A MODEL OF TECHNOLOGICAL UNEMPLOYMENT

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Abstract

The economic literature that explores the consequences of technological change on the labour market tends to support an optimistic view about the threat of automation. In the more recent ‘task-based’ literature, this optimism has typically relied on the existence of firm limits to the capabilities of new systems and machines. Yet these limits have often turned out to be misplaced. In this paper, rather than try to identify new limits, I build a new model based on a very simple claim – that new technologies will be able to perform more types of tasks in the future. I call this ‘task encroachment’. To explore this process, I use a new distinction between two types of capital – ‘complementing’ capital, or ‘c-capital’, and ‘substituting’ capital, or ‘s-capital’. As the quantity and productivity of s-capital increases, it erodes the set of tasks in which labour is complemented by c-capital. In a static version of the model, this process drives down relative wages and the labour share of income. In a dynamic model, as s-capital is accumulated, labour is driven out the economy and wages decline to zero. In the limit, labour is fully immiserated and ‘technological unemployment’ follows. This pessimistic result helps to identify five new channels for optimism that require further research. (JEL: J20, J21, J23)

The economic literature that explores the consequences of technological change on the labour market tends to support an optimistic view about the threat of

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automation. The reasons for optimism, though, have changed over time. In the
traditional literature, the dominant framework is the “canonical” or “textbook”
model (Acemoglu and Autor 2011; Atkinson 2008). Here, the labour of two distinct
skill groups is typically combined in a constant elasticity of substitution production
function to produce output. In this model, by construction, it is assumed that
technology complements all types of worker. In this framework, it is not possible
for any workers to be made worse-off by technological change.

In the new ‘task-based’ literature, the reason for optimism is different. In these
models, technology no longer directly complements all types of workers, but in- stead only complements particular types of workers who are able to perform tasks
that cannot be automated. In turn, technology substitutes for those workers who
perform tasks that can be automated. Now it is possible that certain workers
are made worse-off by technological change. In this framework, optimism instead
relies upon the more nuanced claim that there exists a large set of types of tasks
that cannot be automated. In short, while the canonical model supports an opti-
mistic view about the threat of automation by making the strong assumption that
technology cannot substitute for labour, task-based models tend to support an opti-
mistic view by making the weaker assumption that “the scope for ... substitution
is bounded” (Autor 2015).

The problem, however, is that this weaker assumption is far less compelling
than it used to be. The scope for substitution has proven not to be bounded
in the way that much of the task-based literature expected. Though accurately
forecasting the capabilities of systems and machines is very difficult, this literature
has consistently underestimated them. For instance, Autor, Levy, and Murnane
(2003) noted that the “driving a truck” could not be readily automated, but a type
of driverless vehicle appeared two years later; the same paper argued that “legal
writing” and “medical diagnosis” could not be reading automated, yet document

\[ Y = \left( (A^L L^L)^{\frac{1}{\sigma}} + (A^H L^H)^{\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \]

By construction \( \frac{\partial w^L}{\partial A^H} > 0, \frac{\partial w^L}{\partial A^L} > 0 \) – and similarly for \( w^H \).

1I refer to ‘optimism’ and ‘pessimism’ throughout this paper. I do so as a form of short- hand. Put simply, ‘pessimistic outcomes’ are those that make people feel pessimistic about labour’s future and vice-versa for ‘optimistic outcomes’. More formally, in this paper ‘pessimism’ is captured by a fall in relative, and absolute, wages.

2For instance, in a canonical model with low-skill labour \( L^L \) and high-skill labour \( L^H \), the production function is CES:

3The Society of Automotive Engineers defines five levels of vehicle ‘autonomy’. These early cars were at a low level. Since 2005, further progress has been made. I thank Frank Levy for this point.
automation systems are now widespread in most major legal practices and there are a variety of technologies that can diagnose health problems; Autor and Dorn (2013) noted that order-taking and table-waiting could not be readily automated, but later that year the US restaurants Chili’s and Applebee’s announced they were installing 100,000 tablets to allow customers to order and pay without a human waiter; Autor (2015) noted that the task of identifying a species of bird based on a fleeting glimpse could not be readily automated, but later that year an app was released to do that as well. The set of tasks in which human beings have the comparative advantage over machines is smaller than was commonly supposed.

Each of these papers relied on a particular understanding of how systems and machines operate and the limits to their capabilities that this implied – known as the ‘ALM hypothesis’. Under this hypothesis, these tasks were believed to be out of reach of automation because they were ‘non-routine’ rather than ‘routine’. But, as before, this has proven not to be the case. In Susskind (2017) I explore the nature of recent technological change and explain why the ALM hypothesis no longer holds. In turn, a new literature has emerged that attempts to revise and update the limits to system and machine capabilities. Some areas of this research focus on human faculties that have proven difficult to automate (for instance, Deming 2017 on social skills). Others areas focus on particular features of tasks that make them hard to automate (for instance, Brynjolfsson and Mitchell 2017 on machine learning). Yet the difficulty with trying to identify new limits at all is that, in time, as with the ALM hypothesis, they are likely to fall away as well.

In this paper, I take a different approach. Rather than try and identify new limits, I start instead from a very simple claim – though we cannot know exactly what systems and machines will be able to do in the future, we can be confident that they will be able to do more types of tasks than they can today. I call this ‘task encroachment’. And in what follows, I build a new model to explore the consequences of this process for the labour market, where the set of types of tasks in which labour has the comparative advantage over systems and machines is gradually, but relentlessly, eroded away.

The new model in this paper is based an important conceptual insight – that technological change directly complements tasks not labour. In this paper, I use the term ‘capital’ to refer to the wide variety systems and machines used in the economy. I introduce a new distinction between two types of capital – ‘substitut-

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4 See Pudzer (2016).
5 See http://merlin.allaboutbirds.org/photo-id/.
ing’ capital (‘s-capital’) and ‘complementing’ capital (‘s-capital’). Here, c-capital is a q-complement to a distinct set of types of tasks, and an increase in its quantity or productivity raises the value of those complemented tasks. However, these complemented tasks are not necessarily performed by labour – they are performed either by labour or by s-capital. This implies that labour benefits indirectly from technological change, only if it remains best-placed to perform those complemented tasks. And in this model, an increase in the quantity or productivity of s-capital erodes that comparative advantage of labour in performing these complemented tasks. This is task encroachment at work.

This distinction between two different types of capital, and their distinctive effects on labour, is new and important. The model in this paper is closely related to a set of existing task-based models (for instance, Zeira 1998, Acemoglu and Autor 2011, Hémos and Olsen 2016, Acemoglu and Restrepo 2017; 2018). Yet because of the structure of production in these other models, and their use of only one type of capital, these two effects are entangled. The new model in this paper disentangles them in a revealing way. Each type of capital has only one effect and, as a result, it is possible to explore what happens if the set of tasks in which labour is q-complemented by c-capital shrinks – holding constant the ‘intensity’ of that q-complementarity.

The new model in this paper, and the process of task encroachment that it captures, supports a far more pessimistic view about the threat of automation. As s-capital becomes more productive, labour is forced to specialise in a shrinking set of types of complemented tasks. In a static version of the model, an increase in the quantity or productivity of s-capital drives down relative wages and the labour share of income and forces labour to specialise in a shrinking set of tasks. In a dynamic version, the endogenous accumulation of s-capital drives labour out the economy at an endogenously determined rate, and absolute wages fall towards zero. In the limit, labour is fully immiserated and ‘technological unemployment’ follows. As Autor and Salomons (2017a,b) put it, in their discussion of this new model, labour has “no place left to hide”.

The central argument for optimism in the task-based literature is that people tend to “overstate the extent of machine substitution for labour” and “ignore.

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7 The definition of ‘q-complementarity’ is more nuanced in a many-good setting, as in this paper. This is because in the set-up in which q-complementarity was originally defined – Hicks (1970), and Sato and Koizumi (1973) – the models had only a unique final good. When I use the term ‘q-complement’ I mean that an increase in the allocation of a factor’s task input to the production of a given good, ceteris paribus, causes the marginal product of the other task input involved in the production of that good to increase.
the strong complementarities” (Autor 2015). This paper does not challenge the existence of strong complementarities between certain types of tasks performed by capital and other types of tasks. But it does challenge the idea that labour is uniquely placed to perform those complemented tasks – on the contrary, labour’s comparative advantage over capital in performing those complemented tasks appears to be diminishing. The new model shows that mistaken limits to the capabilities of systems and machines may have created a misleading sense of optimism about the prospects for labour. The new process of ‘task encroachment’ in this paper is a new and important argument for pessimism. Nevertheless, this pessimistic result relies upon a set of restrictive modelling assumptions. In the final part of this paper, I interrogate these assumptions, and from them draw five new reasons for optimism that require further research.

1. A Static Model

In the new model that follows there are two sets of types of tasks. The first is a set of tasks that are performed only by ‘complementing capital’ ('c-capital'). This type of capital cannot perform the same type of tasks as labour. The second is a set of tasks that are either performed by labour or ‘substituting capital’ ('s-capital'). These tasks that are performed by either labour or s-capital are ordered in a line from left to right, going from ‘simple’ to ‘complex’, and the relative productivity of labour with respect to s-capital increases as tasks become more ‘complex’. This feature is similar to that in Acemoglu and Autor (2011). To produce any good in the economy requires a combination of a task performed by c-capital and a task performed by either labour or s-capital.

