VARIATIONS OF HODGE STRUCTURE
NOTES FOR NUMBER THEORY LEARNING SEMINAR ON SHIMURA VARIETIES

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Contents

1. Introduction: Variation of Hodge Structure for Curves

[I'd like to start with the case of a relative curve, where we can see many of the features of the story of variations of Hodge structure, with the benefit that proofs are relatively easy, and that we will be able to work over an arbitrary locally Noetherian base (which we will later require to be smooth).]

1.1. **Hodge Theory for Curves.**

Suppose $\pi : \mathcal{C} \to S$ is a smooth projective morphism of relative dimension 1 with connected geometric fibers, and with S locally Noetherian. We wish to study how the de Rham cohomology of the fibers $\mathcal{C}_s$ of π, with their extra structure coming from Hodge theory, varies. Let $\Omega^\bullet_{dR,\pi}$ be the algebraic de Rham complex:

$$\Omega^\bullet_{dR,\pi} : 0 \to \mathcal{O}_\mathcal{C} \xrightarrow{d} \Omega^1_\pi \to 0.$$

Remark 1. Note that this complex does not live in the category of chain complexes of $\mathcal{O}_\mathcal{C}$-modules, as d is not $\mathcal{O}_\mathcal{C}$-linear; rather, it is $\pi^{-1}\mathcal{O}_S$-linear.

There is a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^1_\pi & \longrightarrow & \Omega^1_\pi & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d} & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_\mathcal{C} & \longrightarrow & \mathcal{O}_\mathcal{C} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which induces a long exact sequence in cohomology:

$$0 \to R^0\pi_*\Omega^\bullet_{dR,\pi} \to R^0\pi_*\mathcal{O}_\mathcal{C} \xrightarrow{\delta^0} R^0\pi_*\Omega^1_\pi \to R^1\pi_*\Omega^\bullet_{dR,\pi} \to R^1\pi_*\mathcal{O}_\mathcal{C} \xrightarrow{\delta^1} R^1\pi_*\Omega^1_\pi \to R^2\pi_*\Omega^\bullet_{dR,\pi} \to 0.$$

The main result of this section is that the boundary maps δ^0 and δ^1 are zero, which immediately implies:

Theorem 2 (Hodge Theory for Curves)**.** *With π as above, there are natural isomorphisms (coming from the long exact sequence)*

$$R^0\pi_*\Omega^\bullet_{dR,\pi} \xrightarrow{\sim} R^0\pi_*\mathcal{O}_\mathcal{C},$$

$$R^2\pi_*\Omega^\bullet_{dR,\pi} \xrightarrow{\sim} R^1\pi_*\Omega^1_\pi,$$

and a short exact sequence

$$0 \to R^0\pi_*\Omega^1_\pi \to R^1\pi_*\Omega^\bullet_{dR,\pi} \to R^1\pi_*\mathcal{O}_\mathcal{C} \to 0.$$

Proof Sketch. We wish to show that the boundary maps δ^0 and δ^1 vanish. Note that

$$\delta^i : R^i\pi_*\mathcal{O}_\mathcal{C} \to R^i\pi_*\Omega^1_\pi$$

is precisely the map on (derived) pushforwards induced by the morphism of abelian sheaves $d : \mathcal{O}_\mathcal{C} \to \Omega^1_\pi$. To check that δ^0 vanishes, we note that the natural map

$$\mathcal{O}_S \to \pi_*\mathcal{O}_\mathcal{C}$$

is an isomorphism, by cohomology and base change. But the composite map

$$\mathcal{O}_S \xrightarrow{\sim} \pi_*\mathcal{O}_\mathcal{C} \to \pi_*\Omega^1_\pi$$

is given at the level of sections by

$$\Gamma(U, \mathcal{O}_S) \xrightarrow{\sim} \Gamma(\pi^{-1}(U), \mathcal{O}_\mathcal{C}) \xrightarrow{d} \Gamma(\pi^{-1}(U), \Omega^1_\pi).$$

And d kills sections coming from S by definition, giving the desired vanishing.

Here are two (sketch) arguments for the vanishing of δ^1:

(1) (Less elementary) By e.g. Poincaré duality for algebraic de Rham cohomology,

$$R^2\pi_*\Omega^\bullet_{dR,\pi} \simeq \operatorname{coker}(\delta^1 : R^1\pi_*\mathcal{O}_\mathcal{C} \to R^1\pi_*\Omega^1_\pi)$$

is a line bundle. [Is there an easier way to see this?] But $R^1\pi_*\Omega^1_\pi$ is a line bundle as well, so $\delta^1 = 0$.

(2) (More elementary) Reduce to the case where S is a point using cohomology and base change. Then identify

$$\delta^1 : H^1(C, \mathcal{O}_C) \to H^1(C, \omega^1_C)$$

with the dual of the map

$$H^1(C, \omega^1_C)^\vee \to H^1(C, \mathcal{O}_C)^\vee$$

induced from δ^0 by Serre duality. As $\delta^0 = 0$ by the first part of this argument, this implies that $\delta^1 = 0$ as well.

□

Remark 3. This theorem should be viewed as an analogue of the fact that for a curve X defined over $\mathbb{C}$, we have

$$H^0_{dR}(X, \mathbb{C}) \simeq H^0(X, \mathcal{O}_X),\ H^2_{dR}(X, \mathbb{C}) \simeq H^1(X, \Omega^1_X),$$

and

$$H^1_{dR}(X, \mathbb{C}) \simeq H^0(X, \Omega^1_X) \oplus H^1(X, \mathcal{O}_X),$$

canonically, with $H^0(X, \Omega^1_X) = \overline{H^1(X, \mathcal{O}_X)}$. Here H^i_{dR} denotes (smooth) de Rham cohomology, or singular cohomology. Note that in the algebraic setting we only have a filtration of H^1_{dR}, not a grading.

Remark 4. Note that the boundary maps δ^0 and δ^1 are precisely the differentials appearing on the E^1 page of the Hodge-to-de-Rham (Fröhlicher) spectral sequence. All differentials on later pages of the spectral sequence vanish for degree reasons.

1.2. **Interlude about Connections, Local Systems, and the Analytic Setting.** For the next several weeks, we will be interested in studying $R^i\pi_*\Omega^\bullet_{dR,\pi}$ and the various structures with which it is endowed. If we are working over $\mathbb{C}$, we would like to have a way of passing back and forth between these structures and analytic data—e.g. comparisons between the algebraic theory and the usual (smooth, or analytic) theory of de Rham or Dolbeault cohomology. In particular, we would like to understand the relationship between the coherent analytic sheaf $R^i\pi_*\Omega^{\bullet,an}_{dR,\pi}$ and the sheaf $R^i\pi_*\underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is a (constant) sheaf in the analytic topology. Let us briefly describe $R^i\pi_*\underline{\mathbb{C}}$.

First, we consider a very general situation. Suppose $f : X \to S$ is a proper (smooth) submersion of smooth manifolds, with S connected. Ehresmann gives a description of such maps:

Theorem 5 (Ehresmann)**.** *f is a locally trivial fibration (a fiber bundle).*

In particular, the fibers X_s of f are all homeomorphic; thus in particular $H^i(X_s, A)$ do not depend on s, where A is any Abelian group.

Theorem 6 (Topological Proper Base Change)**.** *Suppose that $f : X \to S$ is a continuous proper map of Hausdorff topological spaces, with S locally compact, and that $\mathcal{F}$ is a sheaf of Abelian groups on X. Then the formation of (derived) pushforwards through f commutes with base change—e.g. if $g : T \to S$ is any continuous map, and*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

is Cartesian, then the natural map

$$g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$$

is an isomorphism. In particular, taking g to be the inclusion of a point $s \hookrightarrow S$, we have that

$$(R^i f_* \mathcal{F})_s \simeq H^i(X_s, \mathcal{F}|_{X_s}).$$

We conclude that where Ehresmann's theorem applies (when X, S are smooth manifolds, with S connected, and f is a smooth proper submersion), then $R^i f_* \underline{A}$ is a locally constant sheaf with stalk $(R^i f_* \underline{A})_s \simeq H^i(X_s, A)$ (an A-local system). In particular $R^i f_* \underline{\mathbb{C}}$ is a local system of $\mathbb{C}$-vector spaces. Now suppose $f : X \to S$ is a smooth proper map of (smooth) varieties over $\mathbb{C}$. In this case we may identify these stalks via a comparison theorem of Grothendieck:

Theorem 7 (Grothendieck)**.** *Let X be a nonsingular scheme of finite type over the complex numbers $\mathbb{C}$. Then the canonical map*

$$\mathbb{H}^*(\Omega^\bullet_{X,dR}) \to H^*_{dR}(X^{an}) \simeq H^*(X^{an}, \underline{\mathbb{C}})$$

is an isomorphism.

Putting this all together, we have that if $f : X \to S$ is a smooth proper map of smooth algebraic varieties over $\mathbb{C}$, there is a natural isomorphism

$$(R^i f_* \Omega^\bullet_{f,dR})^{an} \xrightarrow{\sim} R^i f^{an}_* \underline{\mathbb{C}} \otimes \mathcal{O}_S^{an}.$$

The object on the left is a fundamentally analytic object, and comes with extra structure which is not apparent in the (essentially algebraic) object on the right—in particular, it comes with a natural map from the local system $R^i f^{an}_* \underline{\mathbb{C}}$, and thus has an integral structure, coming from $R^i f^{an}_* \underline{\mathbb{Z}}$. It is natural to ask what extra data we need to recover $R^i f^{an}_* \underline{\mathbb{C}}$ from $R^i f^{an}_* \underline{\mathbb{C}} \otimes \mathcal{O}_S^{an}$.

