operator tm (see Section 8.5 for the definition of tm).
2. $\mathcal{T}_{\Sigma_{\min(D)}} = \mathcal{T}_{M_{tm(D)}}$; the practice of categorization by which the assertion types in the consistency medium $\min(D)$ are to be interpreted equals the practice used to interpret the models of the truth medium constructed from D by the operator tm .

These equivalences allow us to restate Proposition 108 as follows:

Corollary 112 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\forall p \in \Sigma_D, \mathcal{T}_{\Sigma_{\min(D)}}(Cons_D(p)) = \mathcal{T}_{\Sigma_D}(p)$.*

Given Corollary 112, it is immediate that h as defined is the embedding we desire.

Proving that $\min(D)$ satisfies assumption $D1$ with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.

Proposition 113 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\min(D)$ satisfies assumption $D1$ with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.*

This result is immediate given the definitions.

1. Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$, a practice of categorization, such that D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.

$$2. \left[\begin{array}{l} \text{Show that } \min(D) \text{ satisfies assumption } D1 \text{ with respect to } \mathcal{T}_{\Sigma_{\min(D)}}, \\ \text{that is,} \\ \text{Show } \forall P, Q \in \Sigma_{\min(D)}, P \preceq_{\min(D)} Q \text{ implies } \mathcal{T}_{\Sigma_{\min(D)}}(Q) \subseteq \mathcal{T}_{\Sigma_{\min(D)}}(P), \\ \text{that is,} \\ \text{Show } \forall P, Q \in \Sigma_{\min(D)}, P \subseteq Q \text{ implies } \bigcap \mathcal{T}_{\Sigma_D}[Q] \subseteq \bigcap \mathcal{T}_{\Sigma_D}[P] \end{array} \right]$$

3. This last is immediate.

Proving that $\min(D)$ satisfies assumption D2 with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.

Proposition 114 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies D1, D2, and D3 with respect to $\mathcal{T}_{\Sigma_D}$, then $\min(D)$ satisfies assumption D2 with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.*

1. Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$, a practice of categorization, such that D satisfies D1, D2, and D3 with respect to $\mathcal{T}_{\Sigma_D}$.

2. $\left[\begin{array}{l} \text{Show that } \min(D) \text{ satisfies assumption D2 with respect to } \mathcal{T}_{\Sigma_{\min(D)}}, \text{ that is,} \\ \text{Show } \forall P \in \Sigma_{\min(D)}, P \in C_{\min(D)} \text{ iff } \mathcal{T}_{\Sigma_{\min(D)}}(P) \neq \emptyset \end{array} \right]$

3. Let P be an arbitrary element of $\Sigma_{\min(D)}$.

4. $(\rightarrow)$

(a) Assume $P \in C_{\min(D)}$. $\left[\text{Show } \mathcal{T}_{\Sigma_{\min(D)}}(P) \neq \emptyset \right]$.

(b) $P \in C_{\min(D)}$ iff $P \in \text{Cons}_D[C_D]$, by the definition of $\min$.

(c) So there is a $p \in C_D$, such that $P = \text{Cons}_D(p)$.

(d) By Corollary 112, $\mathcal{T}_{\Sigma_{\min(D)}}(P) = \mathcal{T}_{\Sigma_D}(p)$.

(e) Since $p \in C_D$, and D satisfies D2 with respect to $\mathcal{T}_{\Sigma_D}$, we know that $\mathcal{T}_{\Sigma_D}(p) \neq \emptyset$.

(f) So $\mathcal{T}_{\Sigma_{\min(D)}}(P) \neq \emptyset$. ■ $(\rightarrow)$

5. $(\leftarrow)$

(a) $\left[\text{Show } \mathcal{T}_{\Sigma_{\min(D)}}(P) \neq \emptyset \rightarrow P \in C_{\min(D)} \right]$

(b) Show the contrapositive.

(c) Assume $P \notin C_{\min(D)}$. $\left[\text{Show } \mathcal{T}_{\Sigma_{\min(D)}}(P) = \emptyset \right]$.

(d) Since $P \notin C_{\min(D)}$, we know $P \notin \text{Cons}_D[C_D]$, by the definition of $\min$.

(e) We know that $P \in \text{Cons}_D[\Sigma_D]$, since $P \in \Sigma_{\min(D)}$.

(f) So $P = \text{Cons}_D(p)$ for some $p \in \Sigma_D - C_D$.

- (g) So $p \notin C_D$.
- (h) Since D satisfies $D2$ with respect to $\mathcal{T}_{\Sigma_D}$, we know that $\mathcal{T}_{\Sigma_D}(p) = \emptyset$.
- (i) By Corollary 112, $\mathcal{T}_{\Sigma_{\min(D)}}(P) = \mathcal{T}_{\Sigma_D}(p)$.
- (j) So $\mathcal{T}_{\Sigma_{\min(D)}}(P) = \emptyset$. ■ ($\leftarrow$) ■ Proposition.

Proving that $\min(D)$ satisfies assumption $D3$ with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.

Proposition 115 *Given a consistency medium D , and practice of categorization $\mathcal{T}_{\Sigma_D}$, if D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$, then $\min(D)$ satisfies assumption $D3$ with respect to $\mathcal{T}_{\Sigma_{\min(D)}}$.*

Let D be a consistency medium, and $\mathcal{T}_{\Sigma_D}$, a practice of categorization, such that D satisfies $D1, D2,$ and $D3$ with respect to $\mathcal{T}_{\Sigma_D}$.

$$\left[\begin{array}{l} \text{Show that } \min(D) \text{ satisfies assumption } D3 \text{ with respect to } \mathcal{T}_{\Sigma_{\min(D)}}, \text{ that is,} \\ \text{a) Show } \forall P, Q \in \Sigma_{\min(D)}, \exists \Gamma \subseteq \{P, Q\}_{\min(D)}^u \\ \text{such that } \bigcup \mathcal{T}_{\Sigma_{\min(D)}}[\Gamma] = \mathcal{T}_{\Sigma_{\min(D)}}(P) \cap \mathcal{T}_{\Sigma_{\min(D)}}(Q), \text{ and} \\ \text{b) Show that } \forall P, Q \in \Sigma_{\min(D)}, \exists \Delta \subseteq \uparrow_{\min(D)}(P) \\ \text{such that } \bigcup \mathcal{T}_{\Sigma_{\min(D)}}[\Delta] = \mathcal{T}_{\Sigma_{\min(D)}}(P) - \mathcal{T}_{\Sigma_{\min(D)}}(Q) \end{array} \right]$$

(Part a)

$$1. \left[\begin{array}{l} \text{Show } \forall P, Q \in \Sigma_{\min(D)}, \exists \Gamma \subseteq \{P, Q\}_{\min(D)}^u \\ \text{such that } \bigcup \mathcal{T}_{\Sigma_{\min(D)}}[\Gamma] = \mathcal{T}_{\Sigma_{\min(D)}}(P) \cap \mathcal{T}_{\Sigma_{\min(D)}}(Q) \end{array} \right]$$

2. Let $P, Q \in \Sigma_{\min(D)}$.

3. Then there exists a $p \in \Sigma_D$ such that $\text{Cons}_D(p) = P$, and a $q \in \Sigma_D$ such that $\text{Cons}_D(q) = Q$.