The purpose of this new model is to study the effect of technological progress in the use of the different types of capital. In equilibrium there is a single cut-off and all of the tasks to the left of the cut-off are performed by s-capital with c-capital and all of the tasks to the right are performed by labour with c-capital. When capital is ‘complementing’, with a fixed and distinct role from labour in production, improvements in its capability have a neutral effect on labour. Relative wages are unaffected. This is optimism at work. But when capital is ‘substituting’, increasingly capable s-capital erodes the set of types of tasks in which c-capital

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8 And in turn it shares features with Dornbusch, Fischer, and Samuelson (1977), from which Acemoglu and Autor (2011) also draw. Dornbusch, Fischer, and Samuelson (1977) is a two-country trade model where a continuum of goods are traded and the result is a Ricardian pattern of specialisation.
q-complements labour. The relative return to labour falls and labour is forced to specialise in a shrinking set of types of tasks. This is the new pessimism at work.

1.1. Consumers

There is a spectrum of consumers \( j \in [0, 1] \) who are either high-skilled workers or capitalists. If consumer \( j \) is a high-skilled worker he sells his labour \( l^H_j \) for a wage \( w^H \geq 0 \). If he is a capitalist with s-capital \( k^S_j \) or c-capital \( k^C_j \) he rents them to earn \( r^S \geq 0 \) and \( r^C \geq 0 \) respectively. There is a spectrum of types of goods \( x(i) \) where \( i \in [0, 1] \) and each consumer \( j \) has Cobb-Douglas preferences over those goods:

\[
\ln u(x) = \int_0^1 \theta(i) \ln x_j(i) \, di
\]

Note that, given the Cobb-Douglas utility function, this economy can be captured by a representative consumer who owns all the factors. For simplicity, I assume that all goods have the same expenditure density:

**ASSUMPTION 1:** \( \theta(i) = 1 \forall i \).

1.2. Production and Firms

Goods are produced by combining two different types of tasks, \( z_1(i) \) and \( z_2(i) \), where again \( i \in [0, 1] \). The first set of types of task, \( z_1(i) \), are those that can performed by labour and s-capital. The second set of types of tasks, \( z_2(i) \), can only be performed by c-capital. Perfectly competitive firms must hire factors to perform these tasks. The total stock of available factors is equal to the sum of \( l^H_j \), \( k^S_j \), \( k^C_j \), owned by the consumers – \( L^H \), \( K^S \), and \( K^C \) respectively. The ‘task-based production functions’ for the goods are:

\[
x(i) = z_1(i)^\psi z_2(i)^{1-\psi}
\]

where \( \psi \in (0, 1) \). The ‘factor-based production functions’ for the tasks are:

\[
\begin{align*}
z_1(i) &= a^S(i)K^S(i) + a^H(i)L^H(i) \\
z_2(i) &= a^C(i)K^C(i)
\end{align*}
\]
where $L^H(i)$, $K^S(i)$, and $K^C(i)$ are the allocations of high-skilled labour, s-capital, and c-capital to each type of task, and $a^H(i)$, $a^S(i)$ and $a^C(i)$ are their respective productivities. Again, these factor-based production functions for tasks reflect the fact that c-capital performs its own distinct set of tasks, but s-capital and labour perform the same tasks. The productivities of s-capital and labour combine to form a ‘relative productivity schedule’ over the $z_1(i)$ task-spectrum:

\[
A(i) = \frac{a^H(i)}{a^S(i)}
\]

A second important assumption follows:

**ASSUMPTION 2:** $A(i)$ is continuous, $A(0) > 0$, $A'(i) > 0$, and $A''(i) = 0$.

The assumption that $A'(i) > 0$ is a ‘comparative advantage’ assumption. It reflects two principles. First, as $i$ increases, the task that is performed by labour or s-capital, $z_1(i)$, becomes more ‘complex’. And secondly, that high-skilled labour has an increasing comparative advantage over s-capital at performing more complex tasks. This is the sense in which labour is ‘high’ skilled. This reflects the fact that the most complex tasks draw on creative, problem-solving, and interpersonal faculties of human beings that, as yet, are hardest to automate. Following Susskind (2017), the most complex tasks are relatively hard to routinise.

### 1.3. Equilibrium

**The Supply-Side**

The firms must decide which factors to hire to perform the tasks that will produce each type of good. It is clear that to perform the tasks $z_2(i)$ the firms will need to rent c-capital, $K^C(i)$ – it is the only factor capable of performing those types of task. Less obviously, the firms will hire either labour or s-capital to carry out the tasks $z_1(i)$ – but never both factors together. This is Lemma 1:

**LEMMA 1:** In equilibrium, there exists some cut-off $\tilde{i}$ such that s-capital works with c-capital to produce goods of type $i \in [0, \tilde{i}]$, and labour works with c-capital to produce the goods of type $i \in [\tilde{i}, 1]$.

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*Acemoglu and Autor (2011) also use this approach.*
The proof is intuitive. For any given $w^H$ and $r^S$ in equilibrium, a perfectly competitive firm will hire labour rather than s-capital to perform $z_1(i)$ if:

\[
\frac{w^H}{a^H(i)} \leq \frac{r^S}{a^S(i)}
\]

\[
A(i) \geq \frac{w^H}{r^S}
\]

which takes place to the right of $\tilde{i}$, given the properties of $A(i)$ in Assumption 1. The opposite argument applies for the choice of s-capital over labour. At $\tilde{i}$, the cost of performing a unit of $z_1(\tilde{i})$ to produce that good $\tilde{i}$ is the same for labour and s-capital. This is shown on the left-hand side of Figure 1.

For any $\frac{w^H}{r^S} \in [a, b]$ there is a single ‘cut-off’ type of good $\tilde{i}$. Given that factor-price ratio, firms would want to hire s-capital (to work with c-capital) to produce all goods to the left of $\tilde{i}$ and hire labour (to work with c-capital) to produce all goods to the right of $\tilde{i}$. This is intuitive. $A(i)$ is a factor demand schedule, its shape determined by Assumption 1, and it describes the factor-price ratio for each type of good $i$ that would make a firm indifferent between using either labour or s-capital for a given $z_1(i)$. Moving left to right along the task-spectrum, if a firm is to remain indifferent, the relative price of labour must rise. This is because the relative advantage of labour over s-capital at performing $z_1(i)$ increases.

Given this reasoning and Lemma 1, (2) and (3) combine to form the following factor-based production functions for goods:

\[
x(i) = \left[a^S(i)K^S(i)\right]^{\psi} \left[a^C(i)K^C(i)\right]^{1-\psi} \forall i \in [0, \tilde{i}]
\]

\[
x(i) = \left[a^H(i)L^H(i)\right]^{\psi} \left[a^C(i)K^C(i)\right]^{1-\psi} \forall i \in [\tilde{i}, 1]
\]

\textbf{The Demand-Side}

Call $\gamma(\tilde{i})$ the share of total consumer expenditure on all goods that are produced by s-capital and c-capital i.e. type $i \in [0, \tilde{i}]$. Assumption 1 implies that:

\footnote{Lemma 1 and the proof are similar to Lemma 1 in Acemoglu and Autor (2011).}
\(\gamma(\tilde{i}) = \int_0^{\tilde{i}} \theta(i) \, di = \tilde{i}\) 

In turn, it follows from (2) that the share of total consumer expenditure that is spent on all the tasks performed by s-capital is equal to \(\psi \cdot \tilde{i}\). The same argument applies to the share of total consumer expenditure on tasks performed by labour and c-capital – these are equal to \(\psi \cdot (1 - \tilde{i})\) and \((1 - \psi)\) respectively.

**Equilibrium Factor Prices and Specialisation**

As firms are perfectly competitive, the zero-profit condition requires that the total consumer expenditure on the types of tasks that are performed by each factor is equal to the income that the factor receives in return for carrying out those tasks. As a result, given (7) and the accompanying discussion, the following three conditions must hold:

\begin{align*}
r^S K^S &= \psi \cdot \tilde{i} \cdot Y \\
 w^H L^H &= \psi \cdot (1 - \tilde{i}) \cdot Y \\
 r^C K^C &= (1 - \psi) \cdot Y
\end{align*}

where \(Y = r^S K^S + w^H L^H + r^C K^C\). The first two expressions in (8) implies:

\begin{align*}
\frac{w^H}{r^S} &= \frac{1 - \tilde{i}}{\tilde{i}} \cdot \frac{K^S}{L^H} \\
&= B(\tilde{i})
\end{align*}

\(B(i)\) generates a market equilibrium schedule. This is shown in Figure 2.

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This is because the task-based production function for goods is Cobb-Douglas. To see this formally, call the implicit ‘price’ of \(z_1(i)\), \(p_{z_1}(i)\). Perfectly competitive firms will set this ‘price’ equal to the marginal revenue product of \(z_1(i)\) in producing \(x(i)\), which is \(p(i) \cdot \psi z_1(i)^{\psi - 1} z_2(i)^{1 - \psi}\). As a result the \(z_1(i)\) share of the expenditure on a particular \(x(i)\) is \(\frac{p(i) \cdot \psi z_1(i)^{\psi - 1} z_2(i)^{1 - \psi} \cdot z_1(i)}{p(i) \cdot x(i)} = \psi\).
From inspection, it is clear that for any $\frac{w^H_i}{r^S}$ there is a unique cut-off good of type $\tilde{i}$ that ensures market equilibrium holds. (9) and Assumption 1 imply: $B'(i) < 0$; $B''(i) > 0$; $\lim_{i \to 0} B(i) = \infty$, and $B(1) = 0$.

