

# AN EXPOSITION OF THE TIAN-TODOROV THEOREM

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## 1. INTRODUCTION

We work over a field  $k$  of characteristic zero.

**Definition 1.** A *weak Calabi-Yau variety* is a smooth, connected, projective  $k$ -scheme  $X$  such that the canonical sheaf  $\omega_X \simeq \mathcal{O}_X$ . A *Calabi-Yau variety* is a weak Calabi-Yau variety such that

$$H^i(X, \mathcal{O}_X) = 0$$

for  $0 < i < \dim(X)$ .

The main goal of this note is to describe a relatively cheap proof of the Bogomolov-Tian-Todorov theorem: that Calabi-Yau varieties are unobstructed (i.e. the “moduli space” of Calabi-Yau varieties is smooth). This theorem now has many, many proofs (see e.g. [Bog78, Tod89, Tia87, IM10], ...); the proof contained here is probably not original, and I would guess that it is morally similar to work of Ziv Ran [Ran92]. Unlike the original proofs of Tian and Todorov, it is entirely algebraic.

Let  $\text{Art}_k$  be the category of local Artin  $k$ -algebras with residue field  $k$ . If  $A \in \text{Art}_k$  is an Artin  $k$ -algebra with maximal ideal  $\mathfrak{m}_A$ , and  $X$  is a  $k$ -scheme, a *deformation of  $X$  over  $A$*  is a quasi-coherent sheaf of rings  $\mathcal{O}_{X_A}$  on  $X$  with a short exact sequence

$$0 \rightarrow \mathfrak{m}_A \otimes \mathcal{O}_X \rightarrow \mathcal{O}_{X_A} \rightarrow \mathcal{O}_X \rightarrow 0,$$

where the map  $\mathcal{O}_{X_A} \rightarrow \mathcal{O}_X$  is an algebra map. A morphism of deformations over  $A$  is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}_A \otimes \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_A} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{m}_A \otimes \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'_A} & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \end{array}$$

Then  $\text{Def}_X : \text{Art}_k \rightarrow \text{Sets}$  is the functor sending an Artin  $k$ -algebra  $A$  to the set of deformations of  $X$  over  $A$ , modulo isomorphism.

The main theorem is:

**Theorem 2** (Bogomolov-Tian-Todorov). *Let  $X$  be a weak Calabi-Yau variety such that  $H^0(X, T_X) = 0$ . Then the functor*

$$\mathrm{Def}_X : \mathrm{Art}_k \rightarrow \mathrm{Sets}$$

*is formally smooth, and indeed is pro-represented by the ring*

$$k[[x_1, \dots, x_n]]$$

*where  $n = \dim_k H^1(X, T_X)$ .*

It is a theorem of Grothendieck [Gro95] that if  $X$  is a smooth and proper  $k$ -variety with  $H^0(X, T_X) = 0$ ,  $\mathrm{Def}_X$  is pro-representable. Thus we focus on the smoothness aspect of the theorem.

*Remark 3.* A Calabi-Yau variety  $X$  of dimension  $n \geq 2$  satisfies the condition  $H^0(X, T_X) = 0$ . Indeed, by Serre duality,  $H^0(X, T_X) = H^n(X, \Omega_X^1)^\vee$ , which has the same dimension as  $H^1(X, \omega_X)$  by Hodge theory. But  $\omega_X \simeq \mathcal{O}_X$ , so this is just  $H^1(X, \mathcal{O}_X)$ , which is zero by the definition of a Calabi-Yau variety.

## 2. THE $T^1$ -LIFTING THEOREM

We begin with a commutative algebra lemma, known as the  $T^1$ -lifting theorem, which allows one to test whether or not a complete local Noetherian  $k$ -algebra is smooth.

