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Ideal Color Space

II. The Super-Importance of Hue Differences and Its Bearing on the Geometry of Color Space*

Introduction

The concept of ideal color space has guided, or perhaps we should say misguided, many attempts to write a formula for the perceived size of color differences. Ideal color space is a tridimensional array of points, each representing a color, so located that the length of the straight line between any two points is proportional to the perceived size of the difference between the colors represented by the points. If the location of points in ideal color space is defined by the variables, p, q, r, to be plotted along mutually perpendicular axes, then the basic property of ideal color space is that the perceived size, ΔE, of the difference between the colors corresponding to points p_1, q_1, r_1, and p_2, q_2, r_2, would be simply that of Euclidean geometry:

$$\Delta E = k\,[(p_1 - p_2)^2 + (q_1 - q_2)^2 + (r_1 - r_2)^2]^{1/2} \quad (1)$$

where k is a constant of proportionality connecting the units of the p, q, and r scales with the unit in which the perceived size, ΔE, of the color difference is expressed.

CIE-*XYZ* Color Space

The colorimetric coordinate system recommended by the International Commission on Illumination (CIE) in 1931 is based on the tristimulus values, X, Y, Z, defined

* cf. the article *Curvature of Color Space and Its Implications for Industrial Color Tolerances* by the same author in 'palette' No. 29.

for a specimen of spectral radiance factor $\beta(\lambda)$ viewed under a light source of spectral distribution E_λ as:

$$X = \int_0^\infty E_\lambda \, \beta(\lambda) \, \bar{x}(\lambda) \, d\lambda$$

$$Y = \int_0^\infty E_\lambda \, \beta(\lambda) \, \bar{y}(\lambda) \, d\lambda \qquad (2)$$

$$Z = \int_0^\infty E_\lambda \, \beta(\lambda) \, \bar{z}(\lambda) \, d\lambda$$

where $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ are weighting functions roughly describable as expressing the red, green, or blue sensitivities, respectively, of the normal human eye. By plotting on mutually perpendicular axes the tristimulus values, X, Y, Z, a color space may be formed, CIE-*XYZ* color space, that has some of the properties of ideal color space. Each point in this space corresponds to one, and only one, color. Any desired color range may be specified in terms of the tristimulus values, X, Y, Z. For example, the requirement that the X value of an acceptable color fall between a maximum value, X_x, and a minimum value, X_n; that its Y value fall between Y_x and Y_n; and its Z value, between Z_x and Z_n, specifies a rectangular parallelopiped in this color space. If $X_x - X_n = Y_x - Y_n = Z_x - Z_n$, the acceptable range corresponds to a cube in *XYZ*-space. Automatic color sorters based on photoelectric tristimulus colorimetry have been developed by Ward that will throw into one bin any ceramic tile, or plastic component, whose color corresponds to a point within such a rectangular parallelopiped, and will throw into a reject bin, any tile or components whose color corresponds to a point outside this range. This machine inspection does away with the need for a human inspector, and makes practical the inspection of every component delivered.

An increment of one unit of X, Y, or Z corresponds to a perceived size of color change that diminishes with increase of X, Y, or Z. In CIE-*XYZ* space, just noticeable differences among near-black colors (X, Y, and Z approaching zero) are represented by relatively small distances compared to just noticeable differences among colors (white, yellow, red, green) quite different from black. CIE-*XYZ* color space thus differs drastically from ideal color space. The practical disadvantage is simply that for each color, a separate determination has to be made of how big to make the acceptable range. To a purchaser using a two-swatch method of indicating an acceptable range, this disadvantage may not appear very serious. He is used to making a separate decision for each color when he chooses a tolerance swatch. If, however, an ideal color space could be constructed, the purchaser would have only to decide whether to apply a strict (say, one just noticeable difference), a moderate (say 4 jnd's), or a loose tolerance (say 40 jnd's). Regardless of the desired color, all of the remaining tolerance information could be derived from the ideal space.