The $A(i)$ schedule describes the pattern of specialisation $\tilde{i}$ for each factor-price ratio $\frac{w^H_i}{r^S}$. This is a factor-demand schedule. $B(i)$ describes the wage ratio $\frac{w^H_i}{r^S}$ that ensures market equilibrium for each pattern of specialisation $\tilde{i}$. This is a zero profit schedule. To derive the equilibrium cut-off $\bar{i}$, rather than a hypothetical cut-off $\tilde{i}$, these two schedules are combined. This is shown in Figure 3 and Proposition 1 follows.

**PROPOSITION 1:** Given the properties of $A(i)$ and $B(i)$, there is a unique equilibrium wage ratio, $f$, and a unique equilibrium cut-off type of good, $\bar{i}$.

The uniqueness of the equilibrium can be seen from an inspection of Figure 3.

Since $A(0) > 0$ and $B(1) = 1$, the $A(i)$ schedule must start above the $B(i)$ schedule. As $A'(i) > 0$, $B'(i) < 0$, and $\lim_{i \to 0} B(i) = \infty$ the schedules must cross only once. In equilibrium there is a clear pattern of specialisation. It is ‘Ricardian’.\(^{12}\) $s$-capital specialises in producing the types of goods $i \in [0, \bar{i}]$ with $c$-capital, and labour specialises in producing the goods $i \in [\bar{i}, 1]$ – i.e. $\tilde{i}$ is replaced by $\bar{i}$ in (6).

The full set of relative factor prices follow from (8) and (9) – again, recognising that in equilibrium the actual cut-off $\bar{i}$ replaces the hypothetical cut-off $\tilde{i}$:

\[
\begin{align*}
\frac{w^H}{r^S} &= \frac{1 - \bar{i}}{\tilde{i}} \cdot \frac{K^S}{L^H} \\
\frac{w^H}{r^C} &= \frac{\psi \cdot (1 - \bar{i})}{1 - \psi} \cdot \frac{K^C}{L^H} \\
\frac{r^S}{r^C} &= \frac{\psi \cdot \bar{i}}{1 - \psi} \cdot \frac{K^C}{K^S}
\end{align*}
\]

\(^{12}\)Equilibrium here is similar to Dornbusch, Fischer, and Samuelson (1977), though theirs is an equilibrium in international trade. Acemoglu and Autor (2011) note that their model is ‘isomorphic’ to Dornbusch, Fischer, and Samuelson (1977).
1.4. Comparative Statics

This model can be used to compare the effect of technological progress in the use of the two different types of capital. First consider c-capital, and a rise in $c^C(i)$ across the $z_2(i)$ task-spectrum. This has no effect on the wage relative to s-capital, $w^H_{iS}$, nor on the pattern of labour specialisation, $i$. This follows the definition of the $A(i)$ and $B(i)$ schedules and (10). Figure 3 remains unchanged. Nor does it affect the relative wage with respect to c-capital, $w^H_{iC}$. However, s-capital does have an effect on both relative wages. A uniform rise in the relative productivity of s-capital is shown in Figure 4.

-- FIGURE 4 HERE --

The result is a fall in $w^H_{iS}$ and $w^H_{iC}$ and a rise in $\bar{i}$ – labour is forced to specialise in a shrinking set of types of tasks, and is relatively worse off.\(^\text{13}\)

1.5. Review of the Static Model

The two types of capital have very different consequences for labour. As c-capital becomes more productive, the relative wages $w^H_{iS}$ and $w^H_{iC}$ do not change. This is because c-capital is a q-complement to labour and s-capital in the production of all types of goods and, because the task-based production functions for goods are Cobb-Douglas, this implies a rise in the marginal productivity of c-capital in producing a given good causes an equiproportionate rise in the marginal productivity of either the labour or s-capital producing that good.\(^\text{14}\) This is the traditional channel of optimism – there exists a large set of types of tasks out of reach of automation, in which labour is complemented by capital. However, as s-capital becomes more productive, $w^H_{iS}$ and $w^H_{iC}$ fall and labour is instead relatively worse off. This is because s-capital is a perfect substitute for labour in performing those tasks that are q-complemented by c-capital. Labour’s comparative advantage diminishes, and is forced to specialise in a shrinking set of q-complemented tasks. This is the new pessimism at work.

\(^{13}\)Exploring absolute, rather than relative, prices is more complex are requires the simulations that follow in the dynamic setting.

\(^{14}\)Note again in this many-good setting the traditional definition of ‘q-complementarity’ does not apply straightforwardly. This is because in those settings there is only a unique final good.
2. A Dynamic Model

In the static model, as s-capital becomes more productive, consumers do not respond to any changes in the rate of return to capital. Now I place that new model in a dynamic setting and introduce an endogenous process of s-capital accumulation. The outcome is remorselessly pessimistic – labour is displaced at an endogenously determined rate, is forced into specialising in a shrinking set of tasks, and absolute wages are driven to zero. No steady-state is possible until labour has been entirely driven out the economy by s-capital i.e. the economy must approach a steady-state where $\bar{t}(t) = 1$. labour is fully immiserated, and technological unemployment follows.

To solve this dynamic model, I nest the static analysis in the previous section in a traditional Ramsey growth model framework. In any given $t$, the factor prices and pattern of specialisation are determined instantaneously by that static analysis – the $A(\cdot)$ and $B(\cdot)$ schedules in (5) and (9) are now time dependent such that $A(i, t)$ depends upon $a^S(i, t)$ and $a^H(i, t)$, and $B(i, t)$ on $\bar{t}(t)$, $K^S(t)$, $K^C(t)$, and $\psi$. A further important feature of this model is the innovative use of a numeraire price normalisation using a numeraire good. The price of this numeraire good is set to one, and the prices of all other goods are relative to this good. This numeraire price normalisation makes the dynamic model far more tractable, in two ways. First it allows me to reduce the dimensionality of the many-good model. As I will show, with Cobb-Douglas preferences across the range of goods, the law of motion for the numeraire good is the same as the law of motion for aggregate consumption across all goods – deriving this particular law of motion allows me to focus on aggregate consumption alone, rather than track the full set of laws of motion for each good. Secondly, with Cobb-Douglas production across the range of goods, the numeraire good allows me to derive analytically tractable expressions for the absolute factor prices, $r^C$ and $r^S$. These were not required in the static analysis in the previous section – static equilibrium could be characterised by the relative factor prices alone, as in (10). However, identifying the dynamic equilibrium requires the absolute factor prices as well.

It is also possible to use a simplex price normalisation to solve the dynamic model, where $\int_0^1 p(i) di = 1$. However, this approach can only be solved computationally and is far less tractable than the numeraire price normalisation that I will use.
2.1. Consumers

Consumers have the same preferences as before. Again, the economy can be captured by a representative consumer. Only s-capital $K^S(t)$ is accumulated, and the consumer faces the dynamic maximisation problem:

$$\max_{x(t)} \int_0^\infty e^{-\rho t} \int_0^1 \ln x(i, t) \, di \, dt$$

s.t.

$$\dot{K}^S(t) = r^S(t) \cdot K^S(t) + r^C(t) \cdot K^C + w^H(t) \cdot L^H - c(t)$$

$$K^S(0) = K^S_0$$

$$K^S(t) \geq 0$$

I assume there is exogenous growth in the productivity of s-capital. Any growth process must satisfy the following:

**ASSUMPTION 3:** For any exogenous growth process used, it must be the case that $a^S(\tilde{i}, t) \leq a^S(i, t) \cdot \frac{a^H(\tilde{i}, t)}{a^H(i, t)}$ for $\tilde{i} > \tilde{i}$ $\forall t$.

This is the dynamic version of Assumption 2. It ensures that, as technological progress takes place, s-capital does not become so productive in more complex tasks as to overturn the general principle that labour has the comparative advantage in these more complex tasks i.e. that $A_1(i, t) \geq 0$ $\forall t$. Initially, I assume that the particular growth process is:

$$\frac{\dot{a}^S(i, t)}{a^S(i, t)} = g \quad \forall i, t$$

In the Appendix I show that this satisfies Assumption 3.

2.2. Production and Firms

Production is the same as in the static setting – the task-based production function for goods, and the factor-based production functions for tasks, are as in (2) and (3). For simplicity, I assume in the dynamic setting that there is no depreciation. In closing this section, I explain why depreciation does not change the central results.
The traditional approach to finding the steady-state in a Ramsey growth model with technological progress is to re-define the variables in ‘effective’ terms, dividing each variable by the prevailing level of labour-augmenting technology. The result is that a steady-state is reached not in the actual variables, but instead in these ‘effective’ variables, the variable ‘per efficiency unit of labour’. In exactly the same way, solving this model requires that the s-capital augmenting technological progress I am considering is instead exactly reflected in a process of c-capital-augmenting technological progress. Consider again a good $x(\tilde{i}, t)$ that is produced by s-capital and c-capital. The transformation of the production function is as follows:

$$
x(\tilde{i}, t) = \left[ a^S(\tilde{i}, t) \cdot K^S(\tilde{i}, t) \right]^\psi \left[ a^C(\tilde{i}, t) \cdot K^C(\tilde{i}, t) \right]^{1-\psi} = \left[ K^S(\tilde{i}, t) \right]^\psi \left[ a^S(\tilde{i}, t) \cdot a^C(\tilde{i}, t) \cdot K^C(\tilde{i}, t) \right]^{1-\psi}
$$

(13) implies that a traditional-capital augmenting process of technological change in $a^S(\tilde{i}, t)$ is identical to the s-capital augmenting process of technological change in $a^S(\tilde{i}, t)$ in Assumption 3. This transformation has an important role in solving the dynamic model.