**Definition 4.** Let  $A$  be a ring and  $M$  an  $A$ -module. The ring  $A \oplus M$  is defined so that

$$(a, m) + (a', m') = (a + a', m + m')$$

and

$$(a, m) \cdot (a', m') = (aa', am' + a'm).$$

**Lemma 5** ( $T^1$ -lifting theorem). *Let  $R$  be a local Noetherian  $k$ -algebra with maximal ideal  $\mathfrak{m}_R$  and residue field  $k$ , complete with respect to the  $\mathfrak{m}_R$ -adic topology. Then  $R \simeq k[[x_1, \dots, x_n]]$ , with  $n = \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ , if and only if:*

**t1:** *For each  $A \in \mathrm{Art}_k$ , and each  $M, M' \in A\text{-mod}$  with a surjection  $M' \rightarrow M$ , the induced map*

$$\mathrm{Hom}_{k\text{-Alg}}(R, A \oplus M') \rightarrow \mathrm{Hom}_{k\text{-Alg}}(R, A \oplus M)$$

*is surjective.*

We give two proofs of this lemma: one conceptual and one “hands-on.”

*Remark 6.* This lemma *requires* that  $k$  be characteristic zero. Indeed, if  $k$  is of characteristic  $p > 0$ , the algebra  $k[t]/t^p$  satisfies condition **t1**, but is certainly not smooth.

*Conceptual Proof of Lemma 5.* Consider  $A \in \mathrm{Art}_k$  and a surjection  $f : M' \rightarrow M$  of  $A$ -modules. We have a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{k\text{-Alg}}(R, A \oplus M') & \xrightarrow{h_f} & \mathrm{Hom}_{k\text{-Alg}}(R, A \oplus M) \\ & \searrow \pi_{M'} & \swarrow \pi_M \\ & \mathrm{Hom}_{k\text{-Alg}}(R, A) & \end{array}$$

Condition **t1** holds if and only if for each  $k$ -Algebra map  $g : R \rightarrow A$ , the map  $\pi_{M'}^{-1}(g) \rightarrow \pi_M^{-1}(g)$ , induced by  $h_f$ , is surjective. (That is, we may check surjectivity of the map  $h_f$  on fibers of  $\pi_{M'}, \pi_M$ .) In other words, condition **t1** is equivalent to

**t2:** For each  $A \in \text{Art}_k$ ,  $g : R \rightarrow A$ , and  $M, M' \in A\text{-mod}$  with a surjection  $M' \twoheadrightarrow M$ , the induced map

$$\pi_{M'}^{-1}(g) \rightarrow \pi_M^{-1}(g)$$

is surjective.

Now, (exercise!) if  $B$  is in  $\text{Art}_k$ ,  $N$  is a  $B$ -module,  $g : R \rightarrow B$  is a map, and  $\pi_N : \text{Hom}_{k\text{-Alg}}(R, B \oplus N) \rightarrow \text{Hom}_{k\text{-Alg}}(R, B)$  is the induced surjection, we have that

$$\pi_N^{-1}(g) = \text{Der}_k(R, N) \simeq \text{Hom}_{R\text{-mod}}(\widehat{\Omega}_R^1, N).$$

Thus condition **t2** is the statement that the functor  $\text{Hom}_{R\text{-mod}}(\widehat{\Omega}_R^1, -)$  is right exact, i.e.  $\widehat{\Omega}_R^1$  is projective, thus free.

Now let  $P = k[[x_1, \dots, x_n]]$ , with  $n = \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ . Let  $P \twoheadrightarrow R$  be a surjection with kernel  $I$ . Then the conormal exact sequence gives

$$I/I^2 \rightarrow \widehat{\Omega}_P^1 \otimes_P R \rightarrow \widehat{\Omega}_R^1 \rightarrow 0.$$

But the map  $\widehat{\Omega}_P^1 \otimes_P R \rightarrow \widehat{\Omega}_R^1$  is a surjection of free  $R$ -modules of the same (finite) rank, hence an isomorphism; that is, the map  $I/I^2 \rightarrow \widehat{\Omega}_P^1 \otimes_P R$  is zero. This map is given by

$$f \mapsto df \bmod I.$$

In other words, for each  $f \in I$ , we have that  $df \in I\widehat{\Omega}_P^1$ . Considering  $f \in I$  of minimal degree, this shows that  $I$  must be zero. (This is the step in which characteristic zero is used.)  $\square$

*Remark 7.* The condition **t1** is often replaced with condition **t2** in the statement of the  $T^1$ -lifting theorem.