In order to do this, we need the notion of a connection on a holomorphic vector bundle; we will briefly recall this notion. The goal will be to endow our holomorphic vector bundle with a notion of *differentiation*. The prototypical example will be differentiation itself, which we may view as the composition of a vector field $X \in \Gamma(S, TS)$ with the exterior derivative, e.g.

$$\mathcal{O}_S \xrightarrow{d} \Omega^1_S \xrightarrow{X} \mathcal{O}_S.$$

Thus if $\mathcal{E}$ is any holomorphic vector bundle, we define a connection ∇ to be a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \Omega^1_S$$

which is a derivation over $\mathcal{O}_S$, e.g. if s is a holomorphic function on S, and f is a section to $\mathcal{E}$, we have

$$\nabla(s \cdot f) = s \cdot \nabla f + f \otimes ds.$$

As before, we may use ∇ to differentiate; if X is a vector field, we use ∇_X to denote the composition

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_S \xrightarrow{X} \mathcal{E}.$$

Any connection ∇ admits a unique extension to a sequence $\nabla^0 = \nabla, \nabla^1, \nabla^2, \cdots$

$$\mathcal{E} \xrightarrow{\nabla^0} \mathcal{E} \otimes \Omega^1_S \xrightarrow{\nabla^1} \mathcal{E} \otimes \Omega^2_S \xrightarrow{\nabla^2} \mathcal{E} \otimes \Omega^3_S \xrightarrow{\nabla^4} \cdots$$

so that the ∇^i are derivations over the natural action of the differential-graded algebra $\Omega^\bullet_{dR}$. This sequence of maps is not necessarily a chain complex—if it is, we say ∇ is flat, or integrable. (In the case of the exterior derivative d, this amounts to the fact that mixed partials commute.)

The utility of a connection lies in the notion of parallel transport. If $\gamma : [0, 1] \to S$ is a smooth path, then we may consider the differential equation

$$\nabla_{\dot{\gamma}(t)} \tilde{\gamma}(t) = 0$$

as defining a unique lift of γ to the total space of $\mathcal{E}$ for each initial condition $\tilde{\gamma}(0) = v \in \mathcal{E}_{\gamma(0)}$. By linearity, this defines a map

$$P_\gamma : \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)},$$

which we may easily check is an isomorphism (its inverse is given by the reverse path to γ). ∇ is flat if and only if P_γ depends only on the smooth (based) homotopy class of γ.

In this case, $\mathcal{E}, \nabla$ determine (and, if S is connected, are determined by) a *monodromy representation* $\pi_1(S, s_0) \to GL(\mathcal{E}_{s_0})$, which sends a homotopy class of loops $[\gamma]$ to P_γ, for γ a smooth representative of γ.

Note furthermore that if we have a vector bundle $\mathcal{E}$ with flat connection ∇, we may define a subsheaf of $\mathcal{E}$ consisting of *flat sections*; these are precisely sections s so that $\nabla s = 0$; this will be a local system of dimension equal to the rank of $\mathcal{E}$. If $(\mathcal{E}, \nabla)$ is defined by a monodromy represenation $\pi_1(S, s_0) \to GL(V)$, the flat global sections will be canonically identified with $V^{\pi_1(S,s_0)}$. Indeed, we have the following theorem:

Theorem 8. *Let S be a complex manifold. There is a bijective correspondence, functorial with respect to pullback, between rank n holomorphic vector bundles on S with flat connection, and rank n $\mathbb{C}$-local systems on S, given by sending a vector bundle with flat connection to its sheaf of flat sections. The inverse of this correspondence is given by sending a local system V to $V \otimes \mathcal{O}_S$, with a certain flat connection ∇_V on this vector bundle.*

Remark 9. There are many analogues of this theorem, e.g. in the smooth setting. Indeed, this is the beginning of a very long story going back to work of Riemann and Hilbert.

In particular, the bundle $R^i f^{an}_* \underline{\mathbb{C}} \otimes \mathcal{O}_S$ from above comes with a natural flat connection, uniquely determined by the fact that we may recover $R^i f^{an}_* \underline{\mathbb{C}}$ as the sheaf of flat sections to $R^i f^{an}_* \underline{\mathbb{C}} \otimes \mathcal{O}_S$ relative to this connection; this connection is called the Gauss-Manin connection. In the next section, we will give an *algebraic* construction of this connection.

1.3. **The Gauss-Manin Connection for Curves.** In this section, we will construct a natural integrable connection on the sheaves $R^i \pi_* \Omega^\bullet_{dR,\pi}$. We will assume $S, \mathcal{C}$ are smooth for the rest of this section. Also, as a matter of convention we will view Ω^i_X, for any X, as a complex concentrated in degree i.

Recall that, as π is smooth, there is a short exact sequence (the cotangent exact sequence)

$$0 \to \pi^* \Omega^1_S \to \Omega^1_{\mathcal{C}} \to \Omega^1_\pi \to 0.$$

As Ω^1_π is a line bundle, and thus has vanishing higher exterior powers, taking exterior powers gives short exact sequences

$$0 \to \pi^* \Omega^k_S \to \Omega^k_{\mathcal{C}} \to \pi^* \Omega^{k-1}_S \otimes_{\mathcal{O}_{\mathcal{C}}} \Omega^1_\pi \to 0.$$

Let $F^i(\Omega^\bullet_{dR,\mathcal{C}})$ be the subcomplex given by

$$F^i(\Omega^\bullet_{dR,\mathcal{C}}) := \text{image}(\pi^* \Omega^i_S \otimes \Omega^{\bullet - i}_{dR,\mathcal{C}} \to \Omega^\bullet_{dR,\mathcal{C}}).$$

$F^i(\Omega^\bullet_{dR,\mathcal{C}})$ is the sub-complex of the algebraic de Rham complex of $\mathcal{C}$ whose terms are spanned locally as $\mathcal{O}_X$-modules by sections of the form $\pi^*(ds_{n_1} \wedge \cdots \wedge ds_{n_i}) \wedge \cdots$ where s_j are local coordinates on S. By the cotangent exact sequence, we may identify $F^1(\Omega^\bullet_{dR,\mathcal{C}})$ and $F^2(\Omega^\bullet_{dR,\mathcal{C}})$ as

$$F^1(\Omega^\bullet_{dR,\mathcal{C}}) : 0 \to \pi^*\Omega^1_S \to \Omega^2_{\mathcal{C}} \to \Omega^3_{\mathcal{C}} \to \cdots$$

$$F^2(\Omega^\bullet_{dR,\mathcal{C}}) : 0 \to \pi^*\Omega^2_S \to \Omega^3_{\mathcal{C}} \to \Omega^4_{\mathcal{C}} \to \cdots .$$

Similarly, we may identify $F^i(\Omega^\bullet_{dR,\mathcal{C}})$ as

$$F^i(\Omega^\bullet_{dR,\mathcal{C}}) : 0 \to \pi^*\Omega^i_S \to \Omega^{i+1}_{\mathcal{C}} \to \Omega^{i+2}_{\mathcal{C}} \to \cdots .$$

In particular, the exterior powers of the cotangent exact sequence show that there are short exact sequences

$$0 \to F^{i+1}(\Omega^\bullet_{dR,\mathcal{C}}) \to F^i(\Omega^\bullet_{dR,\mathcal{C}}) \to \pi^*(\Omega^i_S) \otimes_{\mathcal{O}_X} \Omega^\bullet_{dR,\pi} \to 0.$$

Setting $i = 0$, there is a short exact sequence

$$0 \to F^1(\Omega^\bullet_{dR,\mathcal{C}}) \to \Omega^\bullet_{dR,\mathcal{C}} \to \Omega^\bullet_{dR,\pi} \to 0.$$

We may quotient the middle term by $F^2(\Omega^\bullet_{dR,\mathcal{C}})$, giving the short exact sequence

$$0 \to \pi^*(\Omega^1_S) \otimes_{\mathcal{O}_{\mathcal{C}}} \Omega^\bullet_{dR,\pi} \to \Omega^\bullet_{dR,\mathcal{C}}/F^2(\Omega^\bullet_{dR,\mathcal{C}}) \to \Omega^\bullet_{dR,\pi} \to 0.$$

The connecting homomorphisms coming from the derived pushforwards of this sequence will give the Gauss-Manin connection. Indeed, the connecting homomorphisms are, by the projection formula, maps

$$\nabla_{dR,i} : R^i\pi_*\Omega^\bullet_{dR,\pi} \to \Omega^1_S \otimes_{\mathcal{O}_S} R^i\pi_*\Omega_{dR,\pi}.$$

Remark 10. The fact that these morphisms are connections (i.e. that they satisfy a Leibniz rule) follows formally by interpreting them as differentials in the spectral sequence

$$R^{p+q}\pi_*(F^p/F^{p+1}) \implies \Omega^p_S \otimes_{\mathcal{O}_S} R^q\pi_*\Omega^\bullet_{dR,\pi}$$

arising from the filtration $F^\bullet(\Omega^\bullet_{dR,\mathcal{C}})$, with the multiplicative structure on this spectral sequence (for this, we use that the filtration is compatible with the multiplicative structure on the de Rham complex—namely, $F^i \cdot F^j \subset F^{i+j}$).