### 2.3. Dynamic Equilibrium

From the maximisation problem in (11) a current-value Hamiltonian follows:

$$
H = \int_0^1 \ln x(i, t) \, di + \mu(t) \left[ r^C(t) \cdot K^C + r^S(t) \cdot K^S + w^H(t) \cdot L^H - \int_0^1 x(i, t) \cdot p(i, t) \, di \right]
$$

(14)

It is possible to solve this Hamiltonian – with one co-state variable $\mu(t)$ and a spectrum of control variables, $x(i, t)$ where $i \in [0, 1]$ – in the traditional way. I confirm this in the Appendix. A set of first order conditions follow for each good $x(i, t)$:

$$
H_{x(i)} = \frac{1}{x(i, t)} - \mu(t) \cdot p(i, t)
$$

(15)

$$
= 0
$$

14
And for $K^S(t)$, the state variable:

$$H_{K^S} = \mu(t) \cdot r^S(t)$$

$$= \rho \cdot \mu(t) - \dot{\mu}(t)$$

(16)

Together, (15) and (16) imply that for good $x(i, t)$:

$$\frac{\dot{x}(i, t)}{x(i, t)} = -\frac{\dot{p}(i, t)}{p(i, t)} + r^S(t) - \rho$$

(17)

This is derived in the Appendix. In a traditional Ramsey growth model, there is only a unique final output and so there is no need to consider how the price of the good changes over time. But (17) shows that in this many-good setting the rate of growth of demand for $x(i, t)$ will depend upon how its price changes over time.

To maintain tractability, I now use a numeraire price normalisation. In particular, I assume that:

**ASSUMPTION 4:** $\tilde{i} = 0$ such that the numeraire good is $x(0, t)$ and so $p(0, t) = 1 \ \forall t$. Factor productivities and factor stocks are finite such that $\tilde{i}(t) \neq 0 \ \forall t$.

The price of good 0 is set to 1, and the prices of all other goods are in terms of that good. As I will show, this now makes the dynamic model far more tractable.

Given Assumption 4 it follows from (17) that for $x(0, t)$:

$$\frac{\dot{x}(0, t)}{x(0, t)} = r^S(t) - \rho$$

(18)

Given Assumption 1 it follows that the law of motion for the numeraire good $x(0, t)$ is the same as the law of motion for total consumption $c(t)$.15 And so:

Where $c(t) = \int_0^1 x(i, t) \cdot p(i, t) \, di$. To see this note that since $p(\tilde{i}, t) = 1$:

$$c(t) = \frac{x(i, t) \cdot p(i, t)}{\theta(i)} = \frac{x(\tilde{i}, t)}{\theta(\tilde{i})}$$

and so $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{x}(\tilde{i}, t)}{x(\tilde{i}, t)}$ since $\theta(\tilde{i})$ is constant and equal to 1.
(19) \[
\frac{\dot{c}(t)}{c(t)} = r^S(t) - \rho
\]

It follows that if \( x(\hat{i}, t) \) reaches a steady-state then total consumption \( c(t) \) will also be in steady-state. From now, I use the law of motion in (19) and focus on \( c(t) \). The law of motion for \( K^S(t) \) follows from (11):

(20) \[
\dot{K}^S(t) = Y(t) - c(t)
\]

At this point, the dynamic system is expressed in terms of \( c(t) \) and \( K^S(t) \). In order to find the steady-state in this model it is necessary to transform these variables into ‘effective’ terms, as with a traditional Ramsey growth model. For any variable \( v(t) \), I use the following transformations:

(21) \[
\hat{v}(t) = v(t) \quad \hat{\hat{v}}(t) = v(t) \\
\bar{i}(t) \cdot a^S(0, t) \frac{\psi}{\bar{i}(t)}
\]

These transformations are different from those used in a traditional Ramsey growth model. The intuition for the form of these effective variable is revealed once the dynamic equilibrium is derived. But from inspection it is clear that a ‘\( \hat{\hat{\ldots}} \)’ term is ‘effective’ with respect to a term that captures the productivity of s-capital at time \( t \), whereas the ‘\( \hat{\ldots} \)’ term is ‘effective’ with respect to a term that captures the productivity of s-capital and the value of the cut-off at time \( t \). In the analysis that follows I look for an equilibrium in \( \hat{c}(t) - \hat{K}^S(t) \) space i.e. in ‘effective’ consumption and s-capital space.

First, consider \( \hat{c}(t) \). (19) and (21) imply a law of motion for \( \hat{c}(t) \):

(22) \[
\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = r^S(t) - \rho - \frac{\psi}{1 - \bar{i}(t)} \cdot g
\]

Secondly, consider \( \hat{K}^S(t) \). (20) and (21) imply a law of motion for \( \hat{K}^S(t) \):
where \( \dot{g}(t) \) is the growth rate in \( \bar{i}(t) \) (and, given (7), in \( \gamma(t) \) as well). Both the laws of motion in (22) and (23) can be expressed in terms of \( \dot{c}(t) \) and \( \dot{K}^S(t) \) alone. This is what is required to study equilibrium in \( \dot{c}(t) - \dot{K}^S(t) \) space. To re-write (22) in this way requires an expression for \( r^S(t) \) in terms of \( \dot{K}^S(t) \). To re-write (23) in this way requires an expression for \( \dot{Y}(t) \) and \( \dot{g}(t) \) in terms of \( \dot{K}^S(t) \).

First, consider \( r^S(t) \) in terms of \( \dot{K}^S(t) \). The static model implies that \( p(0,t) \) is equal to:

\[
P(0,t) = \left[ \frac{r^S(t)}{\psi \cdot a^S(0,t)} \right]^\psi \left[ \frac{r^C(t)}{(1 - \psi) \cdot a^C(0,t)} \right]^{1-\psi}
\]

This is shown in the Appendix. Substituting the expression for \( r^C(t) \) in terms of \( r^S(t) \) that follows from (10), and using the price normalisation that \( p(0,t) = 1 \), (24) implies that \( r^S(t) \) can be re-written as:

\[
r^S(t) = \left[ a^S(0,t) \right]^\psi \left[ a^C(0) \right]^{1-\psi} \cdot \left[ \bar{i}(t) \right]^{1-\psi} \cdot \left[ \frac{1}{K^S(t)} \right]^{1-\psi} \cdot K^C(1-\psi) \cdot \psi
\]

(25)

where \( D \) is a positive constant equal to \( (a^C(0) \cdot K^C)^{1-\psi} \cdot \psi \).

Now consider \( \dot{Y}(t) \) in terms of \( \dot{K}^S(t) \). Note that the structure of production

\[
(23) \quad \frac{\dot{K}^S(t)}{K^S(t)} = \frac{\dot{Y}(t) - \dot{c}(t)}{K^S(t)} - \left[ \frac{\dot{g}(t)}{1 - \psi} \cdot g \right]
\]

\(16\) implies that the level of \( r^S(t) \) depends on the choice of the numeraire good, \( \bar{i} \) – if a different \( \bar{i} \) is chosen, the level of \( a^S(\bar{i},t) \) and \( a^C(\bar{i},t) \) will differ from those in (25) where \( \bar{i} = 0 \). However, this does not affect the important features of absolute factor prices in equilibrium. In the case where there is a one-off change in the productivity of s-capital, so long as that change is uniform – as in Dornbusch, Fischer, and Samuelson (1977) – (25) implies that, \( \text{ceteris paribus} \), \( r^S(t) \) will always move in the same direction regardless of the choice of numeraire. In the case where there is an increase in the growth rate of the productivity of s-capital, so long as the growth rates remain the same across different tasks, \( \text{ceteris paribus} \), (25) implies that \( r^S(t) \) will always increase at the same rate regardless of the choice of numeraire. To explore non-uniform changes in productivity, or to make the level of \( r^S(t) \) invariant to the normalisation, it is necessary to use a simplex price normalisation. But this leads to an implicit, rather than explicit and tractable, solution to the model. This need for uniformity is an interesting limitation of Dornbusch, Fischer, and Samuelson (1977) that was not explored. Complications involving price normalisations are discussed elsewhere in the literature – Dierker and Grodal (1999), for example, on models of imperfect competition.
in (2) and (3) implies that total s-capital income is equal to:

\[(26) \quad r^S(t) \cdot K^S(t) = \bar{i}(t) \cdot \psi \cdot Y(t)\]

and so substituting in \(r^S(t)\) from (25) it follows that:

\[(27) \quad \dot{\hat{Y}}(t) = \hat{K}^S(t)\psi^{-1} \cdot D \cdot \hat{K}^S(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \]

Finally, to find the expression for \(g^{\bar{i}}(t)\) in terms of \(\hat{K}^S(t)\) note that by definition:

\[(28) \quad g^{\bar{i}}(t) = \frac{\partial \bar{i}(t)}{\partial \hat{K}^S(t)} \cdot \dot{\hat{K}}^S(t) \cdot \frac{1}{\bar{i}(t)}\]

Using the expression for \(r^S(t)\) in (25), \(\dot{Y}(t)\) in (27), and \(g^{\bar{i}}(t)\) in (28), the laws of motion in (22) and (23) can now be re-expressed in terms of \(\hat{c}(t)\) and \(\hat{K}^S(t)\) alone.