CIE-*XYZ* color space has the advantage that it provides for location of the color point corresponding to any sample, swatch, or chip, by means of spectrophotometry. The reduction of the spectrophotometric data is accomplished in accord with equations (2). Other color spaces differ from CIE-*XYZ* space only in the distribution of the points within the space; and the three variables defining such spaces are determinable functions of the tristimulus values, X, Y, Z. It follows that the variables, p, q, r, of equation (1), defining ideal color space, are also, if they exist at all, functions of X, Y, and Z; that is:

$$p = f_p(X, Y, Z)$$

$$q = f_q(X, Y, Z) \qquad (3)$$

$$r = f_r(X, Y, Z)$$

Perhaps the most widely used color space of all is CIE-*Yxy* space, formed by plotting on mutually perpendicular axes, the Y-tristimulus value, and the chromaticity coordinates, x, y, defined as $X/(X + Y + Z)$, and $Y/(X + Y + Z)$, respectively. Other examples of color spaces are the CIE-$U^*V^*W^*$ space recommended provisionally by the CIE in 1964 as an approximation to ideal color space, and the Adams chromatic-value space used by some industrial firms in preference to CIE-$U^*V^*W^*$ space. Still another example is Munsell color space.

Munsell Color Space

The Munsell notation is based on the variables: Munsell hue, Munsell value, and Munsell chroma. The definition of Munsell value V in terms of the ratio, Y/Y_{MgO}, of the Y-tristimulus value of the color to that for magnesium oxide used as the white standard is:

$$100\,Y/Y_{MgO} = 1.2219\,V - 0.23111\,V^2 + 0.23951\,V^3 - 0.021009\,V^4 + 0.0008404\,V^5 \qquad (4)^*$$

In Munsell space, Munsell value is plotted along a vertical scale so that colors whose Munsell values are the same correspond to points in a horizontal plane, and this convention is usual also for CIE-*Yxy* space, CIE-$U^*V^*W^*$ space, and Adams chromatic-value space. The location of points in any one of these planes, instead of being given by rectangular coordinates, is given in Munsell space only by polar coordinates, the angle being specified by Munsell hue, the radius by Munsell chroma. The Munsell variables, hue, value, and chroma, serve to locate one, and only one, point in Munsell space for each color; so the Munsell variables define a color space. Figure 1 indicates the dimensions of Munsell color space.

In Munsell color space, colors having constant hue and chroma correspond to a vertical line, and spacing along such lines is in accord with the Munsell-value function determined experimentally to yield uniform color spacing. This property of Munsell color space conforms to ideal color space.

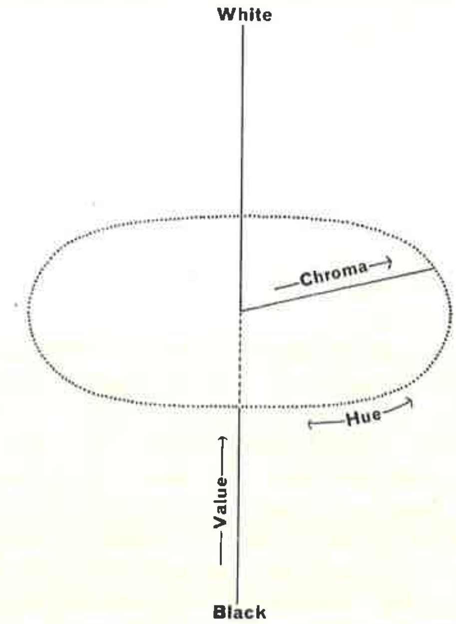
In Munsell color space, the locus of colors of constant value and constant chroma is a circle whose center is the point representing the achromatic, or neutral, or gray color of that Munsell value. Since the spacing along the scales of Munsell chroma was determined experimentally to be perceptually uniform, this property of Munsell color space also conforms closely to ideal color space. Munsell chroma of a color means the departure of the color from the gray of the same value. The locus of colors of constant Munsell chroma regardless of value is a cylinder in Munsell color space centered on the neutral axis, plotted vertical, on which points representing the various grays between black and white are located.