In general, given a connection

$$\nabla_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

we may extend it to a sequence of maps

$$\mathcal{E} \xrightarrow{\nabla_{\mathcal{E}}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla^1_{\mathcal{E}}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X \xrightarrow{\nabla^2_{\mathcal{E}}} \otimes_{\mathcal{O}_X} \Omega^3_X \to \cdots$$

by letting

$$\nabla^i_{\mathcal{E}}(e \otimes \omega) = (-1)^i \nabla_{\mathcal{E}}(e) \otimes \omega + e \otimes d\omega.$$

We say that $\nabla_{\mathcal{E}}$ is integrable (or *flat*) if this is a chain complex. If this is the case, the $\nabla^i_{\mathcal{E}}$ are the unique sequence of maps making $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{dR}$ a differential-graded module over $\Omega^\bullet_{dR}$ with 0-th differential $\nabla_{\mathcal{E}}$.

In this case we may view the $\nabla^i_{dR,j}$ as arising as the connecting homomorphisms δ^j for the pushforward of the short exact sequence

$$0 \to F^{i+1}(\Omega^\bullet_{dR,\mathcal{C}})/F^{i+2}(\Omega^\bullet_{dR,\mathcal{C}}) \to F^i(\Omega^\bullet_{dR,\mathcal{C}})/F^{i+2}(\Omega^\bullet_{dR,\mathcal{C}}) \to \pi^*(\Omega^i_S) \otimes_{\mathcal{O}_X} \Omega^\bullet_{dR,\pi} \to 0$$

where we have

$$F^{i+1}(\Omega^\bullet_{dR,\mathcal{C}})/F^{i+2}(\Omega^\bullet_{dR,\mathcal{C}}) \simeq \pi^*(\Omega^{i+1}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{dR,\pi}$$

canonically.

Identifying these connecting homomorphisms as again arising from the spectral sequence for the filtration $F^\bullet(\Omega^\bullet_{dR,\mathcal{C}})$ shows simultaneously (1) that the δ^j are derivations, and (2) that $\delta^{j+1} \circ \delta^j = 0$. As there is at most one differential-graded module structure on $R^i\pi_*\Omega^\bullet_{dR,\pi} \otimes_{\mathcal{O}_S} \Omega^\bullet_{dR,S}$ over $\Omega^\bullet_{dR,S}$ with 0-th differential $\nabla_{dR,i}$, we see that the connecting homomorphisms are indeed the maps $\nabla^j_{dR,i}$ induced by $\nabla_{dR,i}$, and so $\nabla_{dR,i}$ is integrable. For details see Katz-Oda. [Actually, Katz-Oda do not use the uniqueness of the differential-graded module structure to prove integrability, so their argument is longer but more explicit than the sketch given here.]

Remark 11. It is not difficult to generalize these arguments to give an integrable connection in more general cases, e.g. when $\mathcal{C}/S$ is not a curve. Indeed, we didn't use this fact at all, except in the explicit identifications of $F^i(\Omega^\bullet_{dR,\mathcal{C}})$, so literally every line above except for these explicit identifications is true as written. Later, when we describe the analytic theory, we will give a more pedestrian construction of this connection.

Remark 12. We must check that the connection we have constructed agrees with the connection obtained via Theorem 8 of the previous section. While one may check this directly in our case, it is easier to punt to other sources (though the Katz-Oda paper notably omits this check!). The result follows directly, however, from the compatibility of the Riemann-Hilbert correspondence with direct images through proper morphisms (e.g. in Borel's Algebraic D-Modules, VIII.15, pg. 330), and Grothendieck's comparison theorem between algebraic and analytic de Rham cohomology above.

1.3.1. *Making the Gauss-Manin Connection Explicit.* We first need an acyclic resolution of $\Omega^\bullet_{dR,\pi}$ and the other complexes we use to define the Gauss-Manin connection; in principle this discussion will give an effective way of computing the de Rham cohomology of relative curves. A slight generalization gives an effective method of computing algebraic de Rham cohomology in some generality, but that is beyond the scope of this note. [Perhaps I should work this out carefully, since it gives an interesting interpretation of the Hodge filtration, along the lines of the Griffiths interpretation in terms of "pole order" that Akshay likes.] In general, suppose one has a projective, smooth relative curve $\pi : \mathcal{C} \to S$, with $\mathcal{C}$ and S smooth. As π is projective and everything in sight is smooth, we may choose a relatively ample effective prime divisor D on $\mathcal{C}$. Then for some $N > 0$ and $p = 0, 1$, we have that $\Omega^p_\pi(nD)$ is acyclic for any $n \geq N$. Thus there is a short exact sequence of chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^1_\pi & \longrightarrow & \Omega^1_\pi((N+1)\cdot D) & \longrightarrow & \Omega^1_\pi((N+1)\cdot D)|_{(N+1)D} & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow d & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_\mathcal{C} & \longrightarrow & \mathcal{O}_\mathcal{C}(N\cdot D) & \longrightarrow & \mathcal{O}_\mathcal{C}(N\cdot D)|_{ND} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the horizontal arrows come from the embeddings $\mathcal{O}_\mathcal{C} \to K(\mathcal{C})$ and $\Omega^1_\pi \to K(\Omega^1_\pi)$ (here K denotes the sheaf of meromorphic sections). The middle vertical arrow d is defined by extending the exterior derivative to a $K(\mathcal{O}_\mathcal{C})$-derivation $K(\mathcal{O}_\mathcal{C}) \to K(\Omega^1_\pi)$. Note that the sheaves in the middle column are π_*-acyclic by assumption, and the sheaves on the right are π_*-acyclic for dimension reasons.

Thus the algebraic de Rham complex has an acyclic resolution, given by taking the total complex of the 2×2 double complex on the right, e.g.

$$0 \to \mathcal{O}_\mathcal{C}(N\cdot D) \to \Omega^1_\pi((N+1)\cdot D) \oplus \mathcal{O}_\mathcal{C}(N\cdot D)|_{ND} \to \Omega^1((N+1)\cdot D)|_{(N+1)D} \to 0.$$

We may similarly twist the sheaves appearing in the short exact sequence

$$0 \to \pi^*(\Omega^1_S) \otimes_{\mathcal{O}_\mathcal{C}} \Omega^\bullet_{dR,\pi} \to \Omega^\bullet_{dR,\mathcal{C}}/F^2(\Omega^\bullet_{dR,\mathcal{C}}) \to \Omega^\bullet_{dR,\pi} \to 0$$

to make them all acyclic. The Gauss-Manin connection will then come precisely from the boundary map in the long exact sequence for $R^i\pi_*$ of these complexes. For example, suppose we have some section to $R^1\pi_*\Omega^\bullet_{dR,\pi}$. We may represent this by a section to

$$\Omega^1_\pi((N+1)\cdot D) \oplus \mathcal{O}_\mathcal{C}(N\cdot D)|_{ND}.$$

To evaluate the Gauss-Manin connection, we pick an arbitrary preimage in

$$\Omega^1_\mathcal{C}((N+1)\cdot D) \oplus \mathcal{O}_\mathcal{C}(N\cdot D)|_{ND}$$

and apply the exterior derivative; then we take the (unique) preimage in

$$\pi^*(\Omega^1_S) \otimes (\Omega^1_\pi((N+2)\cdot D) \oplus \mathcal{O}_\mathcal{C}((N+1)\cdot D)|_{(N+1)D}).$$

Note that in characteristic zero, if we choose a section $\omega \in \Omega^1((N+1)D)$, there is at most one section f to $\mathcal{O}(N \cdot D)|_{ND}$ so that the pair (ω, f) is closed; so in the future, we will omit f from the notation, to make our computations simpler.

1.4. **Example: The λ-Family of Elliptic Curves.** Recall that the λ-family of elliptic curves is a family $\pi : \mathcal{C} \to S$, where S is defined by

$$S := \mathbb{A}^1 \setminus \{0, 1\},$$

and $\mathcal{C}$ is defined by

$$y^2 = x(x-1)(x-\lambda),$$

where λ is a coordinate on S. We wish to study $R^1\pi_*\Omega^\bullet_{dR,\mathcal{C}/S}$ (the zeroth and second derived pushforwards are uninteresting). Over $\mathbb{C}$, the analytification of this sheaf is naturally isomorphic to $\mathcal{H}^1 := R^1\pi_*\underline{\mathbb{C}}_{\mathcal{C}} \otimes_{\mathbb{C}} \mathcal{O}_S$, where the pushforward here is taken with respect to the analytic topology. Furthermore, by Theorem 2, $\mathcal{H}^1$ has a natural subsheaf

$$\mathcal{H}^{0,1} := R^0\pi_*\Omega^1_{\mathcal{C}/S} \subset \mathcal{H}^1,$$

which we will also study. For the rest of this section, we will work over $\mathbb{C}$, in the analytic topology.

Let's first explicitly describe the λ-family. Recall that the elliptic curve defined by the equation

$$y^2 = x(x-1)(x-\lambda)$$

with basepoint at ∞ admits a degree 2 map to $\mathbb{P}^1$, given by projection to the x-coordinate, with ramification values $0, 1, \infty$, and λ. The ramification points of this map are precisely the 2-torsion points of this elliptic curve. Furthermore, given *any* elliptic curve E over $\mathbb{C}$, we have that E admits a 2-to-1 covering of $\mathbb{P}^1$ ramified at 4 points (e.g. by quotienting by the inversion map), with ramification values $0, 1, \infty$, and $\lambda(E)$. As such a covering determines the curve, we see E is defined by the equation

$$y^2 = x(x-1)(x-\lambda(E)).$$

Thus, given any elliptic curve, we may find its equation in terms of the ramification values of a 2-to-1 map to $\mathbb{P}^1$.

