Hedging Longevity Risk: Does the Structure of the Financial Instrument Matter?

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February 2018

Abstract

Longevity-linked securities can be constructed either as cash-flow hedging instruments or as value hedging instruments. This article studies the interaction between the structure of longevity-linked securities and shareholder value. Relying on a strand of literature that investigates corporate risk management decisions made in the interests of shareholders, we present a framework that compares cash-flow hedges with value hedges. Both our theoretical model and numerical experiments show that value hedging dominates cash-flow hedging in the context of management decisions being made to maximize shareholder value. This finding provides an explanation for the failure of some attempted issues of longevity risk transfer instruments and suggests efficient alternate structures.

Keywords: Longevity Risk; Longevity Hedging; Cash-flow Hedging; Value Hedging; Shareholder Value

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1 Introduction

Pension Plans and Annuity Providers face financial and longevity risks in their books of business. The management of the financial risk is not unique to this industry but the longevity risk is. Longevity risk is the risk that members of a population may live longer on average than expected when the annuity products were originally formed and priced. Recognizing that risk, pension plans and annuity providers must construct portfolios of financial assets with values sufficient to pay the plan benefits for the duration of the lives covered in the plans or face insolvency. Since management of longevity risk has not been adequately addressed in the literature, this analysis focuses on it by noting how it may be managed with two different financial instruments designed to deal with it directly.

A number of financial instruments have been designed to manage longevity risk. Blake and Burrows (2001) discussed the notion of survivor bonds. Such an instrument would have coupon payments that mimic the stream of payoffs for a pension plan or a life annuity and so the bond's payoff stream would continue until the death of the last plan participant. There have been a number of subsequent proposals and analyses of similar instruments now broadly called longevity bonds, e.g., Blake et al. (2006), Lin and Cox (2008), Chen and Cummins (2010), and Zelenko (2014). Another instrument, the longevity swap, e.g., see Blake et al. (2013), is designed to reduce or eliminate longevity risk; this instrument transfers longevity risk to a counter-party, e.g., insurer, who then makes the stream of pension or annuity payments in return for an agreed-upon stream of payments from the pension fund or annuity provider. Like the longevity bond, this exchange of cash flows can eliminate longevity risk if the swap is a full hedge or reduce it if the hedge is partial. Finally, the q-forward, e.g., see Coughlan et al. (2007), was designed to hedge longevity risk by hedging the value of the annuity liability rather than the actual annuity cash flow. As noted in Blake et al. (2013), "A q-forward is an agreement between two parties in which they agree to exchange an amount proportional to the actual, realized mortality rate of a given population (or subpopulation) in return for an amount proportional to a fixed mortality rate that has been mutually agreed at inception to be payable at a future date (the maturity of the contract)."

The longevity bond represents a cash-flow hedge instrument. There have been several attempts to issue longevity bonds. In 2004 the European Investment Bank introduced a longevity bond designed to allow annuity providers with a way of hedging the cash flows that could occur if longevity improvements were bigger than expected. That bond was ultimately not issued because of insufficient demand. In 2008 and 2009 the World Bank introduced longevity bonds in Chile but both attempted issues also failed due to insufficient demand. The second generic type of cash flow contract is the longevity swap and there is a market for these hedging instruments. In 2011 British Airways Pension Fund and Rolls Royce Pension Fund completed swaps; in 2012 Pilkington

Pension Fund completed a swap. As noted the q-forward is a value hedge instrument and there have been hedges by Friends Provident in 2007, Lucida and Canada Life in 2008, Aviva in 2009 and Pall in 2011, i.e., see Blake et al. (2013).

A number of articles in the literature have proposed rationales for the failure of a longevity bond market. Some of the explanations have been concerned with the design of the bond issue, e.g., Blake et al. (2006) and Chen and Cummins (2010), while others have been concerned with pricing, e.g., Lin and Cox (2008). Zelenko (2014) concluded that the reason for failure was due to a moral hazard problem. MacMinn and Brockett (2017)[1] generalized the argument in Zelenko (2014) and demonstrated that the corporate manager facing insolvency risk and acting in the interest of shareholders would not hedge using a longevity bond, swap or q-forward.[2]

Our analysis in this article is not entirely about market failure. Rather it is about a comparison of hedging tools; it is about risk management and in particular about longevity risk management. Since the corporate manager facing insolvency risk is not born with an incentive to hedge, we introduce a regulatory constraint that provides such an incentive. Here the corporate manager makes decisions not only to maximize shareholder value but also to satisfy a regulatory constraint. The constraint requires the probability of insolvency to be less than or equal to a level set by the regulator. In this setting we are able to consider what type of hedging instrument management should use in order to maximize shareholder value subject to the regulatory constraint. Since longevity occurs due to medical advances, we also introduce a systematic longevity shock to represent the medical advances or to the contrary, a mortality shock. Based on these we compare the two generic forms of longevity risk management. We are able to show that value hedging dominates cash-flow hedging for full hedges; performing some numerical experiments, we are also able to show that value hedging may dominate a partial cash-flow hedging as well, at least for our chosen parameters.

This article is structured as follows: the theoretical model is constructed in the next section. The model is a complete financial market model like Arrow (1963), Debreu (1959) and MacMinn (1987). The model is constructed first without and then with systematic longevity risk and the regulatory constraint is also introduced. The two generic hedging instruments are introduced and the main theorem shows that the stock value of the firm with a value hedge exceeds that of the firm with a full cash flow hedge. The numerical experiment for the value hedge versus partial cash flow hedge is performed in the penultimate section. For the parameters used there the analysis shows that the value hedge continues to dominate the cash flow hedge. The last section concludes.

