PROBLEM SET: THE TATE CURVE AND RIGID GEOMETRY

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1. The Tate curve over $\mathbb{C}$

(1) Let $E_\lambda = \{[x : y : z] \mid y^2z = x(x-z)(x-\lambda z)\} \subset \mathbb{P}^2_{\mathbb{C}}$.
  (a) What is the fundamental group of E_λ, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$?
  (b) What is the fundamental group of E_0?
  (c) What is the universal cover of E_0?

(2) Let $\Delta \subset \mathbb{C}$ be the open unit disk centered at 0; let $\Delta^* = \Delta \setminus \{0\}$. Let $E = \{([x : y : z], \lambda) \in \mathbb{P}^2_{\mathbb{C}} \times \Delta \mid y^2z = x(x-z)(x-\lambda z), \lambda \in \Delta\}$.
  (a) What is the fundamental group of E?
  (b) Let $\lambda \in \Delta^*$. Then there is a natural monodromy representation

$$\mathbb{Z} \simeq \pi_1(\Delta^*, \lambda) \to GL(H_1(E_\lambda, \mathbb{Z})).$$

  compute this representation.
  (c) Fix a point 0 on E_λ. Show that the map $H_1(E_\lambda, \mathbb{Z}) \simeq \pi_1(E_\lambda, 0) \to \pi_1(E, 0)$ factors through the $\pi_1(\Delta^*)$ co-invariants of $H_1(E_\lambda, \mathbb{Z})$.
  (d) Let $\tilde{E}$ be the universal cover of E, and let $\tilde{\pi} : \tilde{E} \to \Delta$ be the natural map. What is $\tilde{\pi}^{-1}(0)$? What is $\tilde{\pi}^{-1}(\lambda)$ for $\lambda \in \Delta^*$?
  (e) Fix $\lambda \in \Delta^*$. Describe the action of $\pi_1(\Delta^*, \lambda)$ on $\tilde{\pi}^{-1}(\lambda)$ by deck transformations.

(3) Let $E_\tau = \mathbb{C}/\langle 1, \tau\rangle$ where τ is in the upper half-plane (i.e. $\mathrm{im}(\tau) > 0$).
  (a) Recall that the Weierstrass $\mathfrak{P}$-function is the unique $\langle 1, \tau\rangle$-periodic meromorphic function on $\mathbb{C}$, holomorphic away from $\langle 1, \tau\rangle$ and with poles of order at most 2 on $\langle 1, \tau\rangle$. Write down a formula for $\mathfrak{P}$.
  (b) What algebraic relations are satisfied by $\mathfrak{P}, \mathfrak{P}'$? That is, can you describe the ring $\mathbb{C}[\mathfrak{P}, \mathfrak{P}']$?
  (c) Writing $\mathbb{C}^* = \mathbb{C}/\mathbb{Z}$ (where the identification is made via the map $z \mapsto e^{2\pi i z}$), $\mathfrak{P}$ descends to a function on $\mathbb{C}^*$. Write down a formula for this function. Note that it holomorphic away from the multiplicative subgroup of $\mathbb{C}^*$ generated by $q = e^{2\pi i \tau}$, and is periodic for this subgroup.

2. Basic algebra and geometry of affinoids

(1) Let $R = \mathbb{Z}_p[x_1, \cdots, x_m]$.
  (a) Let $\widehat{R}$ denote the completion of R at the ideal p. Describe the image of $\widehat{R}[1/p]$ in $\mathbb{Q}_p[[x_1, \cdots, x_m]]$ under the natural map.
  (b) Describe the image of the natural map

$$(\mathbb{Z}_p[[x_1, \cdots, x_m]]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Q}_p[[x_1, \cdots, x_m]].$$

  Is it the same as the answer to part (a)?
  (c) What are the maximal ideals of $\widehat{R}[1/p]$?

We call the ring $\widehat{R}[1/p]$ defined above a *Tate algebra*, and denote it $\mathbb{Q}_p\langle x_1, \cdots, x_m\rangle$.

(2) Consider the map $\mathbb{Q}_p\langle x, y\rangle \to \mathbb{Q}_p\langle x, y\rangle/(xy-p)$ sending x to px and y to py.
   (a) What is the image of the induced map $\mathrm{MaxSpec}(\mathbb{Q}_p\langle x, y\rangle/(xy-p)) \to \mathrm{MaxSpec}(\mathbb{Q}_p\langle x, y\rangle)$?
   (b) Draw a picture of the above map. What does your computation for part (a) mean geometrically? Is the local geometry analogous to the local geometry of problem (1) of Section 1?
   (c) Let $f \in \mathbb{Z}_p[x, y]$ satisfy $f(0,0) = 0$, and suppose $V(f)_{\mathbb{F}_p}$ is smooth at $(0,0)$. Again, let $\mathbb{Q}_p\langle x, y\rangle \to \mathbb{Q}_p\langle x, y\rangle/(f)$ be the map sending x to px and y to py. What is the image of the induced map $\mathrm{MaxSpec}(\mathbb{Q}_p\langle x, y\rangle/(f)) \to \mathrm{MaxSpec}(\mathbb{Q}_p\langle x, y\rangle)$? How does it differ from the picture above? What is the complex-analytic analogue of this situation?