The law of motion for \(\hat{c}(t)\) follows from (22) and (25):

\[(29) \quad \frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \hat{K}^S(t)\psi^{-1} \cdot D - \rho - \frac{\psi}{1 - \psi} \cdot g\]

The law of motion for \(\hat{K}^S(t)\) in (23) is more complex to derive. The full derivation is shown in the Appendix. Using (27) and (28), it can be written as:

\[(30) \quad \frac{\dot{\hat{K}}^S(t)}{\hat{K}^S(t)} = \frac{\hat{K}^S(t)\psi^{-1} \cdot D - \bar{i}(t) \cdot \frac{\psi}{1 - \psi} \cdot g - \frac{\bar{i}(t)}{\hat{K}^S(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{K}^S(t)} \cdot \hat{K}^S(t)}\]

With a traditional Ramsey growth model, the standard approach to identifying steady-state is to consider these schedules in \(\hat{c}(t) - \hat{K}^S(t)\) space by plotting their stable arms. (29) implies that the \(\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0\) schedule, the stable arm for \(\hat{c}(t)\), is:
\[
\dot{\hat{K}}^S(t) = \left[\left(\rho + \frac{\psi}{1-\psi} \cdot g\right) \cdot \frac{1}{D}\right]^{-\frac{1}{\psi-1}}
\]

This expression has an identical form to the stable arm in a traditional Ramsey growth model with a labour-augmenting growth process taking place at rate \(\frac{\psi}{1-\psi} \cdot g\) – the stable arm for \(\dot{c}(t)\) is simply a vertical schedule at some \(\hat{K}^S\) that ensures \(r^S(t) = \rho + \frac{\psi}{1-\psi} \cdot g\) and \(\dot{c}(t)\) is constant. (30) implies that the stable arm for \(\hat{K}^S(t)\), the \(\dot{\hat{K}}^S(t) = 0\) schedule, is equal to:

\[
\dot{\hat{c}}(t) = \hat{K}^S(t)^\psi \cdot D \cdot \left(1 - \frac{\psi}{1-\psi} \cdot g \cdot \hat{K}^S(t)\right)
\]

This is almost identical to the stable arm in a traditional Ramsey growth model, except in one important respect – the presence of \(\bar{i}(t)\). The reason that \(\bar{i}(t)\) appears in the \(\dot{\hat{K}}^S(t) = 0\) schedule in this model is critical. This is because in any period \(t\), s-capital is only used to produce \(\bar{i}(t)\) of the goods in the economy. The remaining \(1 - \bar{i}(t)\) goods are produced by labour whose productivity is unaffected by technological progress. However as \(\bar{i}(t)\) rises, and more goods are produced by s-capital rather than labour, the production of more goods in the economy is affected by the technological progress. As \(\bar{i}(t)\) rises it is as if the ‘effective’ rate of technological progress – this is \(\bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g\) in (32) – rises. Indeed, the increase in \(\bar{i}(t)\) in this new model with many goods has the same consequence as an increase in technological progress in a traditional Ramsey growth model with a unique final good.\(^{17}\) Intuitively, in a traditional Ramsey growth model with a unique final good, the economy ‘feels the full force’ of the technological progress, whereas in this many-good model only \(\bar{i}(t)\) of the economy does.

This role for \(\bar{i}(t)\) emerges because of its importance in determining \(r^S(t)\) in (25). That expression implies that \(r^S(t)\) is increasing, at a diminishing rate, in \(\hat{K}^S(t)\). Given the definition of \(\hat{K}^S(t)\) implied by (21), this means that \(r^S(t)\) is decreasing in \(K^S(t)\) and increasing in \(a^S(0,t)\) – as in a traditional Ramsey growth model – but that it is also increasing in \(\bar{i}(t)\). This is because while there are diminishing returns to the use of \(K^S(t)\) in the production of any given good, increasing the range of goods \(\bar{i}(t)\) that \(K^S(t)\) is used to produce ‘spreads out’ the

\(^{17}\) If the rate of technological progress were to increase a traditional Ramsey growth model, the corresponding ‘effective’ capital schedule would fall, requiring a lower rate of effective consumption to ensure that the effective capital stock is constant.
stock of s-capital across the economy, and offsets those diminishing returns. It is this new role for \( \tilde{i}(t) \) that means the transformations in (21) must be used to find the dynamic equilibrium.

Because of the \( \tilde{i}(t) \) term in the stable arm for \( \hat{K}^S(t) \) in (32), dynamic equilibrium is not analytically tractable. It requires a non-linear simulation of the three equation system:

\[
\begin{align*}
\frac{\dot{c}(t)}{c(t)} &= \hat{K}^S(t)^{\psi-1} \cdot D - \rho - \frac{\psi}{1-\psi} \cdot g \\
\frac{\dot{K}^S(t)}{\hat{K}^S(t)} &= \frac{\hat{K}^S(t)^{\psi-1} \cdot \frac{D}{\psi} - \tilde{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\dot{c}(t)}{\hat{K}^S(t)}}{\tilde{i}(t) + \frac{\partial \tilde{i}(t)}{\partial \hat{K}^S(t)} \cdot \hat{K}^S(t)} \\
\tilde{i}(t) &= f(K^S(t), A(\tilde{i}(t), t), L, H)
\end{align*}
\]

The first two are the familiar differential equations for \( \dot{c}(t) \) and \( \dot{K}^S(t) \). The third is a static equation that determines \( \tilde{i}(t) \) in any given \( t \). In the Appendix, I show a simulation. Nevertheless, it is still possible to identify analytically the unique steady-state that this model approaches and also to provide an intuitive conjecture for transition to this steady-state, without performing the simulation. This is shown in Figure 5.

First, the steady-state itself. This exists where the stable arms for \( \dot{c}(t) \) and \( \dot{K}^S(t) \) intersect. The steady-state is unique and must take place when \( \tilde{i}(t) = 1 \) i.e. when labour is entirely driven out of the economy by s-capital. To see why, first note that the stable arm for \( \dot{c}(t) \) in \( \dot{K}^S(t) - \dot{c}(t) \) space is a vertical schedule at the level of \( \hat{K}^S \) in (31). Now consider the following proof by contradiction. Suppose that the stable arm for \( \dot{K}^S(t) \) crosses the stable arm for \( \dot{c}(t) \) where \( \tilde{i}(t) \neq 1 \). In this potential steady-state the productivity of s-capital grows at a rate \( \frac{\psi}{1-\psi} \cdot g \). If this is to be a steady-state, then \( \tilde{i}(t) \) must remain constant. If \( \tilde{i}(t) \) does not remain constant, then the stable arm for \( \dot{K}^S(t) \) will shift. This is implied by (32).

The analysis of the static equilibrium in the previous section implies that \( \tilde{i}(t) \) will
only stay constant if $K^S(t)$ decreases to offset this increase in productivity. But if $K^S(t)$ decreases such that $\tilde{i}(t)$ remains constant then $\hat{K}^S(t)$ will fall, violating the condition that $\hat{K}^{S*}$ is constant at the steady-state, as required by (31). $K^S(t)$ must therefore rise to ensure that $\hat{K}^{S*}$ remains constant. However, this growth in the stock of s-capital increases $\tilde{i}(t)$ further. From the definition of the stable arm for $\hat{K}^S(t)$ in (32) this will drive down the intersection point of the stable arms for $\hat{c}(t)$ and $\hat{K}^S(t)$. The steady-state value of $\hat{c}^*$ will be driven down. Repeating this argument implies that if steady-state is to exist in $\hat{K}^S(t)$–$\hat{c}(t)$ space, it must take place when $\tilde{i}(t) = 1$.

Now consider the transition path to this steady-state, where $\tilde{i}(t) = 1$. Figure 5 shows this path when the steady-state is approached from the left i.e. $\hat{K}^S(0) < \hat{K}^{S*}$. The transition path drawn is based on three observations. The first is that the stable arm for $\hat{c}(t)$ is known, given in (31). The second is that as $\hat{K}^S(t)$ increases, $\tilde{i}(t)$ also increases – this is intuitive, but proven in the Appendix. This gives the stable arm for $\hat{K}^S(t)$ its hump-shape in Figure 5, given (32). The third is that, supposing the economy reaches the steady-state where $\tilde{i}(t) = 1$, the new model must collapse to have the same general form as a traditional Ramsey growth model in which there is labour and capital, and a labour-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$ – but rather than there being labour and capital, there is instead only two types of capital, with an s-capital-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$. As a result, locally to the steady-state, the new model has the same approximated saddle-path as this traditional Ramsey growth model. This is what is shown in Figure 5.

This transition path is intuitive – during transition effective s-capital, $\hat{K}^S(t)$, effective consumption, $\hat{c}(t)$, and the cut-off $\tilde{i}(t)$ rise and approach the steady state. In effect, the accumulation of s-capital drives up consumption but also drives out labour. This process is driven by the fact that s-capital is accumulated not only to offset the fact that it is becoming more productive, as in transition to steady-state in a traditional Ramsey growth model, but also because s-capital is becoming more intensively used across the economy – $\tilde{i}(t)$ is rising, offsetting the diminishing...

\footnote{Note that it is not possible for a steady-state where $\tilde{i}(t) = 0$ – although $\tilde{i}(t)$ is constant, this would imply there is no s-capital and so the steady-state condition in (31) is violated.}

\footnote{In effect, the $\hat{K}^S(t) = 0$ schedule is bounded below by a hypothetical $\hat{K}^S(t) = 0$ schedule where $\tilde{i}(t)$ is fixed at 1 and above by a hypothetical $\hat{K}^S(t) = 0$ schedule where $\tilde{i}(t)$ is fixed at 0.}

\footnote{By ‘collapse’, I mean the static three-factor analysis is replaced by a two-factor analysis where there is no labour, and s- and c-capital combine to produce all goods.}
returns to s-capital in the production of any particular good. It is as if the economy is chasing a steady-state that is continually slipping out of its grasp – until $\bar{i}(t) = 1$ and labour is fully driven out.