[1]See MacMinn and Brockett (2017) for more on the existing literature on the explanation of the longevity bond failures.

[2]The corporate manager has a different objective function than the manager of another type of organization. If the manager is in charge of a pension fund and makes decisions in the interests of the pensioners then the hedging decisions would be made differently.

2 The Theoretical Model

We separate longevity hedging instruments into two general categories based on their payment structure: The first type of hedging instrument consists of a series of payments that offset or stabilize the liability cash-flow of a life insurer; this instrument is referred to as a cash-flow hedge. The second type of hedging instrument consists of one cash payoff that is contingent on a publicly observable longevity/mortality event; this instrument is referred to here as a value hedge. The name *value hedge* is used because the payoff on this instrument may be designed to recover the loss in value due to a longevity event, e.g., if a new medicine/treatment is developed that extends the lives of individuals then insurers selling annuities may have insufficient reserves to cover the their annuity payments. Similarly, life settlement firms may also have insufficient reserves to cover the life insurance premiums in their books of business.[3] Such an event can also be a mortality-related event, e.g., if a pandemic occurs with many deaths then the reserves of a life insurer or reinsurer might be insufficient to cover the death benefits on its books of life business. Most hedging instruments in the current annuity markets, e.g., longevity swaps, may be broadly categorized as the first type while some hedging instruments in life markets, e.g., mortality bonds, have some similarity with the second type.[4] It is possible to design value hedging products, based either on a mortality shock such as a pandemic or on a longevity shock such as a medical breakthrough. The key difference between a cash-flow hedge and a value hedge is that while the latter only benchmarks its payments based on the underlying systematic shocks, i.e., longevity/mortality events, the former additionally hedges shifts in future mortality that are not caused by, or cannot be explained by such shocks.[5] Therefore, we expect to observe more variation in the frequency and size of the payments for a cash-flow hedge than for a value hedge.

In this section, for simplicity and without loss of generality, we construct a theoretical model with two dates: $t = 0$ (now) and $t = 1$ (then). This model is a direct extension of the one used in MacMinn and Brockett (2017). It does, however, have two key differences. First, we consider a solvency requirement imposed by a regulator as the motivation for the insurer to consider using any hedging instrument. Second, we introduce separately systematic risk as a component of longevity risk and in particular, consider the impact of this systematic risk in the total set of states of nature.

[3]A life settlement firm purchases life contracts and makes the premium payments in return for the death benefit. When an antiretroviral medicine was developed that considerably extended the lives of those with HIV firms in the viatical market—the precursor to the life settlements market—many viatical settlement firms did experience financial distress and a number of them went bankrupt.

[4]Roughly put, a mortality bond is a bond characterized by coupon payments, a principal payment at maturity and an option that is triggered by a change in mortality. If the option is triggered then the insurer receives part or all of the principal rather than the bondholders. The mortality bond may be characterized as a bond plus a mortality-contingent option; it is the option component that resembles a value hedge.

[5]In what follows, we will use the terms *longevity shock/risk* and *mortality shock/risk* interchangeably to denote unexpected change in future mortality experience regardless of the direction of the shock.

Developing the non-systematic and systematic elements of longevity risk enables us to draw a theoretical difference between cash-flow and value hedges from the perspective of shareholders.

2.1 Assumptions and Notations

We start by stating assumptions and notations that will be used throughout the article. Assume that the market is complete. In particular, for our two-date model, the state of nature at time 1 is denoted as ω, where $\omega \in \Omega$. Ω represents the set of all possible states of nature at time 1 that affect the final payoff of the corporation. Under the complete market assumption, for each state ω, there exists a basis stock, i.e., Arrow-Debreu security, with a unit payoff if state ω occurs at time 1 and zero otherwise. The time 0 price of the basis stock is denoted by $p(\omega)$. We assume that the total number of the states is finite.

Here, the corporation of interest is a stylized life insurance company that operates in the life/annuity business. The insurer's time-1 payoff depends on both financial market conditions and its own book of business payoffs. Therefore, with the complete market assumption, Ω contains all possible states of nature from both the financial and mortality perspectives. What is more, as the mortality rates are jointly determined by non-systematic longevity shocks (noises) and by systematic longevity shocks, both are considered and included in the set of states of nature Ω. For simplicity, we assume independence among the financial risk, non-systematic longevity risk, and systematic longevity risk.[6]

Assume that there are $k > 1$ possible outcomes with respect to the systematic longevity shock. Then the set of states Ω can be further divided into k subsets: $\Omega = \{\Omega^{(1)}, \Omega^{(2)}, \dots, \Omega^{(k)}\}$, with $\Omega^{(i)}$ containing all states of nature *conditional on* observing the i-th outcome from the systematic longevity shock. Furthermore, based on the independence assumption, we know that for each $\omega^{(i)} \in \Omega^{(i)}$, we can locate one and only one corresponding state $\omega^{(j)} \in \Omega^{(j)}$, $\forall j \neq i$, such that $\omega^{(i)}$ and $\omega^{(j)}$ only differ in the outcome of the systematic longevity shock, and are identical in both financial and non-systematic longevity shocks. Mathematically speaking, this is equivalent to the existence of a bijective function $f^{(i,j)}(\cdot)$ that maps $\Omega^{(i)}$ to $\Omega^{(j)}$, that is, $\omega^{(j)} = f^{(i,j)}(\omega^{(i)})$, $\forall i, j \in 1, \dots, k, i \neq j$.