*“Hands-on” Proof of Lemma 5.* It is easy to see that if  $R = k[[x_1, \dots, x_n]]$ , condition **t1** is satisfied. We prove the other direction now.

Let  $P = k[[x_1, \dots, x_n]]$ , where  $n = \dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ , and let  $P \twoheadrightarrow R$  be a surjection with kernel  $I$ . By our choice of  $n$ , we have that  $I \subset \mathfrak{m}_P^2$ , where  $\mathfrak{m}_P := (x_1, \dots, x_n)$ . It suffices to show that  $I \subset \mathfrak{m}_P^r$  for all  $r$ . Indeed, suppose that  $I \subset \mathfrak{m}_P^{r-1}$ , so that the natural map  $P \rightarrow A_r := P/\mathfrak{m}_P^{r-1}$  descends to a map

$$g_r : R \rightarrow A_r.$$

Let  $M_r$  be the  $A_r$ -module

$$\bigoplus_{i=1}^n A_{r-1}\epsilon_i.$$

Consider the map  $M_{r+1} \rightarrow M_r$  given by the obvious quotient map in each coordinate; we will show that condition **t1** for this map implies that  $I \subset \mathfrak{m}_P^r$ .

Let  $h_r : P \rightarrow A_r \oplus M_r = A_r \oplus (A_{r-1}\epsilon_1 \oplus \dots \oplus A_{r-1}\epsilon_n)$  be the map induced by

$$x_i \mapsto x_i + \epsilon_i.$$

Then a brief computation shows that for  $f \in P$ ,

$$f \mapsto f + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \epsilon_i.$$

Since  $I \subset \mathfrak{m}_P^{r-1}$  by the induction hypothesis,  $\frac{\partial f}{\partial x_i} \in \mathfrak{m}_P^{r-2}$  for each  $f \in I$ , so the map  $h_r$  descends to a map  $\overline{h}_r : R \rightarrow A_r \oplus M_r$ .

Let  $z_r : R \rightarrow A_r \oplus M_{r+1}$  be a putative lift of  $\overline{h}_r$ . A short computation shows that the existence of  $z_r$  implies that for  $f \in I$ ,

$$\frac{\partial f}{\partial x_i} \in \mathfrak{m}_P^{r-1}$$

for all  $i$ . In characteristic zero, this implies that  $f \in \mathfrak{m}_P^r$ . As  $f$  was arbitrary, we have shown that  $I \subset \mathfrak{m}_P^r$ , as desired.  $\square$

*Remark 8.* Observe that this second proof shows we need only consider Artin rings and modules over them of a very specific type: namely, the  $A_r, M_r$  above.

### 3. PROVING THE THEOREM

We will see that the main input is an important theorem of Deligne and Illusie [DI87, Corollaire 4.1.4]:

**Theorem 9.** *Let  $S$  be a  $k$ -scheme and  $f : X \rightarrow S$  a smooth projective morphism. Then  $R^p f_* \Omega_f^q$  is locally free and its formation commutes with base change.*

It follows from this result that deformations of (weak) Calabi-Yau varieties remain weak Calabi-Yau.