Remark 13. Any degree 2 map $E \to \mathbb{P}^1$ ramified at $0, 1, \infty$ admits exactly 12 automorphisms permuting the set $\{0, 1, \infty\}$, generated by the involution over $\mathbb{P}^1$ and the six fractional linear transformations permuting $0, 1$, and ∞. Thus the map $S \to \mathcal{M}_{1,1}$ induced by $\pi : \mathcal{C} \to S$ is a degree 12 étale cover. This argument only works in characteristic 0, though the statement is true in characteristic $p > 2$. We'll see this over $\mathbb{C}$ in a different way a little bit later.

Let us find the equation of an elliptic curve defined via its period lattice, say

$$E_\tau := \mathbb{C}/\langle 1, \tau \rangle.$$

If $\Lambda \subset \mathbb{C}$ is a lattice, define Weierstrass $\wp$-function for Λ to be

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{r \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+r)^2} - \frac{1}{r^2} \right).$$

We will set $\Lambda = \langle 1, \tau \rangle$ and drop the subscript on $\wp$. Then $\wp$ is a doubly periodic map $\mathbb{C} \to \mathbb{P}^1$ with periods 1 and τ, and so factors through the quotient map $\mathbb{C} \to E_\tau$; the map $E_\tau \to \mathbb{P}^1$ thus induced is a double cover ramified at 4 points, with ramification values

$$\wp(0) = \infty, \wp(\tau/2), \wp(1/2), \wp(1/2 + \tau/2).$$

Consider the fractional linear transformation fixing ∞ and sending $\wp(\tau/2)$ to 0, and $\wp(1/2)$ to 1; that is,

$$z \mapsto \frac{z - \wp(\tau/2)}{\wp(1/2) - \wp(\tau/2)};$$

after composing our double cover with this automorphism of $\mathbb{P}^1$, we see that the ramification values become $0, 1, \infty$, and

$$\lambda(\tau) := \frac{\wp(1/2 + \tau/2) - \wp(\tau/2)}{\wp(1/2) - \wp(\tau/2)}.$$

Thus the elliptic curve E_τ is defined by the equation

$$y^2 = x(x-1)(x-\lambda(\tau)).$$

This will allow us to "uniformize" the λ-family. In particular, consider the universal elliptic curve with chosen (ordered) homology basis $E \to \mathbb{H}$, where $\mathbb{H}$ is the upper-half plane; E is defined by

$$E := \mathrm{coker}_{\mathbb{H}}(\langle 1, \tau\rangle \times \mathbb{H} \to \mathbb{C} \times \mathbb{H}).$$

The fiber of E over a point $\tau \in \mathbb{H}$ is $E_\tau = \mathbb{C}/\langle 1, \tau\rangle$.

Proposition 14. *The map*

$$\lambda : \mathbb{H} \to S, \tau \mapsto \lambda(\tau)$$

is a covering map.

Proof. We first check that λ is surjective—indeed, given $s \in S$, consider the elliptic curve defined by

$$y^2 = x(x-1)(x-s)$$

and write it as $E = \mathbb{C}/\Lambda$. There is a map $r : E \to \mathbb{P}^1$ ramified at $0, 1, \infty$, and s; choose a point a in $\mathbb{C}$ lying above $r^{-1}(s)$. Then any two $\mathbb{Z}$-linearly independent vectors $e_1, e_2 \in \Lambda$ satisfying

$$e_1/2 + e_2/2 = a$$

serve as a homology basis for E and so the point $\pm e_2/e_1 \in \mathbb{H}$ maps to s.

Now we need to check that λ is locally an isomorphism. But given any $\tau \in \mathbb{H}$, there is an open set $U \ni \tau$ so that for any $\tau' \in U \setminus \{\tau\}$, $E_\tau \not\simeq E_{\tau'}$. As $\lambda(\tau)$ determines the isomorphism class of E_τ (since it determines an equation for E_τ!), $\lambda(\tau') \neq \lambda(\tau)$ for any $\tau' \in U \setminus \{\tau\}$. So λ is locally injective; as λ is holomorphic, it is thus a local isomorphism. $\square$

Proposition 15. *The deck transformation group of λ is generated by*

$$\gamma_1 : \tau \mapsto \tau + 2, \gamma_2 : \tau \mapsto \frac{\tau}{1-2\tau}.$$

Choosing a basepoint z_0 of $\mathbb{H}$ near 0, a geodesic $z_0 \to \gamma_2(z_0)$ maps to a small circle in S about 1, and a geodesic $z_0 \to \gamma_1(z_0)$ (namely a horizontal line of length 2) maps to a small circle about ∞ in S.

Proof. If $\Lambda = \langle \omega_1, \omega_2\rangle \subset \mathbb{C}$ is a lattice with ordered basis ω_1, ω_2, such that $\omega_2/\omega_1 \in \mathbb{H}$, we set

$$\lambda(\omega_1, \omega_2) = \frac{\wp_\Lambda(\omega_1/2 + \omega_2/2) - \wp_\Lambda(\omega_2/2)}{\wp_\Lambda(\omega_1/2) - \wp_\Lambda(\omega_2/2)}.$$

The double-periodicity of $\wp_\Lambda$ shows that

$$\lambda(\omega_1, \omega_2) = \lambda(\omega_1, \omega_2 + 2\omega_1)$$

$$\lambda(\omega_1, \omega_2) = \lambda(\omega_1 - 2\omega_2, \omega_2)$$

which, setting $\omega_1 = 1, \omega_2 = \tau$ and normalizing the bases $\langle \omega_1, \omega_2 + 2\omega_1\rangle, \langle \omega_1 - 2\omega_2, \omega_2\rangle$ gives the desired transformations.

Set $\Lambda = \langle 1, \tau\rangle$. The group G generated by these two transformations is (by design) the kernel of the action of $PSL(2, \mathbb{Z})$ on $\frac{1}{2}\Lambda/\Lambda \simeq E_\tau[2]$; as the action of $PSL(2, \mathbb{Z})$ on $(\frac{1}{2}\Lambda/\Lambda) \setminus \{0\}$ is triply transitive, we see that G has index 6 in $PSL(2, \mathbb{Z})$. By explicitly checking on coset representatives we may see that G is precisely the deck transformation group. Indeed, coset representatives for G act by sending λ to

$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}.$$

One may see the claims explicitly identifying the isomorphism $G \simeq \pi_1(S)$ for a basepoint of $\mathbb{H}$ near zero by noting that $\lim_{\tau\to 0} \lambda(\tau) = 1, \lim_{\tau \to 1} \lambda(\tau) = \infty$ (e.g. by drawing lattices and using the joint continuity of $\wp_\Lambda(z)$ in Λ, z). $\square$

Remark 16. As the deck transformation group of λ is index 6 in $PSL(2, \mathbb{Z})$, we again see that $S \to \mathcal{M}_{1,1}$ is étale of degree 12.

1.4.1. *The Gauss-Manin Connection.* We can finally identify the local system of $\mathbb{C}$-vector spaces

$$R^1\pi_*\underline{\mathbb{C}}_{\mathcal{C}} \simeq \{\text{flat sections to } R^1\pi_*\Omega^{\bullet,an}_{dR,\mathcal{C}/S}\}.$$

Consider the homology local system of $\mathbb{Z}$-modules $\mathcal{H}_1(\mathcal{C}/S,\mathbb{Z})$, whose fiber over a point $s \in S$ is canonically identified with $H_1(\mathcal{C}_s,\mathbb{Z})$. Choose a basepoint $s_0 \in S$ and choose generators of $\pi_1(S,s_0)$ (the free group on 2 generators) to be a small loop γ_1 about ∞ and a small loop γ_2 about 1 (for definitness, say the images of the paths chosen in Propositon 9 through the map λ). By propositions 8 and 9 above, the monodromy of this local system is given by the representation

$$\pi_1(S,s_0) \to GL_2(\mathbb{Z})$$

is given by

$$\gamma_1 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \gamma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

By the universal coefficient theorem we may identify

$$R^1\pi_*\underline{\mathbb{C}}_{\mathcal{C}} \simeq (\mathcal{H}_1(\mathcal{C}/S,\mathbb{Z}) \otimes \mathbb{C})^\vee$$

and so the monodromy for $R^1\pi_*\underline{\mathbb{C}}_{\mathcal{C}}$ is given by the dual representation

$$\gamma_1 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \gamma_2 \mapsto \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

This is an explicit description of the Gauss-Manin connection for the λ-family.