4. By Corollary 112, $\mathcal{T}_{\Sigma_{\min(D)}}(P) = \mathcal{T}_{\Sigma_D}(p)$, and $\mathcal{T}_{\Sigma_{\min(D)}}(Q) = \mathcal{T}_{\Sigma_D}(q)$.

5. Since D satisfies $D3a$ with respect to $\mathcal{T}_{\Sigma_D}$,

we know that $\exists R \subseteq \{p, q\}_D^u$, such that $\bigcup \mathcal{T}_{\Sigma_D}[R] = \mathcal{T}_{\Sigma_D}(p) \cap \mathcal{T}_{\Sigma_D}(q)$.

6. So $\bigcup \mathcal{T}_{\Sigma_D}[R] = \mathcal{T}_{\Sigma_{\min(D)}}(P) \cap \mathcal{T}_{\Sigma_{\min(D)}}(Q)$.

7. Let $\Gamma = \text{Cons}_D[R]$
8. Claim: $\text{Cons}_D[R] \subseteq \{P, Q\}_{\min(D)}^u$.
 - (a) [Show $\forall r \in R, P \subseteq \text{Cons}_D(r)$ and $Q \subseteq \text{Cons}_D(r)$]
 - (b) Since $r \in \{p, q\}_D^u$, $r \succeq_D p$ and $r \succeq_D q$, so by Proposition 46, we know $LC_D(r, p)$ and $LC_D(r, q)$.
 - (c) Since LC is transitive (Proposition 49), we know that $\text{Cons}_D(p) \subseteq \text{Cons}_D(r)$ and $\text{Cons}_D(q) \subseteq \text{Cons}_D(r)$, so $P \subseteq \text{Cons}_D(r)$ and $Q \subseteq \text{Cons}_D(r)$. ■Claim.
9. Claim: $\bigcup \mathcal{T}_{\Sigma_{\min(D)}} [\Gamma] = \bigcup \mathcal{T}_{\Sigma_D} [R]$.
 - (a) $\bigcup \mathcal{T}_{\Sigma_D} [R] = \bigcup_{r \in R} \mathcal{T}_{\Sigma_D}(r) = \bigcup_{r \in R} \mathcal{T}_{\Sigma_{\min(D)}}(\text{Cons}_D(r))$ by Corollary 112.
 - (b) $\bigcup_{r \in R} \mathcal{T}_{\Sigma_{\min(D)}}(\text{Cons}_D(r)) = \bigcup \mathcal{T}_{\Sigma_{\min(D)}} [\text{Cons}_D[R]] = \bigcup \mathcal{T}_{\Sigma_{\min(D)}} [\Gamma]$. ■Claim.
10. So $\bigcup \mathcal{T}_{\Sigma_{\min(D)}} [\Gamma] = \mathcal{T}_{\Sigma_{\min(D)}}(P) \cap \mathcal{T}_{\Sigma_{\min(D)}}(Q)$. ■Part a.

(Part b)

 1. $\left[\begin{array}{l} \text{Show that } \forall P, Q \in \Sigma_{\min(D)}, \exists \Delta \subseteq \uparrow_{\min(D)}(P) \\ \text{such that } \bigcup \mathcal{T}_{\Sigma_{\min(D)}} [\Delta] = \mathcal{T}_{\Sigma_{\min(D)}}(P) - \mathcal{T}_{\Sigma_{\min(D)}}(Q) \end{array} \right]$
 2. Let $P, Q \in \Sigma_{\min(D)}$.
 3. Then there exists a $p \in \Sigma_D$ such that $\text{Cons}_D(p) = P$, and a $q \in \Sigma_D$ such that $\text{Cons}_D(q) = Q$.
 4. By Corollary 112, $\mathcal{T}_{\Sigma_{\min(D)}}(P) = \mathcal{T}_{\Sigma_D}(p)$, and $\mathcal{T}_{\Sigma_{\min(D)}}(Q) = \mathcal{T}_{\Sigma_D}(q)$.
 5. Since D satisfies $D3b$ with respect to $\mathcal{T}_{\Sigma_D}$, we know that $\exists S \subseteq \uparrow_D p$, such that $\bigcup \mathcal{T}_{\Sigma_D} [S] = \mathcal{T}_{\Sigma_D}(p) - \mathcal{T}_{\Sigma_D}(q)$.
 6. So $\bigcup \mathcal{T}_{\Sigma_D} [S] = \mathcal{T}_{\Sigma_{\min(D)}}(P) - \mathcal{T}_{\Sigma_{\min(D)}}(Q)$.
 7. Let $\Delta = \text{Cons}_D[S]$
 8. Claim: $\text{Cons}_D[S] \subseteq \uparrow_{\min(D)}(P)$.

- (a) [Show $\forall s \in S, P \subseteq Cons_D(s)$]
- (b) Since $s \in \uparrow_D p$, $s \succeq_D p$, so by Proposition 46, we know $LC_D(s, p)$.
- (c) Since LC is transitive (Proposition 49), we know that $Cons_D(p) \subseteq Cons_D(s)$, so $P \subseteq Cons_D(s)$. ■Claim.

9. Claim: $\bigcup \mathcal{T}_{\Sigma_{min(D)}} [\Delta] = \bigcup \mathcal{T}_{\Sigma_D} [S]$.

- (a) $\bigcup \mathcal{T}_{\Sigma_D} [S] = \bigcup_{s \in S} \mathcal{T}_{\Sigma_D}(s) = \bigcup_{s \in S} \mathcal{T}_{\Sigma_{min(D)}}(Cons_D(s))$ by Corollary 112.
- (b) $\bigcup_{s \in S} \mathcal{T}_{\Sigma_{min(D)}}(Cons_D(s)) = \bigcup \mathcal{T}_{\Sigma_{min(D)}} [Cons_D[S]] = \bigcup \mathcal{T}_{\Sigma_{min(D)}} [\Delta]$. ■Claim.

10. So $\bigcup \mathcal{T}_{\Sigma_{min(D)}} [\Delta] = \mathcal{T}_{\Sigma_{min(D)}}(P) - \mathcal{T}_{\Sigma_{min(D)}}(Q)$. ■Part b. ■Proposition.