Finally, the colors of constant Munsell hue and constant Munsell value are represented by points lying along a horizontal straight line intersecting the neutral axis. The selection of colors to be represented along such a line was made experimentally so as to yield color perceptions of constant hue. Such lines in ideal color space would correspond to geodesics (lines of minimum distance) between the point representing a chromatic color and that representing the nearest gray. Perhaps colors represented by points along such geodesics do yield color perceptions of constant hue. Schrödinger in 1920 in his important treatment of higher color metrology made this assumption. The direction of such a line is identified in Munsell color space by the Munsell hue notation, and along any locus of constant value and constant chroma the colors have been chosen experimentally so as to be perceptually equally spaced for equal increments in Munsell hue notation. In this respect also, Munsell color space corresponds closely to ideal color space.

Munsell color space is defined by giving in CIE-*Yxy* space the loci of constant Munsell hue and constant Munsell chroma for a number of Munsell values ranging between zero for black and 10 for white. Like other color spaces intended to approximate ideal color space, Munsell color space is thus defined by variables that are functions of the tristimulus values, *X*, *Y*, *Z*, of the color, even though there are no explicit formulas for these functions. The estimate of the perceived size of the difference between two colors defined by their Munsell notations: $H_1\,V_1/C_1$, and $H_2\,V_2/C_2$, according to distance in Munsell color space has been derived by Godlove from equation (1):

$$\Delta E = [2\,C_1\,C_2\,(1 - \cos 3.6\,\Delta H) + (\Delta C)^2 + (4\,\Delta V)^2]^{1/2} \qquad (5)$$

* This equation has been developed by the author of this article, Dr. D. B. Judd. Therefore, it is frequently referred to as the 'Judd Polynomial', see S. M. Newshall, D. Nickerson and D. B. Judd, J. Opt. Soc. Am. 33, 385 (1943).



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Dimensions of Munsell color space. Munsell hue varies according to angle about the neutral axis; Munsell value varies along the vertical, and Munsell chroma corresponds to horizontal distance from the vertical neutral axis.

where ΔH, ΔC, ΔV are abbreviations for $H_1 - H_2$, $C_1 - C_2$, and $V_1 - V_2$, respectively, and $3.6\,\Delta H$ gives the angular separation in degrees between the directions from the neutral axis defined by the two Munsell hues, H_1 and H_2. The unit of ΔE in equation (5) is one Munsell chroma step.

Munsell color space is largely responsible for the development of the geometrical concept of ideal color space. Munsell color space conforms within experimental error to the basic idea that the perceived size of the difference between two colors is proportional to the distance between the points representing them in Munsell space for three kinds of loci:

(1) vertical lines of constant Munsell chroma and hue,
(2) horizontal lines of constant Munsell value and hue, and
(3) circles of constant Munsell value and chroma.

Whether the same proportionality holds for loci other than these three kinds depends, as we shall see, on whether or not the geometry of color space is Euclidean.

Munsell color space is exceptionally convenient for color and color-tolerance specification. For swatches viewed against a gray background Munsell hue corresponds closely to the hue of the color perception; Munsell value, to its lightness; and Munsell chroma, to its saturation. The *Munsell Book of Color* is widely distributed, and copies have been continuously available by purchase since 1929. This book shows color chips identified by Munsell hue, value, and chroma ($H\ V/C$); so the purchaser may see by direct inspection the size of a departure from the standard that corresponds to any color tolerance set in Munsell terms.

Evidence for the Curvature of Color Space

In 1942 MacAdam reported the results of an experimental study of the sensitivity of one observer (P.G. Nutting) to chromaticity variations among equiluminous colors specified by points in the (x, y)-plane. The two colors of equal luminance being compared were viewed against a gray surround of luminance equal to one-half the luminance of the colors. MacAdam showed that a map of chromaticities equally spaced in accord with the Nutting data could not be plotted on a plane, but required a surface with a dome (positive curvature) near the center representing near-grays, with ruffles (negative curvature) elsewhere. The implication is that ideal color space does not conform to the facts, and that true color space conforms to Riemannian geometry instead of to Euclidean geometry; that is, it is characterized by Gaussian curvature significantly different from zero.

Although sensitivity to chromaticity variations changes considerably from one observer to another, those who, like myself, have been used to dealing with color differences in terms of ideal color space, have begun to think of other kinds of experimental evidence that this kind of color space does not exist. One such kind of evidence concerns the relative perceptual importance of Munsell hue steps and Munsell chroma steps.