Last, we formalize notations for the asset and liability of the life insurer. For each state ω, denote $\Gamma(\omega)$ as the time-1 accumulated value of the premium income from the book of business, $\Delta(\omega)$ as the time-1 accumulated value of assets held in reserve, and $A(\omega)$ as the time-1 insurance/annuity payoff to the policyholders. The total time-1 payoff to the insurer in state ω is therefore

$$\Pi(\omega) = \Gamma(\omega) + \Delta(\omega) - A(\omega).$$

[6] We note that this independent assumption is reasonable and generally used in the actuarial literature.

Denote the time-0 shareholder (stock) value of the insurer as S, the time-0 liability value of the insurer as L. The time-0 corporate value of the insurer, V, can then be expressed as $V = S + L$. In the following subsections, we investigate and contrast the shareholder and liability values under various scenarios.

2.2 Without Systematic Longevity Risk: The Baseline Case

Before we compare cash-flow hedging with value hedging instruments, we look at the baseline case for the life insurer in a simpler world where there is no systematic longevity risk, but only financial risk and non-systematic longevity risk. In such a world, the total number of states of nature at time 1 is reduced (by a factor of k), and we denote the reduced set of states as $\Omega^{(0)}$. Here, each state $\omega^{(0)} \in \Omega^{(0)}$ would represent a distinct combination of the financial risk and the non-systematic longevity risk. In this baseline case with the complete market assumption, again for each state $\omega^{(0)}$, there exists a basis stock with a unit payoff if state $\omega^{(0)}$ is realized at time 1, and zero otherwise. We denote the time 0 price of such basis stock as $p^0(\omega^{(0)})$.

To draw a direct connection between Ω and $\Omega^{(0)}$, and particularly, between $p(\omega)$ and $p^0(\omega^{(0)})$, we note that since $\Omega^{(0)}$ is the set of all possible states considering only the financial risk and the non-systematic longevity risk, there again exist bijective functions $f^{(0,i)}(\cdot)$, $i = 1, \ldots, k$ that map $\Omega^{(0)}$ to $\Omega^{(i)}$, $i = 1, \ldots, k$. In other words, for any realized state $\omega^{(0)} \in \Omega^{(0)}$, we can find one and only one corresponding state $\omega^{(i)} = f^{(0,i)}(\omega^{(0)}) \in \Omega^{(i)}$, $\forall i = 1, \ldots, k$, that has the same outcomes with respect to both financial and non-systematic risks. This respectively one-on-one mapping further helps us to express the basis stock price $p^0(\omega^{(0)})$ as a function of $p(\omega)$:

$$p^0(\omega^{(0)}) = \sum_{1}^{k} p(f^{(0,i)}(\omega^{(0)})), \quad \forall \omega^{(0)} \in \Omega^{(0)}. \tag{1}$$

The left hand side of Equation (1) is simply the basis stock price for state $\omega^{(0)}$ in the simpler world ($\Omega^{(0)}$) without considering systematic longevity risk. Now what if there exists systematic longevity shock and we are in the more complex world (Ω)? In that case, a unit payment that would be made should state $\omega^{(0)}$ occur, is equivalent to a unit payment that is either made should state $\omega^{(1)} = f^{(0,1)}(\omega^{(0)})$ occur, or should state $\omega^{(2)} = f^{(0,2)}(\omega^{(0)})$ occur, or so on so forth. Because the $\Omega^{(i)}$ are mutually exclusive of each other, one and only one state of $\omega^{(i)} = f^{(0,i)}(\omega^{(0)})$ would eventually occur, and the sum of all basis prices of $p(f^{(0,i)}(\omega^{(0)}))$, which is the right hand side of Equation (1), would have to be the same as $p^0(\omega^{(0)})$ to make the two economies comparable.

We further note that without explicitly considering systematic longevity risk, the model framework essentially reduces to the same one considered in MacMinn and Brockett (2017), so that we can directly borrow their key findings. Specifically, they show that without considering any positive NPV project as a result of hedging, corporate managements that act on behalf of current stock shareholders will always choose not to hedge at all to maximize the shareholder value, and describe this as a *moral hazard* problem. The key insight is that by choosing not to hedge, shareholders retain the implicit put option value arising from possible financial distress (bankruptcy), whereas hedging would effectively shift such value from the shareholders to the policyholders. In particular, when the life insurer chooses no hedging and so has stock market value S^u, we can express that unhedged shareholder value as[7]

$$\begin{aligned} S^u &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)}) \max\{0, \Pi(\omega^{(0)})\}) \\ &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)})\Delta(\omega^{(0)}) \end{aligned} \tag{2}$$

and the unhedged liability value as

$$\begin{aligned} L^u &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)}) \min\{\Gamma(\omega^{(0)}) + \Delta(\omega^{(0)}), A(\omega^{(0)})\} \\ &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)})\Gamma(\omega^{(0)}). \end{aligned} \tag{3}$$

The corporate value is therefore

$$V^u = S^u + L^u = \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)})(\Delta(\omega^{(0)}) + \Gamma(\omega^{(0)})). \tag{4}$$

As noted at the beginning of the section, we explicitly consider solvency requirement imposed on the life insurer by a regulator, as observed in most, if not all, developed countries. In particular, the insurer is required to maintain solvency, i.e., have non-negative payoff $\Pi(\omega)$ with a minimum probability of $1 - \alpha$, where α is the bankruptcy threshold chosen by the regulator.[8] To make the discussion and following comparison easier yet, we assume that the solvency requirement is automatically met based on the insurer's current reserve level in the simpler world.[9] That is, we assume that

$$P_{\Omega^{(0)}}(\Pi(\omega^{(0)}) < 0) = P_{\Omega^{(0)}}(\Gamma(\omega^{(0)}) + \Delta(\omega^{(0)}) < A(\omega^{(0)})) = \alpha. \tag{5}$$

While the addition of the solvency constraint does not change the (no) hedging decision of the life insurer in this baseline case, as shown in the follow subsection, such constraint will materialize

[7] We note that in MacMinn and Brockett (2017) all values are expressed in the form of integral as they assume a continuum state space of Ω.