Pulling it all together

Proposition 116 *Given a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , such that D satisfies Set CG ($D1$, $D2$, and $D3$) with respect to $\mathcal{T}_{\Sigma_D}$, then there is a function $h : \Sigma_D \rightarrow \Sigma_{min(D)}$ such that*

$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{min(D)}, \mathcal{T}_{\Sigma_{min(D)}} \rangle}(h(p), h(q)) \right)$; and consistency medium $min(D)$ satisfies assumption Set CG ($D1$, $D2$, $D3$) with respect to $\mathcal{T}_{\Sigma_{min(D)}}$.

1. Given that we define h as follows:

for all $p \in \Sigma_D$, $h(p) = Cons_D(p)$,

then by Corollary 112, we have

$$\forall p \in \Sigma_D \left(\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_{min(D)}}((h(p))) \right).$$

Applying the definition of ALC gives

$$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_{min(D)}, \mathcal{T}_{\Sigma_{min(D)}} \rangle}(h(p), h(q)) \right).$$

2. Propositions 113, 114, and 115 show that consistency medium $min(D)$ satisfies assumptions $D1$, $D2$, $D3$ with respect to $\mathcal{T}_{\Sigma_{min(D)}}$. ■Proposition.

This shows that $\langle min(D), \mathcal{T}_{\Sigma_{min(D)}} \rangle$ is an application of the Set CG technique, and that the representational relation of logical consequence which it makes intelligible, embeds that made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$.

9.5.2 Showing that no smaller application can make the original relation of consequence intelligible

Proposition 117 *Given an application of the Set CG technique $\langle D, \mathcal{T}_{\Sigma_D} \rangle$, that is, a consistency medium D and a practice of categorization $\mathcal{T}_{\Sigma_D}$ interpreting D , such that D satisfies Set CG (D1, D2, and D3) with respect to $\mathcal{T}_{\Sigma_D}$, there is no application of the Set CG technique $\langle E, \mathcal{T}_{\Sigma_E} \rangle$ such that*

- (a) the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$,
i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$, is embedded within that made intelligible by $\langle E, \mathcal{T}_{\Sigma_E} \rangle$; and
- (b) $|\Sigma_E| < |\Sigma_{\min(D)}|$

To demonstrate this claim, we will assume that there is such an application $\langle E, \mathcal{T}_{\Sigma_E} \rangle$, and show that that assumption leads to a contradiction.

1. Assumptions:

- (a) Since we have assumed that $\langle E, \mathcal{T}_{\Sigma_E} \rangle$ is an application of the Set CG technique; we have assumed that consistency medium E satisfies assumptions D1, D2, and D3 with respect to $\mathcal{T}_{\Sigma_E}$. As an application of the Set CG technique, the relation of consequence which $\langle E, \mathcal{T}_{\Sigma_E} \rangle$ makes intelligible is $ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}$.
- (b) Since we have further assumed that the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$, that is, $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$, is embedded within that made intelligible by $\langle E, \mathcal{T}_{\Sigma_E} \rangle$, we have assumed that there is some function f embedding the relation $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$ within the relation $ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}$. That is, we have assumed that there exists some function $f : \Sigma_D \rightarrow \Sigma_E$ such that
$$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}(f(p), f(q)) \right).$$
- (c) We have also assumed that $|\Sigma_E| < |\Sigma_{\min(D)}|$.

2. [Show that the assumptions given above lead to a contradiction]

- 3. We know that $|\Sigma_{\min(D)}| = |\mathcal{T}_{\Sigma_D}[\Sigma_D]|$ from Proposition 106 and the discussion in Section 8.4.3.

4. Since $|\Sigma_E| < |\Sigma_{\min(D)}|$ (by assumption above), and $|\mathcal{T}_{\Sigma_E}[\Sigma_E]| \leq |\Sigma_E|$ (since $\mathcal{T}_{\Sigma_E}$ is a function, though not necessarily one-to-one), we know that $|\mathcal{T}_{\Sigma_E}[\Sigma_E]| < |\mathcal{T}_{\Sigma_D}[\Sigma_D]|$. Recall that $\mathcal{T}_{\Sigma_E}[\Sigma_E]$ is the set of differentiable sets of possibilities represented by the assertion types in Σ_E via the practice of categorization $\mathcal{T}_{\Sigma_E}$. This claim will be key to the contradiction we derive. In what follows, we will show that $|\mathcal{T}_{\Sigma_E}[\Sigma_E]|$ must be greater than or equal to $|\mathcal{T}_{\Sigma_D}[\Sigma_D]|$.

5. Claim: Given that we have accepted the condition

$$\forall p, q \in \Sigma_D \left(ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ iff } ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}(f(p), f(q)) \right)$$

we are committed to

$$\forall p, q \in \Sigma_D (\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_D}(q) \text{ iff } \mathcal{T}_{\Sigma_E}(f(p)) = \mathcal{T}_{\Sigma_E}(f(q))).$$

(a) Let p, q be arbitrary elements of Σ_D .

(b) $\mathcal{T}_{\Sigma_D}(p) = \mathcal{T}_{\Sigma_D}(q)$ iff

$$\mathcal{T}_{\Sigma_D}(p) \subseteq \mathcal{T}_{\Sigma_D}(q) \text{ and } \mathcal{T}_{\Sigma_D}(q) \subseteq \mathcal{T}_{\Sigma_D}(p) \text{ iff}$$

$$ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(p, q) \text{ and } ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}(q, p) \text{ iff}$$

$$ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}(f(p), f(q)) \text{ and } ALC_{\langle \Sigma_E, \mathcal{T}_{\Sigma_E} \rangle}(f(q), f(p)) \text{ iff}$$

$$\mathcal{T}_{\Sigma_E}(f(p)) \subseteq \mathcal{T}_{\Sigma_E}(f(q)) \text{ and } \mathcal{T}_{\Sigma_E}(f(q)) \subseteq \mathcal{T}_{\Sigma_E}(f(p)) \text{ iff}$$

$$\mathcal{T}_{\Sigma_E}(f(p)) = \mathcal{T}_{\Sigma_E}(f(q)). \blacksquare \text{Claim.}$$

6. Define a family of properties Z_P one for each element P of $\mathcal{T}_{\Sigma_D}[\Sigma_D]$. These properties apply to elements Q of $\mathcal{T}_{\Sigma_E}[\Sigma_E]$, and are defined as follows:

for some element Q of $\mathcal{T}_{\Sigma_E}[\Sigma_E]$, $Z_P(Q)$ is true just in case $Q = \mathcal{T}_{\Sigma_E}(f(p))$ for some $p \in \Sigma_D$ such that $\mathcal{T}_{\Sigma_D}(p) = P$.

7. Claim: $\neg \exists M, N \in \mathcal{T}_{\Sigma_D}[\Sigma_D]$ such that

$$\left(\begin{array}{l} M \neq N \text{ and} \\ \exists Q \in \mathcal{T}_{\Sigma_E}[\Sigma_E] \text{ such that } (Z_M(Q) \text{ and } Z_N(Q)) \end{array} \right).$$

(a) Say that such M and N exist. [Show contradiction.]