Importantly, wages during transition are driven to zero. To see this, first note from (25) that as $\hat{K}^S(t)$ rises during transition, $r^S(t)$ must fall. In turn, consider that the relative wage of labour with respective to s-capital as steady-state is approached is equal to:

$$\lim_{t \to \infty} \frac{w^H(t)}{r^S(t)} = \lim_{t \to \infty} \frac{A(\bar{i}(t), t)}{\hat{r}(t, t)}$$

(34)

and since $\lim_{t \to \infty} \bar{i}(t) = 1$ it follows that:

$$\lim_{t \to \infty} \frac{a^H(\bar{i}(t), t)}{a^S(\bar{i}(t), t)} = \lim_{t \to \infty} \frac{a^H(1, t)}{a^S(1, t)} = 0$$

(35)

given that $a^H(i, t)$ is constant over time $\forall i$, and there is continuous growth in $a^S(i, t)$ $\forall i$, as in (12), so $\lim_{t \to \infty} a^S(1, t) = \infty$. This means that $\frac{w^H(t)}{r^S(t)}$ is driven to zero. And since $r^S(t)$ must fall during transition, this implies $w^H(t)$ is driven to zero as well. The outcome is remorselessly pessimistic for labour. Proposition 2 follows.

**PROPOSITION 2:** When the static model with s-capital is placed in a dynamic setting with endogenous s-capital accumulation, the economy approaches a unique steady-state at $\hat{c}^*$, $\hat{K}^S*$, where $\bar{i}^* = 1$ – in this steady-state, labour is entirely driven out by s-capital. During transition, $w^H(t)$ is driven to zero. The capital share of income rises steadily to 1.

Note that the particular transition path draw in Figure 5 is still a conjecture, and for an important reason – the approximated saddle-path in Figure 5 is only correct locally to the steady-state. It is based on the claim that $\bar{i}(t) = 1$ in steady-state, but as soon as $\hat{K}^S(t)$ deviates from $\hat{K}^S*$ along the saddle-path, $\bar{i}(t) \neq 1$, and this claim no longer holds. In order to find the actual saddle-path to this steady-state, it is necessary to perform a non-linear simulation. I show an example in the
Appendix.

Proposition 2 is also derived without depreciation. This may seem like a significant omission, since the depreciation of the existing stock of s-capital may appear to act as a counterbalance to the accumulation of s-capital and the displacement of labour. But this intuition is incorrect. Suppose depreciation takes place at rate $\delta$. As in a traditional Ramsey growth model, the introduction of depreciation changes the stable arm for $\hat{c}(t)$, $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$:

$$\hat{K}^S(t) = \left[ \rho + \delta + \frac{\psi}{1 - \psi} \cdot g \right] \cdot \frac{1}{D} \cdot \frac{1}{\psi - 1}$$

and also the stable arm for $\hat{K}^S(t)$, $\frac{\dot{\hat{K}}^S(t)}{\hat{K}^S(t)} = 0$:

$$\hat{c}(t) = \hat{K}^S(t) \cdot \psi \cdot \hat{c}(t) - \hat{i}(t) \cdot \left[ \delta + \frac{\psi}{1 - \psi} \cdot g \right] \cdot \hat{K}^S(t)$$

The implication of (36) and (37) is that while the introduction of depreciation will change the level of steady-state $\hat{K}^S$* – implied by (36) – it is still the case that the model must approach a steady-state in $\hat{c}(t) - \hat{K}^S(t)$ space when $\hat{i}(t) = 1$. This is again implied by (37). The outcome again is remorselessly pessimistic for labour.

As a final observation note that, with or without depreciation, the outcome for owners of capital is remorselessly optimistic. The s-capital share of income, $\hat{i}(t) \cdot \psi$, rises over time. And the return to the fixed stock of c-capital, $r^C(t)$, which (10) and (25) imply is equal to:

$$r^C(t) = \hat{K}^S(t) \cdot \psi \cdot a^S(0, t) \cdot \frac{a^C(0)^{1 - \psi} \cdot (1 - \psi)}{(K^C)^{\psi}}$$

This implies $r^C(t)$ rises during transition and in steady-state when $\hat{i}(t) = 1$.

3. Five New Reasons for Optimism

There are five substantive challenges to the pessimistic conclusion in Proposition 2, based on the strict assumptions that the model relies upon. The first three take place on the supply-side, the fourth on the demand-side, and the final challenge is
a general one. I offer reflections on each and their implications for further research.

3.1. Productivity of Complementing Capital

The first challenge to the pessimistic conclusion in this paper concerns the productivity of c-capital. The model relies on the assumption that there is no growth in the productivity of c-capital. But, intuitively, growth in the productivity of c-capital might offset the pessimistic conclusion by raising the value of the shrinking set of types of tasks in which labour retains the comparative advantage, and pushing up the wages of those who are employed to perform them as a result. This can be shown formally by deriving an expression for $w^H$ from (10) and (38):

\[
(39) \quad w^H(t) = \left(a^C(0,t) \cdot K^C \right)^{1-\psi} \cdot \hat{K}^S(t)^\psi \cdot a^S(0,t)^{\frac{\psi}{1-\psi}} \cdot \psi \cdot (1 - \bar{i}(t))
\]

Here, the wage indeed is an increasing function of both the level of c-capital and the productivity of c-capital. This is an important result. In practice, it seems plausible that c-capital will become more productive over time, as well as s-capital. In turn, it is likely that c-capital will be accumulated over time, as well as s-capital. This suggests that the empirical task of understanding the nature of recent technological change – whether it is biased towards ‘s-capital’ or ‘c-capital’ – is increasingly important. The two types of capital have very different consequences for labour.

3.2. Productivity of Substituting Capital

The second challenge to the pessimistic conclusion in this paper concerns the productivity of s-capital. This model relies on the assumption that s-capital becomes continuously more productive across the task-spectrum. One potential criticism is that while it may be right that economists’ expectations about the limits to system and machine capabilities were wrong in the past, this does not mean that all types of tasks can now be automated. Put another way, the problem with the model in this paper is that it does not adjust the limit to what can be automated or not but, instead, removes the limit altogether.

In the short-run, this challenge is likely to be correct. As Autor (2014) notes, “[t]he challenges to substituting machines for workers in tasks requiring flexibility, judgment, and common sense remain immense”. But whether this
will continue in the long-run is not clear. In the medium-run, it seems likely that the boundary between those tasks that can and cannot be done by machines will shift, towards those in which labour has the comparative advantage – as it does in this new model. In the long-run, the technical claim that there exists certain types of tasks that could never be automated is not obvious. Nor is the normative claim that there exists certain types of tasks that ought not to be automated as compelling in a world where machines might perform these tasks vastly more effectively than human beings. It seems inevitable that some form of ‘task encroachment’, where future systems and machines are able to perform a wider range of types of tasks than they can today, will continue.

In this model, I have explored a very simple form of ‘task encroachment’ – where c-capital gradually, but relentlessly, takes on more ‘complex’ tasks. The only dimension in which task encroachment can vary is the speed of this process, determined by the magnitude of $g$ in (12). However, in practice it is unlikely that encroachment takes place in such a deterministic way. Exploring the different forms that this process of task encroachment could take is an increasingly important theoretical and empirical project.

3.3. Productivity of Labour

The third challenge to the pessimistic conclusion in this paper concerns the productivity of labour. The model relies on the assumption that there is no growth in the productivity of labour. But, intuitively, growth in the productivity of labour might offset the pessimistic conclusion by raising the demand for labour to perform the shrinking set of types of tasks in which it retains the comparative advantage. This can be seen formally in (39). From the static model in Section 1, we know that an increase in productivity of labour across the task spectrum causes a fall in $\bar{i}$. This will increase $w^H(t)$ in (39) – both by increasing $\hat{K}^S(t)$ (recall that $\bar{i}$ enters the denominator of the expression for ‘effective’ c-capital in (21) and by increasing the $(1 - \hat{i}(t))$ term.

In the short-run this is an important challenge to the pessimistic conclusion in this paper. The spirit of a large part of the traditional literature on technology and the labour market is that there is a metaphorical “race” between labour and capital – that the former can acquire skills that will allow it to ‘keep up’ with the latter (Goldin and Katz 2008, for instance). But whether this sort of skill accumulation can keep pace in the medium- or long-run is less clear. Consider a recent OECD report (Elliott 2017) comparing the literacy, numeracy, and problem-solving skills
of human beings with those of new systems and machines. (So-called ‘PIAAC skills’.) Their results were striking, finding that:

“[T]here are no examples of education systems that prepare the vast majority of adults to perform better in the three PIAAC skills areas than the level that computers are close to reproducing. Although some education systems do better than others, those differences are not large enough to help most of the population overtake computers with respect to PIAAC skills.” [Emphasis added.]

3.4. The Form of the Production Functions

The fourth challenge to the pessimistic conclusion is this paper concerns the form of the factor-based production function for tasks in (3) and the task-based production function for goods in (2). For instance, the model relies on the assumption that (2) is Cobb-Douglas, which ensures that expenditure shares on the two different types of tasks, \( z_1(i) \) and \( z_2(i) \), remain fixed. However, it is entirely possible in practice that the production function takes a different form. In turn, this would affect the result in Proposition 2.

Take, for instance, a hypothetical production function where expenditure shares on the two different types of tasks do change in response to changes in the relative price of those tasks. Suppose relative price changes caused expenditure shares to increase on tasks performed by labour. This would help to offset the pessimistic conclusion – labour may be confined to a shrinking range of types of tasks, but there would also be a greater relative demand for those residual tasks. However, in the case where relative price changes caused expenditure shares to fall on tasks performed by labour, this would compound the pessimistic conclusion. Both these observations suggest that further work is needed to understand the way in which new technologies combine with labour in production.