[8] See MacMinn and Witt (1987) for such a constraint.

[9] We note this is an innocuous assumption since the life insurer can always adjust its current reserve value so that the solvency requirement is satisfied.

with the introduction of a systematic longevity shock and motivate the insurer to participate in some form of hedge.

2.3 With Systematic Longevity Risk

Now consider the case in which the life insurer faces systematic longevity risk. Equivalently, the insurer faces different payoffs in different states of nature ω realized from the enlarged set Ω. We begin by noting that, without considering the solvency constraint, again the life insurer would prefer no hedging so that the implicit put option value can be fully retained by shareholders. The no-hedging choice, however, will no longer grant an automatic pass of the solvency test in the presence of systematic longevity risk. Indeed, when there is some forms of longevity shock, e.g., when there is a profound medical advancement that considerably reshapes future mortality rates, it is reasonable to believe that this has a considerable impact on the life insurer's book of business, and in particular, *increases* the insurer's likelihood of bankruptcy. To make the discussion meaningful, we therefore assume that the solvency requirement is no longer met with the introduction of systematic risk should the insurer choose not to hedge, i.e.,

$$P_\Omega(\Pi(\omega) < 0) = P_\Omega(\Gamma(\omega) + \Delta(\omega) < A(\omega)) > \alpha. \tag{6}$$

In this case, the insurer has to select a hedging instrument that lowers its probability of bankruptcy. We assume that it can choose either a value hedging instrument, or a cash-flow hedging instrument, with associated payoff structures specified in following subsections.

To keep the discussion simple, we assume that $k = 2$, that is, there are only two possible outcomes with respect to the systematic longevity risk. The first case ($\Omega^{(1)}$) represents that in which there is no systematic longevity shock, with probability q. The second case ($\Omega^{(2)}$) represents that in which there is a longevity shock that systematically changes future population mortality rates with probability $1 - q$, and as a consequent impacts the insurer's insurance/annuity payoff to the policyholders. In particular, we assume that the insurer's payoff to the policyholder increases with the occurrence of such longevity shock. Further recall that we assume the systematic longevity risk is independent of the financial risk as well as the non-systematic risk.[10] Under the assumption that the insurer's investment returns of its premium income and reserve are independent of the systematic longevity risk, we have

$$\Gamma(\omega^{(1)}) = \Gamma(f^{(1,2)}(\omega^{(1)})),\ \forall \omega^{(1)} \in \Omega^{(1)}, \tag{7}$$

[10]If there is a state $\omega^{(0)} \in \Omega^{(0)}$ with a probability of p to occur, then with the additional consideration of systematic longevity risk, we would either observe state $f^{0,1}(\omega^{(0)}) \in \Omega^{(1)}$ with a probability $p \times q$, or state $f^{0,2}(\omega^{(0)}) \in \Omega^{(2)}$ with a probability $p \times (1 - q)$. In other words, state $\omega^{(0)}$ is the simpler case that contains both $f^{0,1}(\omega^{(0)})$ and $f^{0,2}(\omega^{(0)})$, so the probability for state $\omega^{(0)}$ is the *sum* of the two more refined states.

and

$$\Delta(\omega^{(1)}) = \Delta(f^{(1,2)}(\omega^{(1)})),\ \forall\omega^{(1)} \in \Omega^{(1)}. \tag{8}$$

Finally, for the payoff to the policyholder, we assume

$$A(\omega^{(1)}) + C = A(f^{(1,2)}(\omega^{(1)})),\ \forall\omega^{(1)} \in \Omega^{(1)}, \tag{9}$$

with C being the additional payoff as a result of the systematic longevity shock. In order to make fair and direct comparisons between cases with and without systematic longevity shock, we further link the premium, reserve, and liability values between $\Omega^{(0)}$ and Ω by assuming that

$$\Gamma(\omega^{(0)}) = \Gamma(f^{(0,1)}(\omega^{(0)})) = \Gamma(f^{(0,2)}(\omega^{(0)})),\ \forall\omega^{(0)} \in \Omega^{(0)}, \tag{10}$$

$$\Delta(\omega^{(0)}) = \Delta(f^{(0,1)}(\omega^{(0)})) = \Delta(f^{(0,2)}(\omega^{(0)})),\ \forall\omega^{(0)} \in \Omega^{(0)}, \tag{11}$$

and

$$A(\omega^{(0)}) = A(f^{(0,1)}(\omega^{(0)})) + (1-q)C = A(f^{(0,2)}(\omega^{(0)})) - qC,\ \forall\omega^{(0)} \in \Omega^{(0)}. \tag{12}$$

While (10) and (11) are straightforward to understand, Equation (12), in conjunction with (9), essentially means that

$$A(\omega^{(0)}) = qA(f^{(0,1)}(\omega^{(0)})) + (1-q)A(f^{(0,2)}(\omega^{(0)})), \tag{13}$$

that is, in the case without systematic longevity risk, the insurer's payoff to its policyholders in state $\omega^{(0)}$, $A(\omega^{(0)})$, can be seen as an expected value of the corresponding payoffs in the states $f^{(0,1)}(\omega^{(0)})$ and $f^{(0,2)}(\omega^{(0)})$ that have the same financial and non-systematic risk as to $\omega^{(0)}$. This ensures that the insurer has an overall equivalent liability exposure in either $\Omega^{(0)}$ or Ω.

In the following subsections, we investigate the shareholder, liability, and corporate values under the two different structures of longevity hedging.