Munsell Hue and Chroma Steps Compared

For the assessment of the perceptual size of color differences involving both a hue and a chroma component, it is necessary to know the relative importance of the Munsell hue and the Munsell chroma steps.

If Munsell color space is an ideal color space, the perceptual importance of the Munsell hue step relative to that of the Munsell chroma step would be that indicated by equation (5). If we consider two colors of the same value and chroma ($\Delta V = 0$, $\Delta C = 0$, $C_1 = C_2 = C$) that differ by one Munsell hue step ($\Delta H = 1$), equation (5) shows that the perceptual importance of the color difference is:

$$\Delta E_{\Delta H=1} = C\,[2\,(1 - \cos 3.6^\circ)]^{1/2}$$

which indicates correctly that the perceptual importance of one Munsell hue step is proportional to the chroma C of the two colors.

If we now consider two colors of the same hue ($\Delta H = 0$) that differ by one chroma step ($\Delta C = 1$), equation (5) shows that the perceptual importance of the color difference is:

$$\Delta E_{\Delta C=1} = 1,$$

and this corresponds to the statement that the unit of color difference used in equation (5) is one chroma step. The perceptual importance of one Munsell hue step relative to that of one Munsell chroma step is thus:

$$\Delta E_{\Delta H=1}/\Delta E_{\Delta C=1} = C\,[2\,(1-\cos 3.6°)]^{1/2} = C\,[2\,(1-0.998027)]^{1/2} = C\,(0.003946)^{1/2} = 0.0628_2\,C.$$

If the whole 100 Munsell hue steps of the hue circuit at chroma C were stepped off, one at a time, the total color difference would have the size of $6.28_2 C$. This total difference corresponds in ideal color space to the length of the perimeter of a regular polygon of 100 sides inscribed in a circle of radius C. If the hue circuit were stepped off by hue intervals approaching zero, the total difference would correspond to the circumference of the circle itself, whose length is $2\pi C = 6.28_3 C$.

As far as I know, the very first formula intended to yield predictions of the perceptual size of color differences ever published is the 1936 Nickerson index of fading by which was expressed the relative importance of Munsell hue, value, and chroma steps evaluated experimentally for colors viewed with gray surround whose hues differed by less than 10 Munsell hue steps. The Nickerson index, I, of fading is:

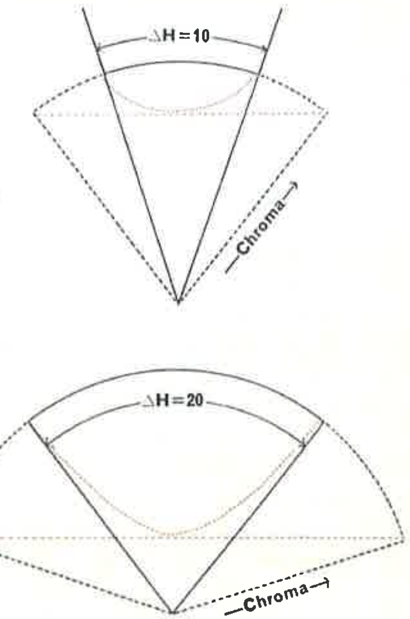
$$I = (C/5)\,(2\,\Delta H) + 6\,\Delta V + 3\,\Delta C, \quad \Delta H < 10. \qquad (6)$$

By this formula, the contribution of one Munsell hue step would be: $I_{\Delta H=1} = 2\,C/5$; and the contribution of one Munsell chroma step would be $I_{\Delta C=1} = 3$. The perceptual importance of one Munsell hue step relative to that of one Munsell chroma step determined experimentally by Nickerson in 1936 for hue intervals of less than 10 steps is thus:

$$I_{\Delta H=1}/I_{\Delta C=1} = 2\,C/15 = 0.1333\,C,$$

and this differs from the maximum ratio ($0.0628_3 C$) permitted by Euclidean geometry by slightly more than a factor of 2. Some adherents of ideal color space, like myself, rejected this violation of Euclidean geometry, and ascribed it to experimental error. They welcomed the Godlove formula (5), because it is in strict accord with ideal color space. Maybe this was a backward step.