1.4.2. *The Picard-Fuchs Equation.* We now wish to study the embedding

$$R^0\pi_*\Omega^1_{\mathcal{C}/S} \hookrightarrow R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S}).$$

In particular, we wish to know how this embedding varies with respect to the Gauss-Manin connection. We first choose a nowhere vanishing section to $R^0\pi_*\Omega^1_{\mathcal{C}/S}$, say the usual invariant differential

$$\omega := \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

The Gauss-Manin connection ∇_{dR} allows us to differentiate ω with respect to λ, giving a new section to $R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S})$, via

$$R^0\pi_*\Omega^1_{\mathcal{C}/S} \hookrightarrow R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S}) \xrightarrow{\nabla_{dR}} R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S}) \otimes \Omega^1_S \xrightarrow{\frac{\partial}{\partial\lambda}} R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S}).$$

We will let

$$\omega^{(1)} = \langle \nabla_{dR}(\omega), \frac{\partial}{\partial\lambda} \rangle$$

be the first derivative of ω in this sense, and

$$\omega^{(2)} = \langle \nabla_{dR}(\omega^{(1)}), \frac{\partial}{\partial\lambda} \rangle$$

be the second derivative. As $R^1\pi_*(\Omega^\bullet_{dR,\mathcal{C}/S})$ is a vector bundle of rank 2, the three sections $\omega, \omega^{(1)}$, and $\omega^{(2)}$ satisfy some relation—namely, ω satisfies a second-order differential equation, called the Picard-Fuchs equation. Let us explicitly determine this differential equation, using the description of the Gauss-Manin connection in the previous section. We choose an explicit resolution of the relevant de Rham complexes, using $V(x-\lambda)$ as our choice of ample divisor D. To make computations easier, we note that $K(\Omega^1_{\mathcal{C}})$ is generated by the meromorphic differentials $dx, dy, d\lambda$ subject to the single relation that

$$2ydy = g(x,\lambda)dx + h(x)d\lambda$$

for certain explicit polynomials g, h. We will use the basis $\{dx, d\lambda\}$ for $K(\Omega^1_{\mathcal{C}})$ throughout, and the dual basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial \lambda}\}$ for $K(T_C)$. The expression $\frac{dy}{d\lambda}$ will mean the meromorphic function $dy(\frac{\partial}{\partial\lambda})$.

We first compute explicit representatives for $\omega^{(1)}$ and $\omega^{(2)}$. Recall that our explicit resolutions were modules consisting of elements of the form (η, f), where η is a meromorphic form and f is a Laurent tail; as we are working in characteristic zero we omit f from our notation as in 1.3.1. First, we choose $dx/y \in \Gamma(\mathcal{C}, \Omega^1_{\mathcal{C}}(2D))$ as a lift of ω. We have that

$$d(dx/y) = dx \wedge dy/y^2 = h(x)dx \otimes d\lambda/2y^3.$$

Having written this as a element of $\Omega^1_{\mathcal{C}} \otimes \pi^*\Omega^1_S$, we may now apply $\frac{\partial}{\partial\lambda}$ to get the meromorphic form

$$\frac{x(x-1)dx}{2y^3}.$$

In more compact notation, we have:

$$\omega^{(1)} = -\frac{dx}{y^2}\frac{dy}{d\lambda} = \frac{x(x-1)dx}{2y^3} = \frac{x(x-1)dx}{2(x(x-1)(x-\lambda))^{3/2}} = \frac{1}{2}\frac{\omega}{x-\lambda}.$$

A similar computation gives:

$$\omega^{(2)} = -3\frac{x(x-1)dx}{2y^4}\frac{dy}{d\lambda} = \frac{3}{4}\frac{x^2(x-1)^2dx}{y^5} = \frac{3}{4}\frac{x^2(x-1)^2dx}{(x(x-1)(x-\lambda))^{5/2}} = \frac{3}{4}\frac{\omega}{(x-\lambda)^2}.$$

Now we are looking for rational functions $A(\lambda)$ and $B(\lambda)$ so that

$$\omega + A(\lambda)\omega^{(1)} + B(\lambda)\omega^{(2)} = \frac{x^2(x-1)^2(x-\lambda)^2 + \frac{1}{2}A(\lambda)x^2(x-1)^2(x-\lambda) + \frac{3}{4}B(\lambda)x^2(x-1)^2}{y^5}dx$$

is exact. We may use the exact forms $d(x^k y^{-3})$ for $k = 0, 1, 2, 3, 4$ to reduce the above equation to be linear in x; then we find that the relevant differential equation is

$$\omega + 4(2\lambda - 1)\omega^{(1)} + 4\lambda(\lambda - 1)\omega^{(2)} = 0,$$

which is called the Picard-Fuchs equation.

Remark 17. Clasically, the Picard-Fuchs equation is a differential equation satisfied by the *periods* of an elliptic curve. To recover this formulation, choose contractible open disc D in S, and a section γ to $\mathcal{H}_1(\mathcal{C}/S, \mathbb{Z})$ over D. Then applying $\int_\gamma$ to the differential equation above (and commuting this operator with $\nabla_{\partial/\partial\lambda}$ gives the classical Picard-Fuchs equation.

Note that this differential equation gives a complete description of the Gauss-Manin connection in this case, as any section to $R^1\pi_*\Omega^\bullet_{dR,\pi}$ may be written unqiuely as an $\mathcal{O}_S$-linear combination of ω and $\omega^{(1)}$—that is, ω is a *cyclic vector* for the Gauss-Manin connection.

Explicitly, any local section to $R^1\pi_*\Omega^\bullet_{dR,\pi}$ may be written as

$$f\omega + g\omega^{(1)}$$

for $f, g \in \mathcal{O}_S$. Flatness is the condition that

$$\nabla_{\frac{\partial}{\partial\lambda}}(f\omega + g\omega^{(1)}) = 0$$

or more explicitly

$$f'\omega + (f + g')\omega^{(1)} + g\omega^{(2)} = f'\omega + (f + g')\omega^{(1)} + \frac{g}{4\lambda(\lambda-1)}\omega + \frac{g(2\lambda-1)}{\lambda(\lambda-1)}\omega^{(1)} = 0.$$

That is, the section is flat if and only if

$$f' + \frac{g}{4\lambda(\lambda-1)} = 0$$

and

$$f + g' + \frac{g(2\lambda-1)}{\lambda(\lambda-1)} = 0.$$

2. Review of Hodge Theory for Kähler Manifolds

Classical Hodge theory may loosely be defined as finding explicit representatives for cohomology classes on Riemannian manifolds, using explicit resolutions like the Dolbeault or de Rham complex. For example, let M be a compact Riemannian manifold with metric g, and $\Lambda^k(M)$ is the sheaf of smooth differential complex-valued k-forms on M; then the complex

$$0 \to C^\infty(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \to \cdots$$

is a fine resolution for the constant sheaf $\underline{\mathbb{R}}$ by the Poincaré lemma, and thus

$$0 \to \Gamma(M, C^\infty(M)) \xrightarrow{d} \Gamma(M, \Lambda^1(M)) \xrightarrow{d} \Gamma(M, \Lambda^2(M)) \to \cdots$$

computes the sheaf cohomology of $\underline{\mathbb{R}}$. The metric g induces an inner product on each $\Gamma(M, \Lambda^1(M))$ via

$$\langle \omega, \eta \rangle = \int_M g(\omega_x, \eta_x) \operatorname{Vol}(M)$$

where $\operatorname{Vol}(M)$ is the volume form induced by g. The naive idea of Hodge theory is to find representatives for elements of the cohomology group

$$H^i(M, \underline{\mathbb{R}}) \simeq \ker(d)/\operatorname{im}(d)$$

by identifying this quotient with the orthogonal complement of $\operatorname{im}(d)$ in $\ker(d)$. This identification is however not tautological, as it is not obvious that $\operatorname{im}(d)$ is closed or (a priori) that the quotient is finite dimensional—one proceeds by identifying this orthogonal complement with the kernel of an elliptic operator, the Laplacian Δ_d on k-forms. The forms in the kernel of the Laplacian are called *harmonic* with respect to the metric g. I will say no more about this topic—a good reference is the early chapters of Voisin's book Hodge Theory and Complex Algebraic Geometry I. I mention this only because it will give a good perspective on later results.

In the context of algebraic or complex geometry, Hodge theory now usually refers to the study of the extra structure with which the cohomology of smooth projective varieties in characteristic zero, or compact Kähler manifolds, are blessed (there are of course extensions of these results to the non-smooth or non-projective case).

Suppose X is a compact complex manifold with Hermitian metric h.

Definition 18 (Kähler)**.** (X, h) is Kähler if the (real) 2-form $\omega := \operatorname{Im}(h)$ is closed. The Kähler class of (X, h) is the cohomology class in $H^2(X, \mathbb{R})$ associated to ω.

The motivation for this definition, as far as I know, comes from the fact that smooth projective varieties are Kähler—to see this, note that the Fubini-Study metric on projective space is Kähler, and that the pullback of a Kähler metric to a closed complex submanifold is Kähler. In the case of a variety embedded in projective space via a very ample line bundle $\mathcal{L}$, the Kähler class associated to the pullback of the Fubini-Study metric is precisely $c_1(\mathcal{L})$ (or equivalently, the cycle class of a hyperplane section). The existence of a Kähler metric has drastic consequences for the cohomology of a complex manifold.

Remark 19. Not all compact Kähler manifolds admit the structure of a projective algebraic variety—for example, there are many non-algebraic compact complex tori (of dimension at least 2). By the Kodaira embedding theorem (combined with Chow's theorem), a compact Kähler manifold is projective algebraic if and only if it admits a Kähler metric with *integral* Kähler class.

2.1. **The Hodge Decomposition.**

Theorem 20 (Hodge Decomposition)**.** *Suppose X is a compact complex manifold admitting a Kähler metric. Then $\mathbb{C}^*$ acts naturally on the cohomology of X, so that*

$$H^n(X, \mathbb{Z}) \otimes \mathbb{C} \simeq H^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X)$$

where the action of $z \in \mathbb{C}^*$ on $H^{p,q}$ is given by multiplication by $z^{-p}\overline{z}^{-q}$. Furthermore, the action respects the natural $\mathbb{R}$-structure on $H^n(X, \mathbb{C})$, i.e.