Longevity Hedging with a Value Hedging Instrument

We begin by considering a value hedging instrument with payoffs dependent only on the occurrence of the systematic longevity risk and compensate the insurer in the event that the shock happens. In particular, consider a value hedging instrument that pays out $B(\omega^{(1)}) = -(1-q)\ C$, $\forall\,\omega^{(1)} \in \Omega^{(1)}$, and $B(\omega^{(2)}) = q\ C$, $\forall\,\omega^{(2)} \in \Omega^{(2)}$ to the insurer.[11] The expected payoff from such instrument can be calculated at $-q\ (1-q)\ C + (1-q)\ q\ C = 0$, which couples with the

[11] Such payment structure could be accomplished in the form of a one-period q-forward, but is not limited in such format. To make the discussion general, we simply assume the existence of such value hedging instrument without assigning a specific name.

independence financial/mortality assumption makes it a no-arbitrage instrument in the complete market.

With the additional payoff $B(\omega)$ from the longevity hedging instrument, the insurer's aggregate time-1 payoff in state ω becomes

$$\begin{aligned}\Pi(\omega) &= \Gamma(\omega)+\Delta(\omega)+B(\omega)-A(\omega)\\ &= \begin{cases}\Gamma(\omega^{(1)})+\Delta(\omega^{(1)})-(1-q)\times C-A(\omega^{(1)}) & \omega^{(1)}\in\Omega^{(1)}\\ \Gamma(\omega^{(2)})+\Delta(\omega^{(2)})+q\times C-A(\omega^{(2)}) & \omega^{(2)}\in\Omega^{(2)}\end{cases}.\end{aligned} \tag{14}$$

It should also be noted that Equations (10) and (11) may be used to demonstrate the relationship between the corporate payoff in the case of systematic risk to that of the case without as the following equation demonstrates:

$$\begin{aligned}\Pi(\omega^{(1)}) &= \Pi(f^{(0,1)}(\omega^{(0)}))\\ &= \Gamma(f^{(0,1)}(\omega^{(0)}))+\Delta(f^{(0,1)}(\omega^{(0)}))+B(f^{(0,1)}(\omega^{(0)}))-A(f^{(0,1)}(\omega^{(0)}))\\ &= \Gamma(\omega^{(0)})+\Delta((\omega^{(0)}))-(1-q)C-A(f^{(0,1)}(\omega))\\ &= \Gamma(\omega^{(0)})+\Delta((\omega^{(0)}))-(1-q)C-(A(\omega^{(0)})-(1-q)C)\\ &= \Pi(\omega^{(0)}).\end{aligned} \tag{15}$$

We first show that by using such a value hedging instrument, the insurer will again meet the solvency constraint. Here, the probability of bankruptcy can be expressed as

$$\begin{aligned}P_{\Omega}(\Pi(\omega)<0) &= P_{\Omega^{(1)}}(\Pi(\omega^{(1)})<0)\,q+P_{\Omega^{(2)}}(\Pi(\omega^{(2)})<0)\times(1-q)\\ &= P_{\Omega^{(1)}}\left(\Gamma(\omega^{(1)})+\Delta(\omega^{(1)})-(1-q)C-A(\omega^{(1)})<0\right)\times q\\ &\quad +P_{\Omega^{(2)}}\left(\Gamma(\omega^{(2)})+\Delta(\omega^{(2)})+qC-A(\omega^{(2)})<0\right)\times(1-q)\\ &= P_{\Omega^{(0)}}\left(\Gamma(\omega^{(0)})+\Delta(\omega^{(0)})-(1-q)C-(A(\omega^{(0)})-(1-q)C)<0\right)\times q\\ &\quad +P_{\Omega^{(0)}}\left(\Gamma(\omega^{(0)})+\Delta(\omega^{(0)})+qC-(A(\omega^{(0)})+qC)<0\right)\times(1-q)\\ &= P_{\Omega^{(0)}}\left(\Gamma(\omega^{(0)})+\Delta(\omega^{(0)})-A(\omega^{(0)})<0\right)\times q\\ &\quad +P_{\Omega^{(0)}}\left(\Gamma(\omega^{(0)})+\Delta(\omega^{(0)})-A(\omega^{(0)})<0\right)\times(1-q)\\ &= P_{\Omega^{(0)}}\left(\Gamma(\omega^{(0)})+\Delta(\omega^{(0)})-A(\omega^{(0)})<0\right)\\ &= \alpha.\end{aligned}$$

Under the assumption that the systematic longevity risk is independent of other risks, a value hedging instrument compensates the insurer in the event of the systematic longevity shock, by making a payment to the insurer that effectively offsets the adverse financial impact from the shock. In other words, using such a value hedging instrument helps the insurer immunize against the systematic shock and restore its payoff structure to the state space with no systematic shock. Therefore, the probability of insolvency is again right at the threshold level α. Using (15), the insurer's current shareholder value given systematic risk and value hedging can be compared to the unhedged value as follows:

$$\begin{aligned} S^v &= \sum_{\omega\in\Omega} p(\omega)\max\{0,\Pi(\omega)\} \\ &= \sum_{\omega^{(1)}\in\Omega^{(1)}} p(\omega^{(1)})\max\{0,\Pi(\omega^{(1)})\} + \sum_{\omega^{(2)}\in\Omega^{(2)}} p(\omega^{(2)})\max\{0,\Pi(\omega^{(2)})\} \\ &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p(f^{(0,1)}(\omega^{(0)}))\max\{0,\Pi(\omega^{(0)})\} + \sum_{\omega^{(0)}\in\Omega^{(0)}} p(f^{(0,2)}(\omega^{(0)}))\max\{0,\Pi(\omega^{(0)})\} \\ &= \sum_{\omega^{(0)}\in\Omega^{(0)}} (p(f^{(0,1)}(\omega^{(0)})) + p(f^{(0,2)}(\omega^{(0)})))\max\{0,\Pi(\omega^{(0)})\} \\ &= \sum_{\omega^{(0)}\in\Omega^{(0)}} p^{(0)}(\omega^{(0)})\max\{0,\Pi(\omega^{(0)})\} \\ &= S^u. \end{aligned} \tag{16}$$