(b) Then $\exists M, N \in \mathcal{T}_{\Sigma_D}[\Sigma_D]$ such that

$$\left(\begin{array}{c} M \neq N \text{ and } \exists Q \in \mathcal{T}_{\Sigma_E} [\Sigma_E] \text{ such that} \\ \left(\begin{array}{c} Q = \mathcal{T}_{\Sigma_E}(f(p)) \text{ for some } p \in \Sigma_D \text{ such that } \mathcal{T}_{\Sigma_D}(p) = M \text{ and} \\ Q = \mathcal{T}_{\Sigma_E}(f(q)) \text{ for some } q \in \Sigma_D \text{ such that } \mathcal{T}_{\Sigma_D}(q) = N \end{array} \right) \end{array} \right)$$

(c) Then $\exists p, q \in \Sigma_D$ such that $\mathcal{T}_{\Sigma_D}(p) \neq \mathcal{T}_{\Sigma_D}(q)$ and $\mathcal{T}_{\Sigma_E}(f(p)) = \mathcal{T}_{\Sigma_E}(f(q))$.

(d) This contradicts the equivalence in Step 5 above. ■ Claim.

8. So every $Q \in \mathcal{T}_{\Sigma_E} [\Sigma_E]$ has the property Z_P (that is $Z_P(Q)$ is true) for at most one element P of $\mathcal{T}_{\Sigma_D} [\Sigma_D]$.

9. Claim: For every P in $\mathcal{T}_{\Sigma_D} [\Sigma_D]$, there is at least one Q that has the property Z_P .

(a) For every $P \in \mathcal{T}_{\Sigma_D} [\Sigma_D]$, there is some $p \in \Sigma_D$ such that $\mathcal{T}_{\Sigma_D}(p) = P$.

(b) Therefore there is some $q \in \Sigma_E$ such that $q = f(p)$, since $f : \Sigma_D \rightarrow \Sigma_E$.

(c) Therefore there is some $Q \in \mathcal{T}_{\Sigma_E} [\Sigma_E]$ such that $Q = \mathcal{T}_{\Sigma_E}(q)$.

(d) That $Q = \mathcal{T}_{\Sigma_E}(f(p))$ for some $p \in \Sigma_D$ such that $\mathcal{T}_{\Sigma_D}(p) = P$, and therefore $Z_P(Q)$ is true.

10. Given the claims in steps 8 and 9 above, we see that the number of elements of $\mathcal{T}_{\Sigma_E} [\Sigma_E]$ must be greater than or equal to the number of elements of $\mathcal{T}_{\Sigma_D} [\Sigma_D]$. This contradicts the claim that $|\mathcal{T}_{\Sigma_E} [\Sigma_E]| < |\mathcal{T}_{\Sigma_D} [\Sigma_D]|$ which was made in Step 4.

■ Contradiction.

9.5.3 Conclusions

We have shown that for any application of the Set *CG* technique $\langle D, \mathcal{T}_{\Sigma_D} \rangle$, the operator *min* constructs an application of the Set *CG* technique $\langle \min(D), \mathcal{T}_{\Sigma_{\min(D)}} \rangle$ such that

1. The relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$,
i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
is embedded within that made intelligible by $\langle \min(D), \mathcal{T}_{\Sigma_{\min(D)}} \rangle$; and
2. There is no application of the Set *CG* technique $\langle E, \mathcal{T}_{\Sigma_E} \rangle$ such that

- (a) the relation of consequence which is made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$,
i.e. $ALC_{\langle \Sigma_D, \mathcal{T}_{\Sigma_D} \rangle}$,
is embedded within that made intelligible by $\langle E, \mathcal{T}_{\Sigma_E} \rangle$; and
- (b) $|\Sigma_E| < |\Sigma_{\min(D)}|$

So for any application of the Set *CG* technique $\langle D, \mathcal{T}_{\Sigma_D} \rangle$, the application of the Set *CG* technique $\langle \min(D), \mathcal{T}_{\Sigma_{\min(D)}} \rangle$ is a minimal equivalent, the “smallest” application of the Set *CG* technique embedding, in the relation of consequence it makes intelligible, the relation of consequence made intelligible by $\langle D, \mathcal{T}_{\Sigma_D} \rangle$.

Further, there is one last interesting thing to notice. Since $\Sigma_{\min(D)}$ just are the models from the *tm* construction of Chapter 8; and the practice by which they are interpreted ($\mathcal{T}_{\Sigma_{\min(D)}}$) is the same as in that case, we can apply the result that $tm(D)$ satisfies *A7* with respect to $\mathcal{T}_{M_{tm(D)}}$ and recognize that the minimal medium is logically simple, with consequence equivalent to set inclusion.

Part V

Reflections and Next Steps

Chapter 10

Reflections and Next Steps

10.1 Introduction

It is now time to review what we have done and look ahead to future work. We will present this discussion in two parts. In the first part, we will review the key results shown in the course of the dissertation. In the second part, we reflect on those results and consider directions for future work from the perspectives of each of the four levels of our investigations. We will work from the “outside” in, starting at the level of applications, and dropping down through the different levels of the representational schema account, to the foundational characterizations at its core. We will begin by considering applications of the representational techniques; then consider the representational techniques themselves; then the explanatory strategies used by the representational techniques; and lastly the foundational characterizations used as a basis for the techniques (including the representational conception of consequence)

10.2 Review of key results

We began with John Etchemendy’s account of how the technique of model-theoretic semantics implements a representational semantics. From that account, we constructed the representational schema, a general form for techniques used to construct theories of logical consequence implementing representational semantics. We set out several tests by which

we could evaluate the proposed schema. To be successful, the schema should (1) be able to assimilate techniques for constructing representational theories of consequence beyond the specific model-theoretic approach described by Etchemendy, (2) help increase our understanding of existing techniques, and (3) help us to create new techniques.

We have shown that the schema is capable of subsuming two broad classes of representational techniques: those using a model-theoretic strategy to explain the consequence relations modelled; and those using an order-consistency strategy. Within those two classes, we have used the schema to describe five different techniques, three of the model-theoretic kind (Set *E*, Set *PP*, and Set *BE*) and two of the order-consistency kind (Set *CL* and Set *CG*). Three of those five techniques (Set *PP*, Set *BE*, and Set *CG*) were created using the structure and vectors of extension provided by the representational schema. The Set *PP* technique reduces the epistemological commitments of the Set *E* technique, the Set *BE* technique is a model-theoretic technique supporting partial models, and the Set *CG* technique is an order-consistency technique which does not require the assumption of Lindenbaum's Lemma (and thus does not require the assumption of maximal extensions for consistent assertion types).