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

There is a natural identification

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p).$$

Furthermore, if we define a decreasing filtration

$$F^p H^n(X, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,n-r}(X),$$

then $F^p H^n(X, \mathbb{C})$ *may be identified with the image of*

$$\mathbb{H}^n(X, \Omega_{X,dR}^{\bullet \geq r}) \to \mathbb{H}^n(X, \Omega_{X,dR}^{\bullet}) \simeq H^n(X, \mathbb{C})$$

where $\Omega_{X,dR}^{\bullet}$ *is the holomorphic de Rham complex. Note that*

$$H^{p,q}(X) = F^p H^n(X, \mathbb{C}) \cap \overline{F^q H^n(X, \mathbb{C})}.$$

The filtration above may be viewed as coming from the natural filtration decreasing of the de Rham complex, $F^p \Omega_{X,dR}^{\bullet} := \Omega_{X,dR}^{\bullet \geq p}$; indeed, the theorem above implies (by a dimension count) that the spectral sequence associated to this filtration degenerates at page E_1 (this spectral sequence is called the Fröhlicher spectral sequence).

Remark 21. In particular, the filtration (and thus the grading) above is independent of the Kähler metric. The Kähler class resides in $H^{1,1}(X)$

Remark 22. One striking consequence of the degeneration of the Fröhlicher spectral sequence is that $d\omega = 0$ for any global holomorphic form ω; this is because d appears as a differential on E_2 of the Fröhlicher spectral sequence.

Remark 23. The Hodge decomposition is a strong restriction on the cohomology of a complex Kähler manifold. For example, the dimension of $H^{2i+1}(X, \mathbb{C})$ must be even.

Now we may see an example of a compact complex manifold which is not Kähler: the quotient Y of $\mathbb{C}^2 \setminus \{0\}$ by the $\mathbb{Z}$ action given by $n \cdot x = 2^n x$. This is certainly compact; but its universal cover has the homotopy type of S^3, and thus $\dim_{\mathbb{C}} H^3(Y, \mathbb{C}) = 1$, which is odd.

In a more pedestrian fashion, we may see this result as arising from an explicit resolution of the holomorphic de Rham complex, namely

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p+1,q-1} & \xrightarrow{\bar\partial} & A^{p+1,q} & \xrightarrow{\bar\partial} & A^{p+1,q+1} & \longrightarrow & \cdots \\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \\
\cdots & \longrightarrow & A^{p,q-1} & \xrightarrow{\bar\partial} & A^{p,q} & \xrightarrow{\bar\partial} & A^{p,q+1} & \longrightarrow & \cdots \\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \\
\cdots & \longrightarrow & A^{p-1,q-1} & \xrightarrow{\bar\partial} & A^{p-1,q} & \xrightarrow{\bar\partial} & A^{p-1,q+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where $A^{p,q}$ is the subsheaf of $\Lambda^{p+q}(M)$ given locally by the the $C^\infty(M)$-linear span of differential forms

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

for local holomorphic coordinates $z_1, ..., z_n$. Note that $\mathrm{Tot}(A^{\bullet,\bullet}) = \Lambda^{\bullet}(X) \otimes \mathbb{C}$. If X is Kähler, we have that the space of harmonic (complex-valued) k-forms $\mathcal{H}^k(X, \mathbb{C})$ is spanned by

$$\mathcal{H}^{p,q}(X) := \mathcal{H}^{p+q}(X, \mathbb{C}) \cap A^{p,q},$$

which we may also identify with $H^q(X, \Omega_X^p)$. That is,

$$\mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X).$$

Again I will omit the details; we use that $A^{p,\bullet}$ is a fine resolution of Ω_X^p (the Dolbeault resolution), and comes equipped with its own Laplacian $\Delta_{\overline{\partial}}$ if X is endowed with a Hermitian metric h. The key point in this argument is that if h is Kähler,

$$2\Delta_{\overline{\partial}} = \Delta_d|_{A^{p,q}}.$$

This point of view has the advantage that we may see the $\mathbb{C}^*$ action explicitly. Namely, the complex structure on X endows each tangent space T_xX with a complex structure, and thus a multiplication action by $\mathbb{C}^*$. Thus the cotangent space obtains the dual action, whence $w \in \mathbb{C}^*$ acts on dz by multiplication by w^{-1}, if z is a holomorphic coordinate on X, and acts on $d\bar{z}$ by multiplication by $\bar{w}^{-1}$.

Remark 24. Note that the grading is not "algebraic," in the sense that it relies on the real structure on $H^k(X, \mathbb{C})$, which cannot be recovered directly from algebraic de Rham cohomology. The filtration, however, is algebraic, via its description in terms of hypercohomology.

Remark 25. Note that the fact that the Fröhlicher spectral sequence degenerates at E_1 for smooth projective varieties/$\mathbb{C}$ implies that the analogous spectral sequence for algebraic de Rham cohomology also degenerates for E_1. The "Lefschetz principle" then implies this degeneration for any smooth projective variety over a field of characteristic zero. This spectral sequence *does not* always degenerate in characteristic $p > 0$.

This extra structure on the cohomology of compact Kähler manifolds motivates the following definition.

Definition 26 (Hodge Structure)**.** An integral Hodge structure of weight n is a finitely generated free abelian group $V_{\mathbb{Z}}$, together with a decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

with

$$V^{p,q} = \overline{V^{q,p}}.$$

Equivalently, we may define an integral Hodge structure of weight n to be a finitely generated free abelian group $V_{\mathbb{Z}}$ together with a decreasing filtration $F^pV_{\mathbb{C}}$ of $V_{\mathbb{C}}$ satisfying

$$F^pV_{\mathbb{C}} \oplus \overline{F^{n-p+1}V_{\mathbb{C}}} = V_{\mathbb{C}}.$$

To pass between these two definitions, we may define

$$F^pV_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,n-r}$$

and

$$V^{p,q} = F^pV_{\mathbb{C}} \cap \overline{F^{n-p}V_{\mathbb{C}}}.$$

There are analogous definitions of rational (resp. real) Hodge structures, etc. Most examples of Hodge structures we will encounter will arise from the cohomology of Kähler manifolds (or possibly as direct summands of the cohomology of Kähler manifolds); there is however one important example we will see later.

Example 1 (The Tate Object). We define $\mathbb{Z}(1) = 2\pi i\mathbb{Z}$ and give it a Hodge structure of weight -2 by declaring $V = \mathbb{C}(1) = \mathbb{Z}(1) \otimes \mathbb{C} = V^{-1,-1}$.

Remark 27. A morphism of integral Hodge structures is a morphism of the underlying free Abelian groups whose complexification respects the grading. Rational Hodge structures naturally are an Abelian category; both integral and rational Hodge structures have a natural tensor structure, with operations such as symmetric and exterior products (which respect, for example, the forgetful functor to graded complex vector spaces).

Example 2. Suppose A is an Abelian variety/$\mathbb{C}$ of complex dimension g. Then $H^1(A,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$, and

$$H^1(A,\mathbb{C}) = H^1(A,\mathbb{Z}) \otimes \mathbb{C} \simeq H^0(A,\Omega^1_A) \oplus \overline{H^1(A,\mathcal{O}_A)}$$

is a Hodge structure of weight 1. The pairs $(H^i(A,\mathbb{Z}), H^{p,i-p}(A))$ are Hodge structures of weight i; indeed,

$$(H^i(A,\mathbb{Z}), H^{p,i-p}) = \bigwedge^i (H^1(A,\mathbb{Z}), H^{j,1-j}).$$

2.2. **The Lefschetz Theorems.** Thus far we have restricted ourselves to studying the structure imposed our cohomology by the existence of a Kähler form; this structure has been independent of our *choice* of form. In this subsection we will describe the Lefschetz theorems, which describe structure that depends very much on our choice of Kähler form ω. Throughout this subsection, X will have complex dimension n.

Let

$$L : H^*(X,\mathbb{R}) \to H^{*+2}(X,\mathbb{R}), L : [\eta] \mapsto [\eta] \cup [\omega]$$

be the operator on cohomology given by cupping with the Kähler class (if $[\omega]$ is integral—i.e. X is projective—this is the same as intersecting with the hyperplane class). L is usually called the Lefschetz operator. Then L has a formal adjoint

$$\Lambda : H^*(X,\mathbb{R}) \to H^{*-2}(X,\mathbb{R})$$

given either by Poincaré duality, or in terms of the Hodge star operator $*$, via $\Lambda = - \cup (*^{-1}\omega *)$. Then a direct computation shows that

$$[L,\Lambda] = (k-n)\,\mathrm{Id}$$

on $H^k(X,\mathbb{R})$. That is, L, Λ generate an $\mathfrak{sl}_2(\mathbb{R})$ representation on $H^*(X,\mathbb{R})$, with weight spaces given by the cohomological grading. This immediately has the following consequences:

Theorem 28 (Hard Lefschetz Theorem)**.** *For $k \leq n$, the map*

$$L^{n-k} : H^k(X,\mathbb{R}) \to H^{2n-k}$$

is an isomorphism.

Corollary 29. *The map*

$$L^{n-k} : H^{p,k-p}(X) \to H^{p+n-k,n-p}(X)$$

is an isomorphism.