3.5. Consumer Expenditure Densities

The fourth challenge to the pessimistic conclusion in this paper concerns the demand-side. A common set of arguments for remaining optimistic about the threat of automation appeal to consumer tastes and preferences. The claim is that although new technologies may replace labor in producing certain types of goods, it is also likely that technological change will cause consumer demand to shift across those goods. That means that labor displaced from producing some
goods can instead be employed to produce other goods that are now in greater relative demand.

The argument can take two particular forms. The first is a claim that demand for goods will rise and displaced workers can be employed to meet that increase in demand for goods. For instance, Autor (2015) notes “I think that people are extremely unduly pessimistic ... as people get wealthier, they tend to consume more, so that also creates demand”, Levy (2017) that, “the lower cost per unit of output will stimulate demand for output and will increase employment”, Summers (2013), that “[t]he stupid people thought that automation was going to make all the jobs go away ... the smart people understood that when more was produced, there would be more income and therefore there would be more demand”, and Kenneth Arrow, that “the economy does find other jobs for workers. When wealth is created, people spend their money on something.” The second form of the argument is a claim that demand for goods will change and displaced works can be employed to meet that change in demand across goods. For instance, Mokyr et al. (2015) “[t]he future will surely bring new products that are currently barely imagined, but will be viewed as necessities by the citizens of 2050 or 2080” and Autor and Dorn (2013a), that the economy will “generate new products and services that raise national income and increase overall demand for labor in the economy”.

These arguments are appealing and intuitive. But at present, the most popular formal models that explore the effect of technological change on the labor market cannot capture them. The problem is that these models tend to only have a unique final good (see, for instance, Autor, Levy and Murnane 2003, Acemoglu and Autor 2011, and Acemoglu and Restrepo 2017;2018). However, the new model in this paper does provide a clear way to explore these arguments in a formal way. Because preferences are Cobb-Douglas in (1), expenditure shares across goods do not change, either due to income or price effects. Yet with more complex preferences, it is possible to show that expenditure shares can change over time in this model. In turn, it is also possible to show that if consumer expenditure shifts towards goods that require types of tasks in which labour retains a comparative advantage, then this indeed can offset the pessimism considered before.

However, there are two problems with this demand-side challenge to the pessimistic conclusion. First, if consumer expenditure does shift in the right direction, towards goods that require types of tasks in which labour retains the comparative advantage, then this indeed can offset the pessimism considered before.

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22The notable exception to this is Autor and Dorn (2013b), which does have a role for the demand-side. However, it is a limited one; the model only has two goods. Bessen (2018) makes a similar observation to this one.
advantage, it is not clear why it would shift at a sufficiently fast rate to offset the growth in the productivity of s-capital. Secondly, it is not clear why consumer expenditure would shift in the right direction in the first place. Consider the other possibility – that consumer expenditure shifts away from goods that require tasks in which labour retains the comparative advantage, towards those in which s-capital holds the comparative advantage. If, for instance, consumer demand in (1) were represented by an ‘elastic’ CES utility function, it can be shown that price effects would cause consumer expenditure to shift towards the increasingly cheaper goods produced by s-capital. Both these problems with the demand-side challenge require further theoretical and empirical work.

3.6. The Task Spectrum

The fifth challenge to the pessimistic conclusion in this paper concerns task spectrum. In the new model, the task spectrum has a fixed upper-bound. This is the assumption that leads Autor and Salomons (2017a,b) to observe that labour has “no place left to hide”. Yet in practice it may be that the task spectrum changes over time. This is captured in Acemoglu and Restrepo (2017; 18). A novel feature of their model is that new tasks are created for displaced workers to perform and that “[t]hese new tasks generated jobs.” This process of task creation is endogenous – downward pressure on wages creates an incentive for entrepreneurs to invest in technologies that create new tasks for this relatively cheaper labour to perform. In a sense, there is a new ‘race’ – between the displacement of labour from old tasks, and the creation of new tasks for them to do. This is an interesting and important challenge to the fixed upper-bound of the task spectrum in this paper.

There are two important comments to make about this challenge. The first is that Acemoglu and Restrepo (2017; 18) is a supply-side account of the new race. Producers create new tasks to produce a unique final good. Yet this is problematic because while the idea of the ‘race’ is often informally appealed to in the economics literature, it is far more often conceptualised as taking place through the demand-side. In these accounts it is not producers who create new types of tasks to produce existing goods, but consumers who demand new goods that in turn require new types of tasks to produce them. For instance, in making a case for optimism, Mokyr et al. (2015) argues “the future will surely bring new products that are currently barely imagined”, Autor and Dorn (2013a) discusses “new products and services that raise national income”, and Autor (2014) appeals to the unforeseen rise of “health care, finance, informational technology, consumer
electronics, hospitality, leisure, and entertainment” to explain why those displaced from farming in the 20th century would not lack for work in the 21st. At present there is no reason to favour one particular version of the race. The new model in this paper, with a variety of goods, and a role for tastes and preferences, can be used to explore this demand-side version of the race. Only a model like the one in this paper, with a variety of goods and a role for tastes and preferences, is capable of exploring this demand-side version. Acemoglu and Restrepo (2017;18), with a unique final good, cannot.

The second comment is to question whether a changing upper-bound would necessarily help labour. In Acemoglu and Restrepo (2017;18) tasks are created in the direction of labour’s comparative advantage. This creates a refuge for displaced labour. Yet it is not clear why this need necessarily be so. Consider another possibility – that tasks are created in the direction of capital’s comparative advantage instead. The reason that tasks are created in the direction of labour’s comparative advantage in Acemoglu and Restrepo (2017;18) is because labour becomes cheaper, and so the incentive is to create tasks that use this labour. But as capital becomes more productive, then this also creates an incentive to create tasks that use this capital. What matters is not only the relative price of a factor, but its relative productivity as well – and if capital become relentlessly more productive, then the incentive to create tasks that benefit labour may fall away.

Acemoglu and Restrepo (2017) wisely quote Leontief (1952) and his mistaken claim that labour in the 21st century would share the fate of horses in the 20th – that they “will become less and less important ... More and more workers will be replaced by machines”. Their model offers a compelling explanation why Leontief was wrong – “the difference between human labour and horses is that humans have a comparative advantage in new and more complex tasks. Horses did not.” But this then raises an interesting question – why does their model not apply equally well to horses, too? If technological change displaced horses and made them more abundant and cheaper, why was there not a surge in the creation of new types of tasks in the 20th century in which horses had the comparative advantage, as the model would suggest? The answer is that it was not simply the relative price of horses that mattered, but their relative productivity too. No matter how cheap horses became it did not make economic sense to create new tasks for them to do. Is this the fate of human beings in the long run – economically redundant, not matter how cheap they become? More theoretical and empirical work is needed on the dynamics of task creation.
4. Conclusion

Autor (2015) captures the case for optimism in the task-based literature:

“These questions underline an economic reality that is as fundamental as it is overlooked: tasks that cannot be substituted by automation are generally complemented by it.” (p. 6)

This is repeated in Autor (2014):

“The fact that a task cannot be computerized does not imply that computerization has no effect on that task. On the contrary: tasks that cannot be substituted by computerization are generally complemented by it.” (p. 8)

In an important sense, the analysis in this paper is in agreement with this claim. Those tasks that cannot be automated are indeed complemented by c-capital. The value of those tasks increases as the quantity or productivity of c-capital increases. However, the new model does challenge the assumption that necessarily labour will indefinitely be best placed to perform those complemented tasks. This is the new role for s-capital. As it increases in quantity or productivity, it erodes the set of types of q-complemented tasks in which labour retains the comparative advantage. This is the new process of task encroachment. Labour is forced to specialise in a diminishing set of types of tasks. Under the ALM hypothesis, the set of types of tasks performed by labour that are q-complemented is protected from this erosion. That the consequences of task encroachment are so pessimistic for labour – a remorseless displacement of labour, driving wages to zero – suggests the traditional literature may have already created a false sense of optimism about the prospects for labour. Note again, however, that for the owners of capital this conclusion is far from pessimistic. All the returns to technological progress flow to them. From an equity standpoint, it follows that who owns and controls capital in this model becomes an increasingly important question over time.

If the process of task encroachment does create a new reason for pessimism, then understanding how we might offset it is an increasingly important task. In this paper, I have used the restrictive modelling assumptions to identify possible countervailing reasons for optimism. Each of these require further theoretical and empirical research.