The representational schema, by assimilating the various techniques to a common framework, made it possible for us to compare and contrast differing techniques of representational semantics with respect to four criteria: the mode by which they explain the consequence relation; the range of interpreted languages to which they are applicable; their degree of epistemological commitment; and the ease with which they can be used in particular applications. We were able to describe, in terms of the schema, a standard methodology for comparing the ranges of interpreted languages to which different techniques are applicable. That methodology showed how we could “move” relations of logical consequence from the context of one technique to that of another. Using that methodology, we were able to establish that the ranges of applicability for the five techniques discussed could be arranged in a linear order: $ra(\text{Set } E) \subseteq ra(\text{Set } PP) \subseteq ra(\text{Set } CL) \subseteq ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$.

We used the schema to describe a general methodology for applying representational techniques in the construction of theories of logical consequence for given interpreted languages. We then used that methodology to apply (or outline the application of) techniques

of both kinds to a number of interpreted languages, including propositional logic, feature logics (sentential languages with feature structures as models), and languages in which feature structures are considered as assertions in their own right.

We further used the schema to prove an interesting result, that for any application of the Set *CG* technique, we could construct an application of that same technique, such that the relation of logical consequence made intelligible by the given application was embedded in the relation of logical consequence made intelligible by the constructed application, and that no “smaller” application (in the sense of a consistency medium with a set of assertion types of lesser cardinality) could do the same.

10.3 Reflections and next steps

Now we look both back and ahead. With respect to each of four levels of our investigation (applications of representational techniques, representational techniques, explanatory strategies, and foundational characterizations) we shall consider integrating concepts and directions for future work.

10.3.1 Applications of representational techniques

Two levels of semantic analysis

We can notice that in all of our applications of the representational schema, there were two levels to the semantic analysis: the level of characterizing semantic conventions and the level of applying a representational technique. The same semantic conventions could be sufficient to imply the technique-specific assumptions of multiple techniques (e.g. the semantic conventions governing acceptable interpretations of the $\mathcal{P}(WFF)$ -language of propositional logic imply both the Set *E* or Set *CG* assumptions), or different semantic conventions could be sufficient to imply the technique-specific assumptions of a single technique (e.g. the semantic conventions governing acceptable interpretations of both the *WFF* or $\mathcal{P}(WFF)$ -languages of propositional logic imply the Set *E* assumptions).

The argument is not that these two levels were not present in traditional semantic analyses. For example, many accounts of the semantics of propositional logic (though not all!

See Barwise and Etchemendy 1999) include the convention “the sentence symbols are logically independent.” What we would like to point out is that the process of applying the representational schema makes the separate existence of these levels more clear, and gives us a method for formally specifying both these semantic conventions, and the representational techniques whose technique-specific assumptions they imply; and then considering the interaction between the two levels.

For example, the two-level analysis allows us to give an explanation of why the corollary to Lindenbaum’s Lemma works as a proxy for consequence in the case of propositional logic. Given that the semantic conventions for interpretations of a language imply monotonic extension of commitment and weak extensibility, then if further given that *LL* (the representational version of Lindenbaum’s Lemma) holds of a consistency medium modelling that language, it is the case that *LLC* (the representational version of the corollary to Lindenbaum’s Lemma) is equivalent to the representational conception of logical consequence.

It is interesting to consider that the two levels of analysis have different purposes. The level of representational techniques is geared to the purpose of making the representational relation of logical consequence explanatorily intelligible. The level of semantic conventions has (at least) two purposes. One purpose is that the semantic conventions of the language should make possible an explanatory account of the relation of logical consequence for the language (and one way to do that is by implying the technique-specific assumptions of some representational technique). But another purpose is to ensure the usability of the language by the subjects (human, machine, ...) who will be using it to think and communicate. So in creating new representational techniques, our focus is more on making the process of modelling consequence easier or more theoretically productive. Whereas in creating languages with new semantic conventions, our focus includes making the languages created easier to speak, manipulate, and understand.

We saw, for instance, that the semantic conventions of propositional logic imply that acceptable interpretations of that language satisfy the conditions of monotonic extension of commitment and weak extensibility. We further saw reasons, from a practical, applied perspective, why it would be useful for a language to support those properties. Yet we also saw that those properties are not necessary for a model-theoretic explanatory account

of consequence. For example, we were able to give a model-theoretic explanatory account of consequence for the simple language *Roll*, even though its acceptable interpretations satisfied neither property. We could argue that the semantic conventions of propositional logic evolved the way they did because those conventions make the language more usable.

Thinking about the level of semantic conventions opens up an important level of variation among languages. For example, it allows us to consider the multiple ways in which different languages make it possible for their users to extend their commitments. Contrast the powerset construction of the $\mathcal{P}(WFF)$ -language of propositional logic with the subsumption relation for feature structures. There are two very different ways of extending commitments, and yet they both imply monotonic extension of commitment. The two approaches are more or less usable than each other in different situations.

Something which is sometimes hard to remember is that our languages did not spring from Gargantua's ear fully formed (Rabelais 1990). They were created and evolved over time. Feature structures are an obvious example. In the process of creating new (or evolving existing) languages, having a clear distinction between the semantic conventions of a language and the properties required for making consequence intelligible, may help designers in the process of constructing new modes of expression. For example, perhaps there are other ways of extending commitments than those we have already seen? In designing a language with a new approach to extending commitments, one might want to keep in mind that if the semantic conventions created for the language implied the conditions of monotonic extension of commitment and weak extensibility, then one of the order-consistency techniques could be applied to make the consequence relation for that language explanatorily intelligible.

Importance of justificatory accounts for proposed semantic relations of logical consequence

An important message from Etchemendy's work in the "Concept of Logical Consequence" (Etchemendy 1990) is that any proposed semantic model of the relation of logical consequence needs a principled justification, and that Tarski's interpretational justification of the model-theoretic approach is flawed. Etchemendy proposes representational semantics

as a means of constructing such justifications. If we accept Etchemendy’s position, that means that we need to carry out such justificatory constructions for every proposed model of consequence. It is not sufficient to construct a semantic account which is “analogous to Tarski” and consider the account, for that reason alone, to be complete. We saw this in the case of the feature logic application above (Sections 3.3.3 and 3.6). There, we claim that for the account of consequence between sentences in a feature logic to be complete, we should construct a justificatory account showing that the proposed proxy for consequence relation is in fact equivalent to the representational conception of logical consequence.