Proof. The Kähler form ω is in $H^{1,1}(X)$. □

Corollary 30. *If $k < n$,*

$$L : H^k(M,\mathbb{R}) \to H^{k+2}(M,\mathbb{R})$$

is injective. Thus the odd degree betti numbers b_{2i-1} increase for $2i-1 \leq n$ and the even degree betti numbers b_{2i} increase for $2i \leq n$.

Furthermore, the representation theory of $\mathfrak{sl}_2$ implies that if we define the primitive part of the cohomology of X via

$$P^k(X) := \ker(L^{n-k+1} : H^k(X,\mathbb{R}) \to H^{2n-k+2}(X,\mathbb{R})$$

for $k \leq n$ (these are precisely the "lowest weight" vectors), one has

Theorem 31 (Lefschetz Decomposition)**.** *The natural map*

$$i : \bigoplus_{k-2r\geq 0} P^{k-2r}(X) \xrightarrow{L^r} H^k(X,\mathbb{R})$$

is an isomorphism. Furthermore, the P^k are compatible with the Hodge structure in the sense that setting

$$P^{p,q}(X) := P^k(X) \otimes \mathbb{C} \cap H^{p,q}(X)$$

we have

$$P^k(X) \otimes \mathbb{C} \simeq \bigoplus_{p+q=k} P^{p,q}(X)$$

and

$$\overline{P^{p,q}(X)} \simeq P^{q,p}(X).$$

Remark 32. The $P^k(M)$ thus have induced (real) Hodge structures of weight k. These pieces of the cohomology of X are called "primitive" because the Lefschetz Decomposition shows how the rest of the cohomology of X may be built from them; in the case X is a projective variety, these are precisely the classes that don't come from intersecting a class of lower weight with the hyperplane class, so they are the "new" pieces of cohomology.

Remark 33. It is these sub-Hodge structures of the cohomology of X we will use to define period mappings.

2.3. **The Hodge Index Theorem and Polarizations.** The choice of a Kähler class allows us to extend the cup product to an "intersection form" on the cohomology of X. Namely, for $\alpha, \beta \in H^k(X, \mathbb{R}), k \leq n$, we define

$$Q(\alpha, \beta) = L^{n-k}\alpha \cup \beta \in H^{2n}(X, \mathbb{R}) \simeq \mathbb{R}.$$

Note that if the Kähler class $[\omega]$ is integral, this pairing is defined integrally. If k is even, this form is symmetric; if k is odd, it is anti-symmetric. We may extend this to a Hermitian form $H_k(\alpha, \beta) := i^k Q(\alpha, \bar{\beta})$ on $H^k(X, \mathbb{C})$. The Hodge Index Theorem describes how this form interacts with the Hodge decomposition:

Theorem 34. *The decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

is orthogonal with respect to H_k. Furthermore, $(-1)^{k(k-1)/2} i^{p-q-k} H_k$ is positive definite on $P^{p,q}(X)$.

Corollary 35 (Hodge Index Theorem)**.** *The signature of Q on $H^n(X, \mathbb{R})$ is equal to*

$$\sum_{a,b} (-1)^a \dim H^{a,b}(X).$$

These facts motivate the following definition.

Definition 36. A polarized compact Kähler manifold (X, ω) is a compact complex manifold equipped with a choice of integral Kähler class ω.

Remark 37. By Kodaira's embedding theorem, such a manifold is always a complex projective variety.

Definition 38 (Polarized Hodge Structure)**.** An (integral) polarized Hodge structure of weight k is an integral Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$ of weight k so endowed with a bilinear form Q on $V_{\mathbb{Z}}$. Q is symmetric if k is even, and antisymmetric if k is odd. Furthermore, the form H on $V_{\mathbb{C}}$,

$$H(\alpha, \beta) := i^k Q(\alpha, \bar{\beta})$$

satisfies:

(1) The decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

is orthogonal for H.

(2) The form

$$i^{p-q-k}(-1)^{k(k-1)/2} H$$

is positive definite on $V^{p,q}$. As $p - q$ has the same parity as k, this just says that H alternates between being positive- and negative- definite.

A rational (resp. real) polarized Hodge structure is defined analogously.

The prototypical example of a real polarized Hodge structure comes from the primitive part of the cohomology of a complex variety.

Remark 39. We have throughout this section worked with compact Kahler manifolds—we may, however, deduce from these results many similar results about smooth projective varieties X over a field k of characteristic zero. In particular, the (algebraic) Fröhlicher spectral sequence for such varieties degenerates. Furthermore, upon choosing an embedding $k \hookrightarrow \mathbb{C}$, one obtains all of the results above for the (analytic de Rham) cohomology of $X^{an}_{\mathbb{C}}$.

3. Variation of Hodge Structure

3.1. **Hodge Bundles and Griffiths Transversality.** In this section we work with a smooth, proper holomorphic map $\pi : \mathcal{X} \to S$ of relative dimension n, such that the Fröhlicher spectral sequence of each fiber X_s degenerates at E_1. This happens e.g. if all the fibers X_s admit Kähler structures.

Recall that our original goal was to study how the structures present on the cohomology of algebraic varieties vary in families—that is, we wish to abstract away the properties satisfied by the bundles $R^i\pi_*\Omega^\bullet_{dR,\pi}$. Note that the fibers of these bundles have a natural grading, by Theorem 19. However, this splitting of $R^i\pi_*\Omega^\bullet_{dR,\pi}$, as, say, a continuous vector bundle, is *not* holomorphic. This should be apparent from Theorem 19 itself—if one has a non-constant family of elliptic curves, for example, we saw earlier that $H^{1,0}$ varies holomorphically within $R^1\pi_*\Omega^\bullet_{dR,\pi}$; but $H^{0,1}$ is *conjugate* to $H^{1,0}$ and thus varies anti-holomorphically in this case. We will see that the filtration defined in Theorem 19 does vary holomorphically, however.

Let us briefly consider the structure present on the bundles $R^i\pi_*\Omega^\bullet_{dR,\pi}$. Recall that the filtration

$$\cdots \hookrightarrow \Omega^{\bullet\geq k+1}_{dR,\pi} \hookrightarrow \Omega^{\bullet\geq k}_{dR,\pi} \hookrightarrow \Omega^{\bullet\geq k-1}_{dR,\pi} \hookrightarrow \cdots \hookrightarrow \Omega^\bullet_{dR,\pi}$$

induces maps

$$R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi} \to R^i\pi_*\Omega^{\bullet\geq k-1}_{dR,\pi}.$$

By our earlier remarks, the sheaves $R^i\pi_*\Omega^\bullet_{dR,\pi}$ are vector bundles (this followed from Ehresmann's theorem and proper base change.) We claim the same is true for $R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi}$.

Proposition 40. *Let $\pi : \mathcal{X} \to S$ be as above. Then the Hodge numbers of $\mathcal{X}_s$ are locally constant.*

Proof. By the upper semi-continuity theorem, we have that $\dim_{\mathbb{C}} H^{p,q}(X_s)$ is upper semi-continuous in S. But the k-th Betti number

$$b_k(X_s) = \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}(X_s)$$

is constant (by our remarks in section 1.2), so the $H^{p,q}(X_s)$ must be constant in s as well. □

Corollary 41. *$R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi}$ is a holomorphic vector bundle for all i and k. Furthermore, the map*

$$R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi} \to R^i\pi_*\Omega^{\bullet\geq k-1}_{dR,\pi}$$

is an injective map with constant fibral rank (i.e. the cokernel is a vector bundle). Thus the $R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi}$ form a decreasing filtration

$$F^pR^i\pi_*\Omega^\bullet_{dR,\pi} := \operatorname{im}(R^i\pi_*\Omega^{\bullet\geq p}_{dR,\pi} \to R^i\pi_*\Omega^\bullet_{dR,\pi}).$$

Proof. We first check that the $R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi}$ are vector bundles, using backward induction on k. By Proposition 40 and cohomology and base change, the result is true for $k = n$. Consider the long exact sequence obtained by applying $R\pi_*$ to the short exact sequence

$$0 \to \Omega^{\bullet\geq k}_{dR,\pi} \to \Omega^{\bullet\geq k-1}_{dR,\pi} \to \Omega^{k-1}_{dR,\pi} \to 0.$$

By applying cohomology and base change to $\Omega^{k-1}_{dR,\pi}, \Omega^{\bullet\geq k}_{dR,\pi}$ (which are vector bundles by the proposition, and the induction hypothesis, respectively), we see that the boundary maps are all 0 (using the degeneration of the Fröhlicher spectral sequence on the fibers). Thus the $R^i\pi_*\Omega^{\bullet\geq k-1}_{dR,\pi}$ is an extension of vector bundles and is thus a vector bundle itself.

Note that our observation about the boundary maps being zero shows that

$$R^i\pi_*\Omega^{\bullet\geq k}_{dR,\pi} \to R^i\pi_*\Omega^{\bullet\geq k-1}_{dR,\pi}$$

is injective as claimed; the claim that the cokernel is a vector bundle follows from Proposition 40 and cohomology and base change by identifying it with $R^i\pi_*\Omega^{k-1}_{dR,\pi}$. [Brian remarks that this result also follows by applying cohomology and base for hypercohomology.] □

We may now apply cohomology and base change to deduce:

Corollary 42.

$$R^i\pi_*\Omega^\bullet_{dR,\pi} = F^pR^i\pi_*\Omega^\bullet_{dR,\pi} \oplus \overline{F^{i-p+1}R^i\pi_*\Omega^\bullet_{dR,\pi}}.$$

Proof. This is immediate from the analogous result on fibers. ☐

Remark 43. The interaction between the filtration we've observed above and the integral structure on $R^i\pi_*\underline{\mathbb{C}}$ is very subtle; indeed, it is the subject of much current research. In particular, while the Hodge numbers and the ranks of the $F^pR^i\pi_*\underline{\mathbb{C}} \otimes \mathcal{O}_S$ are locally constant, the rank of

$$H^{2p}(X_s, \mathbb{Z}) \cap H^{p,p}(X_s)$$

is not constant, in general.