**Lemma 10** (Deformations of Calabi-Yau varieties are Calabi-Yau). *Let  $X/k$  be a weak Calabi-Yau variety. Let  $A \in \text{Art}_k$  be an Artin  $k$ -algebra, and let  $f : \mathcal{X}_A \rightarrow A$  be a deformation of  $X$  over  $A$ . Then  $\omega_{\mathcal{X}_A/A} \simeq \mathcal{O}_X$ .*

*Proof.* By Theorem 9,  $H^0(\mathcal{X}_A, \omega_{\mathcal{X}_A/A})$  is a locally free  $A$ -module whose formation commutes with base change. Changing base along the map  $\text{Spec}(k) \rightarrow \text{Spec}(A)$ , we find that  $H^0(\mathcal{X}_A, \omega_{\mathcal{X}_A/A})$  is free of rank one, that is,  $H^0(\mathcal{X}_A, \omega_{\mathcal{X}_A/A}) \simeq A$ . Hence  $\omega_{\mathcal{X}_A/A}$  is trivialized by the global section  $1 \in A$ .  $\square$

Finally, we turn to the proof of the main theorem.

*Proof of Theorem 2.* Let  $X$  be a Calabi-Yau variety of dimension  $n$ , as in the statement of the theorem. As  $\text{Def}_X$  is prorepresentable [Gro95], it suffices to check that  $\text{Def}_X$  satisfies the  $T^1$ -lifting condition, i.e. that for every  $A \in \text{Art}_k$  and every surjection of  $A$ -modules  $M' \rightarrow M$ , the map

$$\text{Def}_X(A \oplus M') \rightarrow \text{Def}_X(A \oplus M)$$

is surjective.

Consider the diagram

$$\begin{array}{ccc} \text{Def}_X(A \oplus M') & \xrightarrow{h} & \text{Def}_X(A \oplus M) \\ & \searrow \pi_{M'} & \swarrow \pi_M \\ & \text{Def}_X(A) & \end{array}$$

We wish to show surjectivity of the top arrow  $h$ . Let  $f : \mathcal{X}_A \rightarrow \mathrm{Spec}(A)$  be an element of  $\mathrm{Def}_X(A)$ . It suffices to show that every deformation of  $\mathcal{X}_A$  over  $A \oplus M$  lifts to  $A \oplus M'$ , i.e. that the map  $h$  is surjective when restricted to a map  $\pi_{M'}^{-1}(f) \rightarrow \pi_M^{-1}(f)$ . But if  $N$  is an  $A$ -module, the fiber of the map  $\mathrm{Def}_X(A \oplus N) \rightarrow \mathrm{Def}_X(A)$  over  $f : \mathcal{X}_A \rightarrow \mathrm{Spec}(A)$  is identified with

$$H^1(\mathcal{X}_A, T_{\mathcal{X}_A/A} \otimes f^*N) = \mathbb{H}^1(R\Gamma(T_{\mathcal{X}_A/A}) \otimes_A^{\mathbb{L}} N),$$

where this last equality follows from the projection formula.

We claim that this module is in fact  $H^1(\mathcal{X}_A, T_{\mathcal{X}_A/A}) \otimes N$ . Indeed, it suffices to show that each  $H^i(\mathcal{X}_A, T_{\mathcal{X}_A/A})$  is projective. But

$$H^i(\mathcal{X}_A, T_{\mathcal{X}_A/A}) = H^{n-i}(\mathcal{X}_A, \Omega_{\mathcal{X}_A/A}^1 \otimes \omega_{\mathcal{X}_A/A})^\vee = H^{n-i}(\mathcal{X}_A, \Omega_{\mathcal{X}_A/A}^1)^\vee,$$

where the first equality is Serre duality and the last equality follows from Lemma 10. Now this last module is projective by Theorem 9.

So we wish to show that the map

$$H^1(\mathcal{X}_A, T_{\mathcal{X}_A/A}) \otimes_A M' \rightarrow H^1(\mathcal{X}_A, T_{\mathcal{X}_A/A}) \otimes_A M$$

is surjective, where this map is just induced by tensoring the map  $M' \rightarrow M$  with  $H^1(\mathcal{X}_A, T_{\mathcal{X}_A/A})$ .

But tensoring with a module is right exact, so the proof is complete.  $\square$

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