If we ask of the MacAdam surface, on which equally important chromaticity differences correspond to equal distances, what would be the perceptual importance of one Munsell hue step relative to that of one Munsell chroma step, we get the answer that it depends on the chroma. For colors of very small departures from gray (very small Munsell chromas) the ratio should be $0.0628\,C$, because a strictly local map accords with ideal color space. For somewhat higher chromas, those corresponding to the end of the dome-shaped portion of the MacAdam surface, the ratio should be less than $0.0628\,C$; and for high chromas extending



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···· Color geodesics (dotted lines) implied by weighting Munsell hue steps twice as much, relative to Munsell chroma steps, as that implied by the Euclidean geometry of ideal color space, shown on the Munsell hue-chroma diagram, and derived from the fan-crinkled surface.
——— Half-extended fan
– – – – Fully extended fan
Note that for a hue interval of 20 Munsell steps the length of the geodesic is not much less than $2\,C$; even for the 10-step hue interval the geodesic departs considerably from the chord corresponding to Euclidian geometry [Godlove formula (5)].

into the ruffled portions of the MacAdam surface the ratio might be greater than $0.0628\,C$. The Nickerson index of fading, as we have seen, implies for colors differing in hue by less than 10 Munsell hue steps a ratio of $0.1333\,C$ regardless of chroma.

The Nickerson index of fading does not imply that the chromaticity surface has to be curved to make the distance between two points proportional to the perceived size of the difference between the equiluminous colors represented by the points. It suggests for each segment of 10 Munsell hue steps (one-tenth of the hue circuit), the kind of surface produced by a half-extended fan that, when fully extended, covers about one-fifth of the circle. The number of ribs in the fan may be increased indefinitely without altering the property that peripheral distances along the surface are about double those on a plane surface for the same angular interval. Such a surface might be called a half-extended fan-shaped surface, or a fan-crinkled surface.

It is composed of a large number of triangles, the plane of each triangle making a dihedral angle of about 60° with its neighbor. This surface is not curved because it may be made into a plane surface simply by extending the fan. All points of a fan-crinkled surface may be made to come as close to the central plane as desired simply by increasing the number of ribs in the fan.

Increasing the hue range of a fan-crinkled surface of this kind even moderately beyond the 10 hue steps representing the maximum hue interval for which the Nickerson index of fading was experimentally verified produces some rather surprising results. For example, if the hue range of the fan be increased to 20 Munsell hue steps (one-fifth of the hue circuit), extending the fan-crinkled surface to form a plane surface makes it subtend two-fifths of the hue circuit (see Fig. 2). Straight lines on this extended surface conform to geodesics according to the Nickerson index of fading for colors of constant Munsell value. The dashed straight line on Figure 2 shows the path of minimum distance from a color of one hue to that of another of the same chroma differing by ΔH hue steps. Projecting this straight line back onto the fan-crinkled surface yields the curved dotted line. For $\Delta H = 10$, Figure 2 shows a considerable curvature, and for $\Delta H = 20$, the curvature is extreme. Extending a fan-crinkled surface to $\Delta H = 25$ (one quarter of the hue circuit) yields a semi-circle. The straight line bridging the interval of 25 Munsell hue steps is simply the diameter of this semi-circle. The increase in the hue range of the Nickerson index of fading from 10 to 25 thus implies that the shortest path (geodesic) from one hue, say 5*R*, to another at the same chroma differing by 25 Munsell hue steps, say 10*Y*, passes through gray, and that the total length of the geodesic is 2*C*. Increasing the hue range of the fan-crinkled surface to 50 Munsell hue steps makes the extended plane surface into a circle. The shortest distance on this circle bridging a hue interval of 50 is likewise 2*C*. It is, of course, not legitimate to jump directly from one end of the fan to the other; so the indicated geodesic would pass from, say 5*R*, to *N* (gray) with a length of *C*, and then back up the same line, which now refers to the other end of the extended fan, to the opposite hue, 5*BG*, with a length of *C*, making a total of 2*C*. This result agrees with ideal color space, and can be obtained from equation (5) by setting $\Delta C = \Delta V = 0$, and $\Delta H = 50$. The implication is that gray is both precisely between 5*R* and 10*Y*, and precisely between 5*R* and 5*BG* (see Fig. 3). But can we believe this? Inspection of Figure 3 makes me think that we cannot. It would seem that the super-importance of a Munsell hue step found to apply to small hue intervals progressively diminishes as the