Example 3. Let $\mathcal{C} \to S$ be a family of elliptic curves (with varying moduli), and let $\mathcal{X} = \mathcal{C} \times_S \mathcal{C}$. Then for "most" $s \in S$,

$$\operatorname{rk}\ H^2(X_s, \mathbb{Z}) \cap H^{1,1}(X_s)$$

(the Picard rank of X_s) is 3. But for points s with $\mathcal{C}_s$ a CM elliptic curve, the Picard rank is 4. Note that the set of points where the Picard rank jumps is "small" (it is countable, for example) but it is dense in both the Zariski and analytic topologies.

Finally, let's examine the local structure of the decreasing filtration $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$. Suppose S is contractible, so the Gauss-Manin connection ∇ gives a natural isomorphicm between any two fibers of $R^i\pi_*\Omega^\bullet_{dR,\pi}$; let V be such a fiber. This lets us view the fibers of $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$ as defining a flag on V. In the next lecture, Akshay will explain how the period mappings encode the variation of such flags. Regardless, supposing $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$ has rank r, we may ask how $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$ varies in $\mathrm{Gr}(r, V)$.

Recall that the if $W \subset V$ is a rank r subspace of W, the tangent space to $[W] \in \mathrm{Gr}(r, V)$ is naturally identified with $\mathrm{Hom}(W, V/W)$. Consider the following abstract situation: $\mathcal{E}$ and $\mathcal{F}$ are holomorphic vector bundles on a contractible complex manifold, S with $\mathcal{E} \subset \mathcal{F}$ an injective bundle map, and ∇ a flat connection on $\mathcal{F}$. Suppose $\mathcal{E}$ has rank r and let V be a fiber of $\mathcal{F}$. Then ∇ induces an isomorphism of the bundle $\mathrm{Gr}_S(r, \mathcal{F})$ with the trivial bundle $\mathrm{Gr}(r, V) \times S$. As $\mathcal{E}$ gives a section to the bundle $\mathrm{Gr}(r, \mathcal{F})$ this trivialization induces a map $P : S \to \mathrm{Gr}(r, V)$. A direct computation (see e.g. Voisin's Hodge Theory and Complex Algebraic Geometry I, page 244) shows that the derivative of this map

$$dP_s : T_sS \to T_{[\mathcal{E}_s]}\,\mathrm{Gr}(r, V)$$

is given by the composition

$$\frac{\partial}{\partial z} \mapsto \left(\mathcal{E}_s \xrightarrow{\nabla_{\frac{\partial}{\partial z}}} \mathcal{F}_s \simeq V \to V/\mathcal{E}_s \right)$$

where we view the parenthesized map as an element of $T_{[\mathcal{E}_s]}\,\mathrm{Gr}(r, V) \simeq \mathrm{Hom}(\mathcal{E}_s, V/\mathcal{E}_s)$.

Griffiths famously observed that the variation of the $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$ within $H^i(X_s, \mathbb{C})$ is not arbitrary—instead, it satisfies a very restrictive condition, usually called Griffiths transversality.

Proposition 44 (Griffiths Transversality). *Let ∇ be the Gauss-Manin connection on $R^i\pi_*\Omega^\bullet_{dR,\pi}$. Then*

$$\nabla(F^pR^i\pi_*\Omega^\bullet_{dR,\pi}) \subset (F^{p-1}R^i\pi_*\Omega^\bullet_{dR,\pi}) \otimes \Omega^1_S.$$

In terms of the discussion in the previous paragraph, the differential of the map parametrizing the variation of $F^pR^i\pi_*\Omega^\bullet_{dR,\pi}$ inside of $H^i(X_s, \mathbb{C})$ lands inside of

$$\mathrm{Hom}(F^p, F^{p-1}/F^p) \subset \mathrm{Hom}(F^p, H^i(X_s, \mathbb{C})/F^pH^i(X_s, \mathbb{C})).$$

Proof. Recall that the Gauss-Manin connection was defined as the connecting homomorphism in the long exact sequence obtained by applying $R\pi_*$ to the short exact sequence

$$0 \to \pi^*(\Omega^1_S) \otimes_{\mathcal{O}_X} \Omega^\bullet_{dR,\pi} \to \Omega^\bullet_{dR,X}/G^2(\Omega^\bullet_{dR,X}) \to \Omega^\bullet_{dR,\pi} \to 0.$$

Here

$$G^i\Omega^\bullet_{dR,X} = \mathrm{image}(\pi^*\Omega^i_S \otimes \Omega^{\bullet-i}_{dR,X} \to \Omega^\bullet_{dR,X}).$$

Let us compute this boundary map via resolutions; choose injective resolutions $I_1^{p,\bullet}$ of $\pi^*(\Omega^1_S) \otimes_{\mathcal{O}_X} \Omega^p_{dR,\pi}$, $I_2^{p,\bullet}$ of $\Omega^p_{dR,X}/G^2(\Omega^p_{dR,X})$, and $I_3^{p,\bullet}$ of $\Omega^p_{dR,\pi}$ fitting into a short exact sequence of double complexes. Suppose we have some section

$$\omega \in \Gamma(U, F^pR^i\pi_*\Omega^\bullet_{dR,\pi}).$$

We may represent ω by a section to
$$\pi_* \operatorname{Tot}(I_3^{\bullet \geq p, \bullet})^i,$$
say
$$\omega = \left[\sum_{r \geq p} \omega^{r,i-r} \right].$$
with $\omega^{r,i-r} \in \Gamma(\pi^{-1}(U), I_3^{r,i-r})$. To compute the Gauss-Manin connection, we choose a preimage
$$\eta = \sum_{r \geq p} \eta^{r,i-r} \in \Gamma(U, \pi_* \operatorname{Tot}(I_2^{\bullet \geq p, \bullet})^i), \eta^{r,i-r} \in \Gamma(U, \pi_* I_2^{r,i-r})$$
If d is the differential in $\Gamma(\pi_* \operatorname{Tot}(I_2^{\bullet \geq p, \bullet})^i)$, note that $d(\eta^{r,i-r}) \in \Gamma(U, \pi_*(I_2^{r+1,i-r} \oplus I_2^{r,i-r+1}))$! Taking the (unique) preimage ξ of $d\eta$ in $\pi_*(\operatorname{Tot}(I_1^{\bullet \geq r, \bullet}))$, we see that ξ is a representative for $\nabla \omega$. But by the remark on grading in the previous sentence (and the degeneration of the Fröhlicher spectral sequence), $\nabla \eta = [\xi] \in (F^{p-1} R^i \pi_* \Omega^\bullet_{dR,\pi}) \otimes \Omega^1_S$ as desired. □

Remark 45. Nothing was special about the choice of injective resolution above

Thus, we are led to the definition of an abstract variation of Hodge structure.

3.2. **Variation of Hodge Structure in the Abstract.**

We wish to formalize the properties above—we do so in the following definition.

Definition 46 (Variation of Hodge Structure)**.** An (integral) variation of Hodge structure of weight n on S is a $\mathbb{Z}$-local system $V_\mathbb{Z}$ on S, together with a decreasing filtration
$$F^p V_S \subset V_S := V_\mathbb{Z} \otimes \mathcal{O}_S$$
by holomorphic vector bundles, such that

(1) The F^p define a Hodge structure on each fiber of V_S, that is,
$$V_S = F^p V_s \oplus \overline{F^{n-p+1} V_s}.$$

(2) (Griffiths Transversality) Let $V_\mathbb{C} = V_\mathbb{Z} \otimes \mathbb{C}$. Then $V_\mathbb{C}$ defines an integrable holomorphic connection ∇ on V_S. We require that the F^p satisfy Griffiths transversality with respect to ∇, that is,
$$\nabla(F^p V_S) \subset F^{p-1} V_S \otimes \Omega^1_S.$$

There are analogous notions of rational or real variations of Hodge structure. We have shown above that the cohomology of a family of Kähler manifolds, $R^i \pi_* \underline{\mathbb{C}} \otimes \mathcal{O}_S$ naturally admits the structure of an integral variation of Hodge structure.

For Akshay's talk we will need a slightly stronger notion—the prototypical example will be the primitive cohomology of a (polarized) family of projective varieties.

Definition 47 (Polarized Variation of Hodge Structure)**.** An (integral) polarized variation of Hodge structure of weight n on S is a variation of Hodge structure $(V_\mathbb{Z}, F^p V_S)$ on S along with a bilinear form $Q : V_\mathbb{Z} \otimes V_\mathbb{Z} \to \mathbb{Z}$, so that the restriction of $(V_\mathbb{Z}, F^p V_S, Q)$ to each fiber is a polarized Hodge structure.

Again, there are entirely analogous definitions for rational or real polarized variations of Hodge structures. The primitive cohomology of a (polarized) family of projective varieties is a real polarized Hodge structure.