This should not be surprising, since with the additional payoff received from the value hedge, the insurer would be indifferent between a no-shock state $\omega^{(1)} \in \Omega^{(1)}$ and the corresponding with-shock state $\omega^{(2)} = f^{(1,2)}(\omega^{(1)}) \in \Omega^{(2)}$. In particular, if the insurer would receive an overall post-hedging negative profit $\Pi(\omega^{(1)})$ from state $\omega^{(1)}$, then it would receive the same post-hedging negative profit in state $\omega^{(2)} = f^{(1,2)}(\omega^{(1)})$. This, together with the mapping between the two systems of basis prices as described in (1), ensures that the shareholder value with value hedging is the same as the shareholder value in the baseline case described in Section 2.2. Following the same argument, we can derive the liability value under value hedging to be the same as the liability value in the baseline case, $L^v = L^u$, and finally the equivalence of the corporate values, $V^v = V^u$.

Longevity Hedging with a Full Cash-flow Hedging Instrument

Next, we turn to a cash-flow hedging instrument. Generally, cash-flow instruments are longevity products with a series of payments that depend on realized future mortality experiences. The underlying demographic group that the rates are based on can be either a large-scaled population, or a company-specific insured cohort. Again, in this section we do not intend to distinguish based on the underlying group, nor define a cash-flow hedging instrument in a specific form. We note that the key difference between a value hedging instrument and a cash-flow hedging instrument is that the former provides hedging opportunities only to some systematic shock, whereas the latter typically makes mortality-related payments that are affected by or correlated to the non-systematic longevity risk as well as the systematic longevity risk. Hence, we expect to observe more variation in payments from a cash-flow hedging instrument.

Bearing this in mind, we define the payout structure from a cash-flow hedging instrument as $B(\omega) = A(\omega) - a$, $\forall \omega \in \Omega$. This may be described as the swap of an uncertain insurance payoff $A(\omega)$ with a certain payoff a. Particularly, this provides a *full* hedge to the insurer's future book of business. Suppose a is selected so that

$$\sum_{\omega \in \Omega} p(\omega) a = \sum_{\omega \in \Omega} p(\omega) A(\omega). \tag{17}$$

Follow the same assumption as in footnote 16 of MacMinn and Brockett (2017), we assume that a is less than the value of the premium income and other assets, otherwise the firm would not hedge with such a contract since it would result in the corporate payoff being negative.

The total time-1 payoff to the insurer under such full cash-flow hedge can then be represented as

$$\Pi(\omega) = \Gamma(\omega) + \Delta(\omega) + B(\omega) - A(\omega) = \Gamma(\omega) + \Delta(\omega) - a.$$

It immediately follows that given $\Gamma(\omega) + \Delta(\omega) > a$, $\forall \omega \in \Omega$, the probability of bankruptcy is reduced to zero in this case, that is, by choosing a full cash-flow hedge, the insurer can completely avoid an insolvency situation. We further derive the shareholder value of the firm with such cash-flow hedging as

$$\begin{aligned} S^c &= \sum_{\omega \in \Omega} p(\omega) \max\{0, \Pi(\omega)\} \\ &= \sum_{\omega \in \Omega} p(\omega) \left(\Gamma(\omega) + \Delta(\omega) - a\right) \\ &= \sum_{\omega \in \Omega} p(\omega) \left(\Gamma(\omega) + \Delta(\omega) - A(\omega)\right) \\ &= \sum_{\omega^{(0)} \in \Omega^{(0)}} p^{(0)}(\omega^{(0)}) \left(\Gamma(\omega^{(0)}) + \Delta(\omega^{(0)}) - A(\omega^{(0)})\right)) \\ &= S^u - \sum_{\omega^{(0)} \in \Omega^{(0)}} p^{(0)}(\omega^{(0)}) \max\{0, -\Pi(\omega^{(0)})\} \\ &= S^u - P^u, \end{aligned} \tag{18}$$

where again P^u is the implicit put option value created from the possibility of a bankruptcy. It should be noted here that both the systematic and non-systematic longevity risk is hedged with the cash-flow instrument. A hedge, cash-flow or value, must be selected to satisfy the regulatory constraint. The cash-flow hedge not only hedges the longevity risk but also transfers value from shareholders to policyholders and therefore hedges too much.[12] The following Theorem 2.1 compares the two longevity hedging instruments in their full-hedge form, from a corporate executive's perspective:

Theorem 2.1. *The corporate executive acting in the interests of shareholders will select a value hedge rather than a cash-flow hedge to satisfy the solvency constraint since $S^v - S^c = P^u > 0$.*

By comparing the shareholder values with value hedging (Equation (16)) and with cash-flow hedging (Equation (18)), we see that the value hedging instrument relieves the insolvency issue by offering compensation that is directly related to the systematic longevity shock, while at the same time allowing the shareholders to retain the implicit put option value (P^u). On the other hand, a full cash-flow hedge provides excessive protection by eliminating insolvency risk, and therefore redistributing the put option value from the shareholders to the policyholders. This *over hedge* of the cash-flow instrument therefore leaves shareholders with less value, which is undesirable from the corporate executive's perspective.