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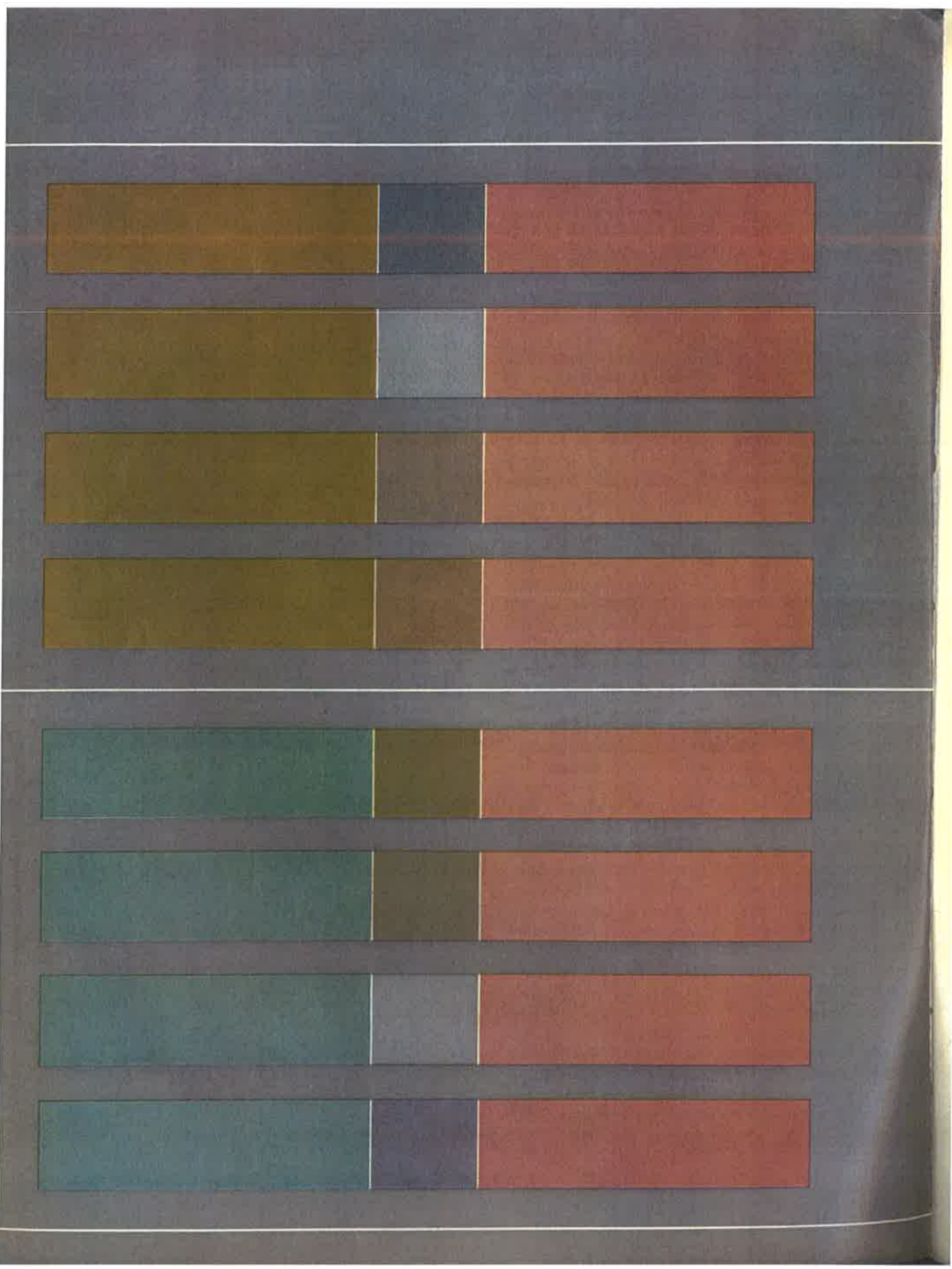
Can gray be precisely between red and blue-green, and also precisely between red and greenish yellow? The eight colors on the left are intended to be identical light reds (Munsell notation 5*R* 6/6). The upper four colors on the left are intended to be identical blue-greens (5*BG* 6/6). The lower four colors on the left are intended to be identical greenish yellows (10*Y* 6/6). The central swatches have colors intended to cover the range within which the geodesic (path of shortest distance) between the adjacent colors (left and right) might pass. Most people judge that gray is precisely ‹between› red and blue-green because they are complementary hues; that is, to pass from red to blue-green by way of the other colors of the intermediate swatches (intended to be 10*Y* 6/4, 10*Y* 6/2, and 10*PB* 6/2) corresponds to a larger total color difference than to pass from red to blue-green by way of gray. The path by way of gray is thus the direct route, and other paths are detours. On the other hand most people judge that gray is not the precise intermediate between red and greenish yellow, but that the most direct path goes through grayish orange somewhere between the upper central swatch (intended to be 7.5*YR* 6/4) and the one just below it (intended to be 7.5*YR* 6/2). The lowest central swatch (intended to have the color 7.5*B* 6/2) is usually judged to be on a rather long detour from red to greenish yellow. Which central swatch appears to you to have a color precisely ‹between› red and greenish yellow?