Completing the feature structure accounts

One of the next steps in the project, then, is to see if we can use the representational schema to construct justificatory accounts for models of logical consequence for some language of feature logic, and some language in which feature structures are used as assertion types. To complete these accounts, we would need to characterize the semantic conventions for acceptable interpretations of feature structures. We have seen that we should expect these conventions to be different when feature structures are used as models (for feature logics) and when feature structures are used as assertion types. In the case of a feature logic, we will seek to develop a proof that the semantic conventions for acceptable interpretations of the sentences in the logic and the feature structures (used as models) imply the technique-specific assumptions for some representational technique. Our candidate technique for this proof is the model-theoretic Set *BE* technique. In the case of a language in which feature structures are used as assertion types, we will seek to develop a proof that the semantic conventions for feature structures (used as assertion types) imply the technique-specific assumptions for some representational technique. The candidate technique we propose for this proof is the order-consistency Set *CG* technique.

10.3.2 Representational techniques

Multiple dimensions of flexibility in constructing techniques

We have seen that there are many dimensions of flexibility in constructing representational techniques. Here we will consider three: flexibility in the roles of specific technique-specific

assumptions; flexibility in the kind of technique-specific assumptions used; and flexibility in the “division of labor” between the technique-specific assumptions and the proxy for consequence relation.

First, let’s consider flexibility in the roles of specific technique-specific assumptions. We saw in many cases, that the technique-specific assumptions for a technique played a dual role. Each assumption was used in two ways, both to ensure the intelligibility of consequence and to support the explanatory strategy. In the case of the Set *BE* technique, we saw something different. There, two assumptions (*B1* and *B2*) were used to ensure the intelligibility of consequence, and a third (*A7*) was used to support the explanatory strategy.

Secondly, we have seen flexibility in the kind of technique-specific assumptions used. The technique-specific assumptions used by the model-theoretic representational techniques placed no requirements on assertion types involving the possibilities represented by some other type or types. Whereas the order-consistency techniques did involve such assumptions (such as monotonic extension of commitment and weak extensibility). Those assumptions introduced expressibility constraints into the account of consequence. For the languages we were considering, those expressibility constraints were implied by the semantic conventions of the language.

Third, we saw flexibility in how the work of making consequence intelligible could be split across the technique-specific assumptions and the proxy for consequence relation. In all of the model-theoretic techniques we considered, the only difference between the techniques was in the set of technique-specific assumptions which they used. Whereas in the order-consistency case, we constructed the Set *CG* technique from the Set *CL* technique by deleting the technique-specific assumption *LL* (the representational version of Lindenbaum’s Lemma), and then changing the proxy for consequence relation to “make up the difference.” This highlights important degree of freedom which can be used in constructing new techniques. Not only can we change the technique-specific assumptions, we can also change the proxy for consequence relation. This flexibility reflects that which we see in logical semantics generally, where a combination of (1) truth definition and (2) special constraints on the class of models determines what you get as a logic, and setting those two semantic “parameters” to different combinations often results in the same “yield.”

Completing our picture of the relations between the ranges of applicability of the various techniques

We have gone a long way toward constructing a picture of how the relative ranges of applicability of representational techniques compare with one another. We have been able to arrange the techniques discussed into a linear order in terms of relative range of applicability. However, it is important to note that our understanding of the whole picture is partial, and that potentially valuable understanding awaits our completing this picture. The first way in which our current picture is partial is that we have demonstrated a series of inclusions, but in each case, we do not know whether the inclusion is proper, or in fact an equality. In a number of the cases (particular $ra(\text{Set } PP) \subseteq ra(\text{Set } CL)$ and $ra(\text{Set } CG) \subseteq ra(\text{Set } BE)$) we have tried to prove the reverse inclusions but have so far been unsuccessful. The next steps in this area are to continue working on those proofs in the hopes of either finding a way to make them go through or being able to characterize counterexamples. Either result should add to our understanding. A second way in which our picture is partial is that it only reflects those techniques which we have already defined.

Looking ahead to more refined comparisons of ranges of applicability

It is one thing for a technique to be technically capable of modelling the relation of logical consequence for a particular interpreted set of assertion types; it is another thing for that modelling process to be easy to carry out, or for the constructed models to be easy to work with. We should look ahead to developing more refined categories and comparisons with respect to ranges of applicability. For example, it may prove useful to be able to answer the question, for a particular interpreted set of assertion types: “Can the representational relation of logical consequence for this interpreted set of assertion types be modelled by technique X using an intelligible medium in which all of the assertion types are expressible by a finite structure?” Asking such questions, and then being able to compare techniques in terms of their relative ranges with respect to such questions, should be a source of deeper understanding.

Open to new techniques

A natural next step is to continue seeking to define new techniques. Nothing in our investigations suggests that the list of techniques we have considered is complete.

10.3.3 Explanatory strategies

The relation of model-theoretic and order-consistency semantics

Let us consider two high-level relations between model-theoretic and order-consistency explanatory strategies. First, we will look at the overlap in terms of the relative ranges of applicability of techniques using the two strategies; and then we will consider the way in which order-consistency models of consequence contain “implicit models.”

From our analysis of the relative ranges of applicability of representational techniques, we see that the ranges of applicability of the two model-theoretic and order-consistency strategies overlap significantly. Many interpreted sets of assertion types can have their representational relation of logical consequence modelled from both perspectives. We have seen that any interpreted set of assertion types whose relation of logical consequence can be modelled by an application of the Set *E* or Set *PP* model-theoretic techniques can be modelled by either the Set *CL* or Set *CG* order-consistency techniques; and that any interpreted set of assertion types whose relation of logical consequence can be modelled by the Set *CL* or Set *CG* order-consistency techniques can be modelled by the Set *BE* model-theoretic technique.

One thing that is interesting about the overlap is that it enables us to use techniques from both strategies in a particular application. For example, we could use a technique of one strategy to model a relation of consequence for a language (perhaps because that strategy was easier to apply in the case of that language); and then use the methods of the proofs of range inclusion to move that description into the context of the other explanatory strategy; perhaps to take advantage of results or techniques available only there.

Another thing interesting point results from the proofs in Chapters 8 and 9 (the proof showing the inclusion of the range of the Set *CG* technique within the range of the Set *BE* technique, and the construction of a “minimal equivalent” for any application of the

Set *CG* technique). While a consistency medium does not explicitly use models to model the relation of logical consequence; there is a set of models implicit in any consistency medium satisfying the Set *CG* technique-specific assumptions with respect to the practice of categorization interpreting its assertion types. That set of models is characterized by the operator *tm* in the proof of Chapter 8. From Chapter 9, we also see that the set of those models are themselves the assertion types of a consistency medium satisfying the Set *CG* technique specific assumptions with respect to the practice of categorization interpreting them. This concept of implicit models is related to the notion of a “core” from Barwise and Seligman 1997, Lecture 4 and Lecture 6. There those authors present results showing that one can always construct a core from any family of classifications and informorphisms.

Considering the relation to information systems

We expect that the theory of information systems, algebraic $\bigcap$ -structures, and domains is related to order-consistency semantics. The work on minimal media presented in Chapter 9 is a prospective first step toward building a connection between the concepts developed in this dissertation and that theory.