Longevity Hedging with a Partial Cash-flow Hedging Instrument

We continue our discussion by noting that Theorem 2.1 compares the two hedging instruments in their respectively full form. In particular, the cash-flow hedging instrument is designed so that it *completely* offsets any variations from the insurer's realized portfolio payoff, which effectively eliminates any possibility of bankruptcy. Given that the insurer is allowed to be insolvent with a positive probability α, one might, however, rightfully ask if such full cash-flow hedging is necessary. In other words, should it suffice for the insurer to undertake a *partial* cash flow hedging approach that meets the regulatory constraint, while at the same time allows the shareholder to retain some of the implicit put option value? In that case, will there still be any difference between the shareholder values under the two hedging instruments?

Consider now a partial cash-flow hedge with its payout to the insurer as

$$B(\omega) = k \times (A(\omega) - a),$$

in which $k \in [0, 1]$ indicates the partial hedging fraction. The final time-1 payoff to the insurer can then be expressed as

$$\begin{aligned} \Pi_k(\omega) &= \Gamma(\omega) + \Delta(\omega) - A(\omega) + k \times (A(\omega) - a) \\ &= \Gamma(\omega) + \Delta(\omega) - (1 - k) \times A(\omega) - k \times a. \end{aligned}$$

[12] We can, of course, also show that $L^c = L^u + P^u$.

When $k = 0$, that is, the insurer takes no hedging, we know from Equation (6) that $P_\Omega(\Pi_0(\omega) < 0) > \alpha$. On the other hand, with $k = 1$ indicating a full cash-flow hedging, we have $P_\Omega(\Pi_1(\omega) < 0) = 0$ as discussed in the previous subsection. Assuming continuity, there exists $k^* \in (0, 1)$ such that $P_\Omega(\Pi_{k^*}(\omega) < 0) = \alpha$. With such optimal partial hedging fraction k^*, the shareholder value can then be further expressed as:

$$\begin{aligned}
S^c_{k^*} &= \sum_{\omega\in\Omega} p(\omega)\max\{0, \Pi(\omega)\} \\
&= \sum_{\omega^{(1)}\in\Omega^{(1)}} p(\omega^{(1)})\max\{0, \Pi(\omega^{(1)})\} + \sum_{\omega^{(2)}\in\Omega^{(2)}} p(\omega^{(2)})\max\{0, \Pi(\omega^{(2)})\} \\
&= \sum_{\omega^{(0)}\in\Omega^{(0)}} p(f^{(0,1)}(\omega^{(0)}))\max\{0, \Pi(\omega^{(0)}) + (1-k^*)(1-q)C + k^*(A(\omega^{(0)}) - a)\} \\
&\quad + \sum_{\omega^{(0)}\in\Omega^{(0)}} p(f^{(0,2)}(\omega^{(0)}))\max\{0, \Pi(\omega^{(0)}) - (1-k^*)qC + k^*(A(\omega^{(0)}) - a)\}. \qquad (19)
\end{aligned}$$

Consider an insurer that chooses between a full value hedge and a partial (k^*) cash-flow hedge. With both hedges achieving the same bankruptcy probability (α), the insurer would be *indifferent* from the regulatory perspective. However, with different expressions of the shareholder values (S^v and $S^c_{k^*}$) as contrasted in Equations (16) and (19) respectively, the insurer would prefer the one with a higher shareholder value. While it is analytically difficult to make comparisons of S^v and $S^c_{k^*}$ without relying on additional assumptions, we note that the two hedges work conceptually differently in terms of which part of longevity risk is hedged. Particularly, although both hedges guarantee the same bankruptcy probability, this is done in the value hedge by eliminating only systematic longevity risk, whereas the partial cash flow hedge take an alternate route by targeting a mix of systematic and non-systematic longevity risk. In the next section, we run numerical experiments to show such difference in the hedging mechanism does result in variations in the shareholder values.

3 Numerical Experiments

3.1 Specification

In this section, we numerically implement the model framework as studied in the previous section. Here, the financial market risk is represented by a non-dividend paying market fund that evolves based on a Geometric Brownian motion. We consider 120 representative discrete states for the fund values at time 1. The basis stock price for each of the 120 *financial states* can then be solved based on the difference between various put option values with different strike prices under the Black-Scholes framework, following the strand of literature originating from Breeden and Litzenberger

(1978).

We consider a hypothetical insurer with a portfolio composed of both life insurance and annuity products. The non-systematic longevity shock is modeled by a gamma distribution on the insurer's time-1 payoff to the policyholders. We assume an aggregate nominal expected payoff at $\$2,000$ and a dispersion parameter for the gamma distribution at $\theta = 0.1$.[13] Similarly, we discretize the gamma distribution to consider only 39 representative states of the time-1 realized portfolio payouts. Combining both financial and non-systematic longevity risk, for the hypothetical life insurer, the reduced set of state space, $\Omega^{(0)}$, would include $120 \times 39 = 4680$ discrete states. For each of the state $\omega^{(0)}$, we further obtain the related basis stock price $p^0(\omega^{(0)})$, which—given the independency between financial and mortality risk—is simply the product of the financial-state basis stock price and the probability of the realized portfolio payouts.

The systematic longevity risk is further introduced as an adverse random event that could occur within the time period with probability $q = 0.5$. When it happens, the insurer suffers an additional loss $C = \$1,000$ to its time-1 portfolio payout. The total set of state space Ω would then contain $120 \times 39 \times 2 = 9360$ discrete states, and once again for each state ω, we obtain the associated basis stock price $p(\omega)$ that will help us obtain various time-0 values.