hue interval becomes larger. A formula extending the Nickerson index of fading to hue intervals of any size by applying this idea can easily be written by adjustment of equation (5). We need only multiply the first term by a function of ΔH varying smoothly from a numerical value of 4 for ΔH approaching zero, to a value of unity for ΔH approaching 50. One such function is $[4/(3 - \cos 3.6\Delta H)]^2$, and if we use the symbol, f_h, for this function, equation (5) becomes:

$$I = [2f_h\, C_1\, C_2(1 - \cos 3.6\Delta H) + (\Delta C)^2 + (4\Delta V)^2]^{1/2} \quad (7)$$

Study of this formula was recommended by CIE Committee E-1.3.1, Colorimetry, at its June 1967 meeting in Washington. There does not seem to be a geometrical model agreeing with equation (7); at least I have not been able to think of one. If this formula were found to be exactly in accord with experimental determinations of the perceived size of the difference between any two colors viewed with a gray surround, I think it would have to be concluded that there is no way to arrange points in space, one point for each color, such that the perceived size of the difference between any two colors is proportional to the distance between the points representing the colors. In other words, the super-importance of hue differences implied by either the Nickerson index of fading, or by the corresponding modification of the Godlove formula given by equation (7), renders impossible the existence of ideal color space.

References

E. Q. Adams, *X-Z Planes in the 1931 ICI System of Colorimetry*, J. Opt. Soc. Am. *32*, 168 (1942).

Commission Internationale de l'Eclairage (CIE), Proc. 8th Session, Cambridge 1931 (Cambridge University Press, 1932), p.19.

I. H. Godlove, *Improved Color-difference Formula, with Applications to the Perceptibility and Acceptability of Fadings*, J. Opt. Soc. Am. *41*, 760 (1951).

D. L. MacAdam, *Visual Sensitivity to Color Differences in Daylight*, J. Opt. Soc. Am. *32*, 247 (1942).

A. H. Munsell, *A Color Notation*, 1st Ed. (Ellis, Boston 1905); 11th Ed. (Munsell Color Company, Baltimore, 2441 North Calvert St., 1961).

D. Nickerson, *The Specification of Color Tolerances*, Text. Res. *6*, 509 (1936).

S. M. Newhall, D. Nickerson, and D. B. Judd, *Final Report of the O.S.A. Subcommittee on the Spacing of the Munsell Colors*, J. Opt. Soc. Am. *33*, 385 (1943).

E. Schrödinger, *Grundlinien einer Theorie der Farbenmetrik im Tagessehen*, Ann. Physik *63*, 481 (1920).

J. W. Ward, *An Automatic Digital Colorimeter for Laboratory and Process Control Applications*, Proc. Am. Assoc. Text. Chem. Col., Am. Dyestuff Rept. *55*, 55 (November 21, 1966).

A final contribution by Dr. D. B. Judd on the 'Ideal Color Space' will appear in the next number of 'palette'.

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