The two explanatory strategies need not be the only ones

We have shown there to be two explanatory strategies capable of grounding representational models of logical consequence. It is interesting to think that there may be others. It could be that in changing the foundational characterizations (see below) that other explanatory strategies may become possible.

10.3.4 Foundational characterizations

Every technique which we considered in this dissertation used the same foundational characterization of the space of possibility, assertion types, practices of categorization interpreting assertion types, and the representational conception of logical consequence. These assumptions are not the only possible ones. First we consider a reflection on the notion of “intelligibility,” then we turn to consider variations on the foundational assumptions.

Intelligibility is a general idea

One of the key ideas underlying the representational schema is the notion of “intelligibility,” the way in which a relation defined in terms of the structure of an “intelligible medium” can, given certain assumptions, act as a proxy for some other relation assumed to exist in the space of possibility. The kind of intelligibility we were most concerned with was the “intelligibility of consequence.” In that case, the representational relation of logical consequence (as we had defined it) was the relation assumed to exist in the space of possibility for which we sought a proxy. But interestingly, in the course of our explorations, we saw that “intelligibility” is a general idea, and put it to use in many forms, including the intelligibility of consistency, the intelligibility of compossibility, and the intelligibility of logical truth.

Three-valued practices of categorization

If we are seeking to model the interpretation of assertion types as carried out by actual agents in the world, it may be of more value to use a three-valued approach to practices of categorization. On such an approach, the practice by which an assertion type is interpreted divides the space of possibility into three parts: those possibilities for which the practice holds the assertion type true; those possibilities for which the practice holds the assertion type false; and those possibilities for which the practice is uncommitted or neutral. This would require a change in the assumed representational conception of logical consequence; and those changes would ripple through the entire theoretical structure developed.

Non-monotonic reasoning

The representational relation of logical consequence we assumed for the purposes of our discussion was standard consequence. A direction for future study would be to consider how the concepts and techniques presented here could apply to other kinds of consequence relations, e.g. of the kind modelled in studies of non-monotonic reasoning. Such phenomena are considered in Barwise and Seligman 1997, Lecture 19.

10.4 Conclusion

Our work in this dissertation, together with these reflections and potential next steps all point in a common direction. That direction is one which sees logic as a dynamic field, seeking to create principled accounts of concepts of logical consequence. In this work, we have constructed and explored a framework for understanding, comparing, and creating such accounts on the basis of the principles of intelligibility and explanatory strategy.

This orientation toward “principled creativity” is perhaps the most important lesson my teachers have given me. They have taught me that the way is open, and that the possibilities for creativity are there; but that as we create, we should keep our principles, the ways in which we choose to order our creations, in mind. Those principles are more than measures, they are a source of creativity itself.

Part VI

Appendices

Appendix A

Logical Truth and “Analytic Consequence”

A.1 The concept of analytic consequence in the informal setting

For the purposes of this section, we need to recall the way we were thinking of model-theoretic semantics before we began the process of formalization, i.e. as we considered it in section 1.3. We made certain assumptions about how we were choosing to think about logical consequence and logical truth: namely, that a sentence p was a logical truth iff it was true in all possibilities; and that a sentence p was a logical consequence of a set of sentences Σ just in case the argument $\langle \Sigma, p \rangle$ preserved truth in every possibility. In terms of our chosen interpretation of logical consequence, we can define a new concept, namely “**analytic consequence**.” We can say that a sentence p is an analytic consequence iff p is a logical consequence of any set of sentences. Given our chosen interpretations, it is a simple argument that the following claims are equivalent: p is an analytic consequence, p is a logical consequence of $\emptyset$, and p is logically true. That analytic consequence and logical truth are equivalent is a point is made by Tarski (1956, 418). The proof is trivial but just to ensure that the way we are using the various terms is clear, we give it below.

A.1.1 Equivalence of “ p is an analytic consequence” and “ p is a logical consequence of $\emptyset$ ”

1. By definition, “ p is an analytic consequence” is equivalent to “For any set of sentences Σ , the argument $\langle \Sigma, p \rangle$ preserves truth in all possibilities.”
2. Claim: “For any set of sentences Σ , the argument $\langle \Sigma, p \rangle$ preserves truth in all possibilities.” is equivalent to “the argument $\langle \emptyset, p \rangle$ preserves truth in all possibilities.”
 - (a) $(\rightarrow)$ Immediate since $\emptyset$ is a set of sentences. ■ $(\rightarrow)$
 - (b) $(\leftarrow)$ If $\langle \emptyset, p \rangle$ preserves truth in all possibilities, then p is true in every possibility. So regardless of Σ , the argument $\langle \Sigma, p \rangle$ preserves truth in all possibilities. ■ $(\leftarrow)$ ■ Claim.

A.1.2 Equivalence of “ p is a logical consequence of $\emptyset$ ” and “ p is logically true”

$\langle \emptyset, p \rangle$ preserves truth in all possibilities iff
 in every possibility, either some element of $\emptyset$ is false, or p is true iff
 p is true in every possibility iff
 p is logically true.

A.2 Formalizing the concept of analytic consequence

Our informal understanding of analytic consequence is: a sentence p is an analytic consequence iff p is a logical consequence of any set of sentences. In our formal account of the model-theoretic framework, we have characterized arguments more generally as consisting of pairs of assertion types: the first assertion type as premise, and the second as conclusion. So we could say that relative to set of assertion types Σ and practice of categorization $\mathcal{T}_\Sigma$ interpreting Σ ,

for any assertion type $p \in \Sigma$, p is an analytic consequence iff $\forall q \in \Sigma (ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(q, p))$.

We will define a predicate *AAC* (Assumed Analytic Consequence) on Σ as follows:

$$AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p) \text{ iff } \forall q \in \Sigma \left(ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(q, p) \right).$$

We show now a simple result about the analytic consequence predicate.

Proposition 118 *For any set of assertion types Σ interpreted by any practice of categorization $\mathcal{T}_\Sigma$, it is the case that*

$$\forall p \in \Sigma, AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p) \text{ iff } \mathcal{T}_\Sigma(p) = \bigcup \mathcal{T}_\Sigma[\Sigma]$$

($\rightarrow$)

1. Let p be an arbitrary element of Σ such that $AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p)$.
2. So $\forall q \in \Sigma \left(ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(q, p) \right)$ by definition of AAC .
3. So $\forall q \in \Sigma \left(\mathcal{T}_\Sigma(q) \subseteq \mathcal{T}_\Sigma(p) \right)$ by definition of ALC .
4. So $\bigcup \mathcal{T}_\Sigma[\Sigma] \subseteq \mathcal{T}_\Sigma(p)$.
5. Since $p \in \Sigma$, $\mathcal{T}_\Sigma(p) \subseteq \bigcup \mathcal{T}_\Sigma[\Sigma]$.
6. So $\mathcal{T}_\Sigma(p) = \bigcup \mathcal{T}_\Sigma[\Sigma]$. ■ ($\rightarrow$)

($\leftarrow$)

1. Let p be an arbitrary element of Σ such that $\mathcal{T}_\Sigma(p) = \bigcup \mathcal{T}_\Sigma[\Sigma]$.
2. Let q be an arbitrary element of Σ .
3. Then $\mathcal{T}_\Sigma(q) \subseteq \bigcup \mathcal{T}_\Sigma[\Sigma]$.
4. So $\mathcal{T}_\Sigma(q) \subseteq \mathcal{T}_\Sigma(p)$.
5. So $ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(q, p)$.
6. Since q was arbitrary, $\forall q \in \Sigma \left(ALC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(q, p) \right)$
7. So $AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}(p)$. ■ ($\leftarrow$) ■ Proposition.