With regard to the insurer's asset side, we begin by noting that the total premium collected at time 0 should not simply be the "actuarially fair" one calculated as $\$2,000 + 0.5 \times \$1,000 = \$2,500$. Due to the bankruptcy risk and particularly the possibility that policyholders might not get paid in full, they will rationally demand to pay a fair premium that is *lower* than $\$2,500$, which reflects the loss of value in the case of the insurer's insolvency. Here, we note that there are two alternative yet equivalent expressions for the insurer's liability value without considering systematic longevity risk (see MacMinn and Brockett (2017) for similar continuum-state expressions):

$$L^u = \sum_{\omega^{(0)} \in \Omega^{(0)}} \min(\Gamma + \Delta, A) \times p^{(0)}(\omega^{(0)}),$$

and

$$L^u = \sum_{\omega^{(0)} \in \Omega^{(0)}} \Gamma \times p^{(0)}(\omega^{(0)}).$$

Therefore, the fair (and unique) premium Γ_0 can be solved by satisfying both representations. In particular, Γ_0 depends on the time-0 reserve amount Δ_0 that is set exogenously. Naturally, the larger the reserve, the smaller the bankruptcy probability, and the larger the premium amount that the policyholders are willing to pay. We assume that the total premium Γ_0 will be fully invested in the market fund, whereas the reserve Δ_0 is deposited in a risk-free account. Table 1 summarizes

[13] The dispersion parameter of a gamma distribution controls the volatility of the realized payoff. A higher θ would imply a more volatile payoff to the policyholders.

the parametric specifications for our numerical experiments.

Table 1: Numerical Experiment Parameter Values

Description	Parameter	Value
Interest rate	r	0.04
Market fund volatility	σ	0.05
Sys. mortality shock additional payout	C	$\$1,000$
Gamma distribution dispersion parameter	θ	0.05
Solvency requirement	α	0.05

3.2 Implementation

Based on the parameter values in Table 1, we first obtain the necessary reserve amount and the fair premium when assuming there is no systematic mortality risk. The reserve amount is solved at $\Delta_0 = \$714.1$ with $\alpha = 5\%$. That is, with this reserve the insurer automatically passes the solvency requirement without having to undertake any hedges. The fair premium is further solved at $\Gamma_0 = \$2,385.2$ $(< \$2,500)$. In other words, without systematic longevity risk and any hedges, the shareholder value $S^u = \Delta_0 = \$714.1$, and the liability (policyholder) value $L^u = \Gamma_0 = \$2,385.2$.

With systematic longevity shock, however, the insurer's future portfolio payout becomes more volatile. This implies a higher implicit put option value retained by the shareholder, at the cost of the policyholder. In particular, with the reserve and premium values remain unchanged and without any hedging instrument, the new shareholder value can be calculated at $S^s = \$745.5$, whereas the new policyholder value is reduced to $L^s = \$2,353.8$. While this seems as a good news to the insurer at a first glance, it also faces an increased bankruptcy probability at 12.2%, which considerably exceeds the regulatory requirement.

To once again meet the regulatory solvency requirement, the insurer chooses between a value hedging instrument and a partial cash-flow hedging instrument. As shown in the previous section, the value hedge would completely eliminate the additional volatility in the insurer's portfolio payout created by the systematic longevity shock, so we immediately have $S^v = S^u = \$714.1$, and $L^v = L^u = \$2,385.2$. For the partial cash-flow hedging, we first solve the optimal hedging fraction $k^* = 29.6\%$ as the one that reduces the insurer's bankruptcy probability from 12.2% back to exactly 5%. Under this specific partial cash-flow hedging, the shareholder and liability values can be further obtained at $S^c = \$710.8$ and $L^c = \$2,388.5$. Comparing the two shareholder values S^v and S^c, we find that here the insurer would again prefer a value hedge over a partial cash-flow hedge. We argue this is due to the underlying difference in the hedging mechanism: the value hedge works only on the systematic risk, whereas the cash-flow hedge works on both systematic and non-systematic risks simultaneously and indifferently. While both deliver the same bankruptcy probability, such difference in risk targeting will materialize on the shareholder value, particularly affecting the implicit put option value embedded in it. More specifically, in our experiment the partial cash-flow hedging seems to *over-smooth* the insurer's future portfolio payoff, which results in a *reduced* put option value as compared with the value hedge, and should be therefore less preferred by the insurer.

We perform a simple sensitivity test by varying the bankruptcy probability α, and the results are displayed in Table 2. From the table, we observe that the difference of the shareholder values between the value hedge and partial cash-flow hedge generally exists across different values of α, with the value hedge always dominates. However, such difference reduces as the initial reserve value increases, or equivalently, as the solvency requirement tightens.

Table 2: Comparison of Shareholder and Liability Values

α	$\Delta(=S^u=S^v)$	k^*	S^c
1%	$\$1,085.0$	22.6%	$\$1,084.7$
5%	$\$714.1$	29.6%	$\$710.8$
10%	$\$540.2$	31.9%	$\$535.3$

4 Conclusion

In this article, we connect a hypothetical life insurer's shareholder value with the structure of longevity hedging instrument that it chooses. Using a simple complete financial market model, we demonstrate that the structure of the instrument does matter in the context of corporate management decisions, when a regulatory solvency requirement is explicitly imposed. In particular, we are able to show that a (full) value hedging instrument will always dominate a full cash-flow hedging instrument by generating a higher stock value to the company's shareholders, and should therefore be preferred by corporate executives acting in the interests of shareholders. Our numerical experiments further suggest that the value hedge over-performs a partial cash-flow hedge under chosen parametric specifications.

Our findings may help explain the popularity (or unpopularity) of certain longevity-linked securities offered in the new life market from the perspective of the demand side. More importantly, it sheds light on the potential innovations of new longevity products by suggesting structures that better align corporate interests and risk appetites.

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