(3) Let $f_i : \mathrm{MaxSpec}\, \mathbb{Q}_p\langle x, y\rangle \to \mathrm{MaxSpec}(\mathbb{Q}_p[x, y])$ be the map given by sending x to x/p^i and y to y/p^i. Show that $\mathrm{MaxSpec}(\mathbb{Q}_p[x, y])$ is the rising union of the images of the f_i. Can you give a similar description of $\mathbb{G}_m$ as a rising union of the images of $\mathrm{MaxSpec}(A_i)$, where each A_i is a Tate algebra?

3. Basic deformation theory

(1) Let A be a commutative ring and B an smooth A-algebra. Let $A' \to A \to 0$ be a surjection of rings with kernel I, and suppose that $I^2 = 0$. (Hint: it might help to do this problem and the next simultaneously.)
   (a) Show that there exists a smooth A' algebra B' and an isomorphism $B' \otimes_{A'} A \xrightarrow{\sim} B$.
   (b) Show that any two such B' are isomorphic.
   (c) Give an example of the following: X is a scheme, $Y \to X$ is a smooth morphism, and X' is a scheme with X embedded as a closed subscheme $X \hookrightarrow X'$ defined by an ideal sheaf $\mathscr{I}$ with $\mathscr{I}^2 = 0$, but there exists no smooth $Y' \to X'$ such that $Y' \times_{X'} X \simeq Y$. (By part (a), there are no examples with X, Y affine.)
   (d) Now suppose that B is an étale A-algebra. Show that any two lifts B' to A' are *canonically* isomorphic.
   (e) Suppose $Y \to X$ is an étale morphism and X' is a scheme with X embedded as a closed subscheme $X \hookrightarrow X'$ defined by an ideal sheaf $\mathscr{I}$ with $\mathscr{I}^2 = 0$. Show that there exists Y' étale over X' with $Y' \times_{X'} X \simeq Y$.

(2) Again, suppose X is a scheme, $f : Y \to X$ is a smooth morphism, and X' is a scheme with X embedded as a closed subscheme $X \hookrightarrow X'$ defined by an ideal sheaf $\mathscr{I}$ with $\mathscr{I}^2 = 0$. As above, a deformation of Y over X' is a flat (necessarily smooth) X' scheme Y' along with a choice of isomorphism $\phi : Y' \times_X X' \tilde{Y}$. A morphism of deformations $(Y_1, \phi_1) \to (Y_2, \phi_2)$ is a morphism $Y_1 \to Y_2$ over X', intertwining the ϕ_i's.
   (a) Show that automorphism group of a deformation Y' is isomorphic to $\mathrm{Hom}(\Omega^1_{Y/X}, f^*\mathscr{I})$.
   (b) Show that if the set of deformations of Y over X' is non-empty, it is naturally a torsor for $\mathrm{Ext}^1(\Omega^1_{Y/X}, f^*\mathscr{I})$.

(c) Show that there is a natural (functorial) class

$$\mathrm{ob}_{Y/X} \in \mathrm{Ext}^2(\Omega^1_{Y/X}, f^*\mathscr{I})$$

whose vanishing is equivalent to the existence of a deformation of Y over X'.

(d) * Now let $X = \mathrm{Spec}(k)$ and let Y be a (proper) nodal curve over X. Is there a sheaf on Y which controls its deformation theory, as Ω^1 controls the deformation theory of smooth schemes? What is the deformation-obstruction theory of an irreducible nodal cubic? What is the deformation theory of a stable curve, each of whose components is isomorphic to $\mathbb{P}^1$?

4. The Tate curve

(1) Let $q \in \mathbb{Q}_p^\times$ be such that $|q|_p < 1$.

(a) Describe the Galois action on the Tate module of $\mathbb{G}_m/q^{\mathbb{Z}}$.

(b) Let $E/\mathbb{Q}_p$ be a curve with split multiplicative reduction with origin 0; let E' be the special fiber of a semistable model. Describe the specialization map $\pi_1^{\text{ét}}(E_{\overline{\mathbb{Q}_p}}, 0) \to \pi_1^{\text{ét}}(E'_{\overline{\mathbb{F}_p}}, 0)$. Compare this to the complex-analytic situation.

(c) * Show that the quotient $\mathbb{G}_m/q^{\mathbb{Z}}$ exists in the category of rigid spaces over $\mathbb{Q}_p$.

(d) What is the endomorphism group of the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$, viewed as a rigid space?

(2) * Imitate the complex-analytic construction (and the construction in the lecture) to make a version of the Tate curve as a formal scheme over $\mathbb{Z}[[q]]$. (That is, construct a formal scheme over $\mathbb{Z}[[q]]$, such that after base change along any continuous homomorphism $\mathbb{Z}[[q]] \to \mathscr{O}_{K_v}$, with K_v a local field, one obtains a formal model of the Tate curve over $\mathscr{O}_{K_v}$ with uniformizing parameter q.)