A.3 A distinction between analytic consequence and logical truth

Now we can notice something very interesting. The equivalence of analytic consequence and logical truth which held in the informal account of model-theoretic semantics does not carry across into our formalization. For arbitrary set of assertion types Σ and practice of categorization $\mathcal{T}_\Sigma$, we have characterized logical truth ($ALT_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$) for assertion type $p \in \Sigma$ by $\mathcal{T}_\Sigma(p) = \mathcal{U}$. Whereas we have characterized analytic consequence ($AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$) for assertion type $p \in \Sigma$ by $\mathcal{T}_\Sigma(p) = \bigcup \mathcal{T}_\Sigma[\Sigma]$. When the possibilities represented by the assertion types of Σ (that is, $\bigcup \mathcal{T}_\Sigma[\Sigma]$) do not equal to whole space of possibility ($\mathcal{U}$), then assessments of logical truth and analytic consequence may diverge. Some analytic consequences may not be analytic truths.

Where did this difference between the informal and formal accounts come from? It has its source in the fact that the form of arguments which we are using in our formalization (premise/conclusion pairs of assertion types) is more general than the form of arguments used in the informal representational account (set of sentences as premises and single sentence as conclusion). Our formalization makes clear something built into the informal account by making us consider the empty set of premises as an assertion in its own right. Because of the way in which preservation of truth across all possibilities is defined, the empty set of premises functions as if it were an assertion true in every possibility, that is, it functions as a logical truth. For any set of assertion types Σ , if Σ contains an assertion type p which is a logical truth, then $\bigcup \mathcal{T}_\Sigma[\Sigma] = \mathcal{U}$, and hence, logical truth ($ALT_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$) and analytic consequence ($AAC_{\langle \Sigma, \mathcal{T}_\Sigma \rangle}$) are equivalent. Another case in which $\bigcup \mathcal{T}_\Sigma[\Sigma] = \mathcal{U}$, and thus one in which logical truth and analytic consequence are equivalent, is when $\langle \Sigma, \mathcal{T}_\Sigma \rangle$ is a conventional application of a language with negation and a semantic convention of excluded middle, like the $\mathcal{P}(WFF)$ -language of propositional logic.

We can consider the condition that $\bigcup \mathcal{T}_\Sigma[\Sigma] = \mathcal{U}$ with respect to both model-theoretic and order-consistency techniques.

A.3.1 Model-theoretic techniques

We show below that neither the intelligibility of consequence nor the intelligibility of logical truth for truth media depend upon the condition that $\bigcup \mathcal{T}_\Sigma [\Sigma] = \mathcal{U}$. So the existence of a built in assertion which functions as a logical truth is not a requirement for using model-theoretic semantics. Instead it is an artifact of a specific way of constructing a model-theoretic semantics. At the more general level at which our formalization is functioning, logical truth and analytic consequence may be extensionally distinct.

Proposition 119 *It is not the case that if truth medium A satisfies assumption set E with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, that $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] = \mathcal{U}$.*

We construct a case in which truth medium A satisfies assumption set E ($B5, A4, A5, B8$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, and $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] \neq \mathcal{U}$.

$$\Sigma_A = \{p\}$$

$$M_A = \{m, n, o\}$$

$$\mathcal{U} = \{1, 2, 3, 4, 5\}$$

Define $\mathcal{T}_{\Sigma_A}$ as follows:

x	$\mathcal{T}_{\Sigma_A}(x)$
p	$\{1, 2, 3\}$

So $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] = \{1, 2, 3\}$.

Define $\mathcal{T}_{M_A}$ as follows:

x	$\mathcal{T}_{M_A}(x)$
m	$\{1\}$
n	$\{2, 3\}$
o	$\{4, 5\}$

The above definitions satisfy assumptions $A4$ (the models are disjoint) and $A5$ (the models are consistent).

We use the statement of $B5$ to construct t_A .

Construct t_A as follows:

x	$t_A(x)$
p	$\{m, n\}$

Given the above definitions, assumptions $B5$ and $B8$ are satisfied. (For $B8$ take $S = \{m, n\}$, then $\bigcup \mathcal{T}_{M_A} [S] = \{1, 2, 3\} = \mathcal{T}_{\Sigma_A}(p)$, and take $R = \{o\}$, then $\bigcup \mathcal{T}_{M_A} [R] = \{4, 5\} = \mathcal{U} - \mathcal{T}_{\Sigma_A}(p)$).

In our example, A satisfies assumption set E ($B5, A4, A5, B8$) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, yet $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] \neq \mathcal{U}$. ■Proposition.

From this result, we see that none of the assumption sets implied by E imply that $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] = \mathcal{U}$.

Corollary 120 *It is not the case that if truth medium A satisfies assumption set TP (or B or LT) with respect to $\mathcal{T}_{\Sigma_A}$ and $\mathcal{T}_{M_A}$, that $\bigcup \mathcal{T}_{\Sigma_A} [\Sigma_A] = \mathcal{U}$.*

This is immediate from Proposition 119 and Proposition 13. ■Corollary.

A.3.2 Order-consistency techniques

We show below that the intelligibility of consequence for consistency media also does not depend upon the condition that $\bigcup \mathcal{T}_{\Sigma} [\Sigma] = \mathcal{U}$. So this assumption is not a requirement for using order-consistency semantics either.

Looking at the example in Section 5.4, we see that we could consistently supplement the conditions OEX with an additional constraint that there be some possibility in which none of $odd, even, oddOrEven, oddAndEven$, is true. Under those expanded conditions, $\bigcup \mathcal{T}_{\Sigma_{OE}} [\Sigma_{OE}] \neq \mathcal{U}$, but it would still be the case that the consistency medium OE satisfies $D1, D2, D3$ with respect to $\mathcal{T}_{\Sigma_{OE}}$, and LC_{OE} makes the assumed relation of consequence $ALC_{\langle \Sigma_{OE}, \mathcal{T}_{\Sigma_{OE}} \rangle}$ intelligible.

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