For example, the return to c-capital, $r_C(t)$, rises over time. See (38).
5. Appendix: For Online Publication

5.1. Deriving the Hamiltonian

To find this steady-state, I used a Hamiltonian. But this is an unconventional Hamiltonian – there is one state variable, $K_S(t)$, one co-state variable, $\mu(t)$, and a range of choice variables, $x(i,t)$. To show that the traditional Hamiltonian approach applies in this setting, I can derive the Hamiltonian and accompanying first order conditions explicitly. The aim is to maximise:

$$\int_0^\infty e^{-\rho t} \left[ \int_0^1 \theta(i) \ln x(i,t) \, di + \mu(t) \left[ Y(t) - \int_0^1 x(i,t) \cdot p(i,t) \, di - K^S(t) \right] \right] \, dt$$

where $Y(t)$ is defined as before. Integrating by parts implies:

$$\int_0^\infty e^{-\rho t} \cdot \mu(t) \cdot K^S(t) \, dt = \left[ e^{-\rho t} \cdot \mu(t) \cdot K^S(t) \right]_{t=0}^{t=T} - \int_0^\infty K^S(t) \frac{\partial [\mu(t) \cdot e^{-\rho t}]}{\partial t} \, dt$$

$$= e^{-\rho T} \cdot \mu(T) \cdot K^S(T) - \mu(0) \cdot K^S(0) - \int_0^T K^S(t) \cdot e^{-\rho t} \left[ \dot{\mu}(t) - \rho \cdot \mu(t) \right] \, dt$$

and since:

$$A = e^{-\rho T} \cdot \mu(T) \cdot K^S(T) - \mu(0) \cdot K^S(0)$$

where $A$ is a constant, it follows that (40) can be re-written:

$$\int_0^\infty e^{-\rho t} \left[ \int_0^1 \theta(i) \ln x(i,t) \, di + \mu(t) \left[ Y(t) - \int_0^1 x(i,t) \cdot p(i,t) \, di \right] + K^S(t) \cdot [\dot{\mu}(t) - \rho \cdot \mu(t)] \right] \, dt - A$$

It follows that to maximise (43) over time the following expression must be maximised for each given $t$:

$$\int_0^1 \theta(i) \ln x(i,t) \, di + \mu(t) \left[ Y(t) - \int_0^1 x(i,t) \cdot p(i,t) \, di \right] + K^S(t) \cdot [\dot{\mu}(t) - \rho \cdot \mu(t)]$$

or more concisely the following must be maximised in each period $t$:

$$H + K^S(t) \cdot [\dot{\mu}(t) - \rho \cdot \mu(t)]$$
where $H$ is the current value Hamiltonian. And so, in the traditional way, to maximise the expressions in (44) and (45) a traditional set of first order conditions follow:

\[
\frac{\partial H}{\partial x(i,t)} = 0 \quad \forall i
\]

\[
\frac{\partial H}{\partial K_S(t)} + \dot{\mu}(t) - \rho \cdot \mu(t) = 0
\]

These are the first order conditions that I use in this paper. This derivation is based on Wren-Lewis (2012).

5.2. Assumption 3

To derive the condition in Assumption 3, take two arbitrary levels of $i$ such that $\tilde{i} < \tilde{\tilde{i}}$. Suppose Assumption 2 initially holds so that at $t = 0$:

\[
A(\tilde{i}, 0) \leq A(\tilde{\tilde{i}}, 0)
\]

Given the definition of $A(i,t)$ this implies:

\[
a_S(\tilde{i}, 0) \leq a_S(\tilde{\tilde{i}}, 0) \cdot \frac{a_H(\tilde{i}, 0)}{a_H(\tilde{\tilde{i}}, 0)}
\]

For Assumption 2 to continue hold over time, over time it must be that $\forall t$:

\[
a_S(\tilde{i}, t) \leq a_S(\tilde{\tilde{i}}, t) \cdot \frac{a_H(\tilde{i}, t)}{a_H(\tilde{\tilde{i}}, t)}
\]

This is Assumption 3. There are two growth processes used in the dynamic model. The first is used when the model is solved analytically. This is (12) and implies:

\[
a_S(i, t) = a_S(i, 0) \cdot e^{\alpha t}
\]

To see that this maintains Assumption 3, note that (48) and (50) imply:
\[ a^S(\tilde{i}, 0) \cdot e^{gt} \leq a^S(\tilde{i}, 0) \cdot e^{gt} \cdot \frac{a^H(\tilde{i}, 0)}{a^H(\tilde{i}, 0)} \]

\[ a^S(\tilde{i}, t) \leq a^S(\tilde{i}, t) \cdot \frac{a^H(\tilde{i}, 0)}{a^H(\tilde{i}, 0)} \]

and so Assumption 3 holds, since the productivities of labour do not change over time. The second growth process is used when the model is solved computationally, later in the Appendix. This is implied by (86):

\[ a^S(i, t) = a^S(0, 0) \cdot e^{gt} - b \cdot i \]

To see that this maintains Assumption 3, note that (52) and (48) imply:

\[ a^S(\tilde{i}, t) = a^S(0, 0) \cdot e^{gt} - b \cdot \tilde{i} \leq a^S(0, 0) \cdot e^{gt} - b \cdot \tilde{i} \cdot e^{gt} \]

\[ \leq \left( a^S(0, 0) \cdot e^{gt} - b \cdot \tilde{i} \cdot e^{gt} \right) \cdot \frac{a^H(\tilde{i}, 0)}{a^H(\tilde{i}, 0)} \]

\[ \leq \left( a^S(0, 0) \cdot e^{gt} - b \cdot \tilde{i} \right) \cdot \frac{a^H(\tilde{i}, 0)}{a^H(\tilde{i}, 0)} = a^S(\tilde{i}, t) \cdot \frac{a^H(\tilde{i}, 0)}{a^H(\tilde{i}, 0)} \]

Again, since (47) will hold \( \forall t \) so long as:

\[ \dot{a}^S(\tilde{i}, t) \leq \dot{a}^S(\tilde{i}, t) \cdot \frac{a^H(\tilde{i}, t)}{a^H(\tilde{i}, t)} \]

it follows from (53) that Assumption 3 again holds, since the productivities of labour do not change over time.

### 5.3. Law of Motion for \( x(i, t) \)

From (15) it follows that:

\[ x(i, t) = \frac{1}{\mu(t) \cdot p(i, t)} \]
And from (16) that:

\[ \frac{\dot{\mu}(t)}{\mu(t)} = -r^S(t) + \rho \]

It follows from (55) that:

\[ \frac{\dot{x}(i, t)}{x(i, t)} = -\frac{\dot{p}(i, t)}{p(i, t)} - \frac{\dot{\mu}(t)}{\mu(t)} \]

(56) and (57) therefore imply that:

\[ \frac{\dot{x}(i, t)}{x(i, t)} = -\frac{\dot{p}(i, t)}{p(i, t)} + r^S(t) - \rho \]

5.4. Expression for \( p(0, t) \)

To find any \( p(i, t) \), suppose that \( i(t) \in [0, \bar{i}(t)] \). This implies that \( x(i, t) \) is produced by s-capital and c-capital:

\[ x(i, t) = [a^S(i, t)K^S(i, t)]^\psi [a^C(i, t)K^C(i, t)]^{1-\psi} \forall i(t) \in [0, \bar{i}(t)] \]

This implies that the marginal product of s-capital in producing \( x(0, t) \) is:

\[ MPK^S(i, t) = a^S(i, t) \cdot \psi [a^S(i, t)K^S(i, t)]^{\psi-1} [a^C(i, t)K^C(i, t)]^{1-\psi} \]

\[ = \psi \cdot \frac{x(i, t)}{K^S(i, t)} \]

and the marginal product of c-capital in producing \( x(i, t) \) is:

\[ MPK^C(i, t) = a^C(i, t) \cdot (1 - \psi) \cdot [a^L(i, t)L^L(i, t)]^\psi [a^C(i, t)K^C(i, t)]^{-\psi} \]

\[ = (1 - \psi) \cdot \frac{x(i, t)}{K^C(i, t)} \]

Given perfectly competitive profit-maximising firms, the price of each of these factors – \( r^S(t) \) and \( r^C(t) \) – must be equal to their respective marginal revenue
products:

\[ r^S(t) = p(i, t) \cdot \psi \cdot \frac{x(i, t)}{K^S(i, t)} \quad \forall i(t) \in [0, \bar{i}(t)] \]

(62)

\[ r^C(t) = p(i, t) \cdot (1 - \psi) \cdot \frac{x(i, t)}{K^C(i, t)} \quad \forall i(t) \in [0, 1] \]

These can be re-arranged:

\[ K^S(i, t) = p(i, t) \cdot \psi \cdot \frac{x(i, t)}{r^S(t)} \quad \forall i(t) \in [0, \bar{i}(t)] \]

(63)

\[ K^C(i, t) = p(i, t) \cdot (1 - \psi) \cdot \frac{x(i, t)}{r^C(t)} \quad \forall i(t) \in [0, 1] \]

and substituting these expressions for \( K^S(i, t) \) and \( K^C(i, t) \) into (59) implies:

(64)

\[
\begin{align*}
    x(i, t) &= \left[ a^S(i, t) \cdot p(i, t) \cdot \psi \cdot \frac{x(i, t)}{r^S(t)} \right]^{\psi} \left[ a^C(i, t) \cdot p(i, t) \cdot (1 - \psi) \cdot \frac{x(i, t)}{r^C(t)} \right]^{1-\psi} \\
    p(i) &= \left[ \frac{r^S}{\psi \cdot a^S(i)} \right]^\psi \left[ \frac{r^C}{(1 - \psi) \cdot a^C(i)} \right]^{1-\psi}
\end{align*}
\]

(64) is therefore the \( p(i, t) \) for any good \( i(t) \in [0, \bar{i}(t)] \). A similar exercise to derive \( L^H(i, t) \) provides \( p(i, t) \) for those remaining goods \( i(t) \in [\bar{i}(t), 1] \) produced by labour with \( c \)-capital.

### 5.5. Law of Motion for \( \dot{K}^S(i, t) \)

Using (27) and (28), (23) can be written as:

(65)

\[
\begin{align*}
    \frac{\dot{K}^S(t)}{\dot{K}^S(t)} &= \frac{\dot{K}^S(t)\psi \cdot \frac{D}{\psi} - \dot{c}(t)}{\dot{K}^S(t)} - \left[ \frac{\partial \tilde{i}(t)}{\partial \dot{K}^S(t)} \cdot \dot{K}^S(t) \cdot \frac{1}{i(t)} + \frac{\psi}{1 - \psi} \cdot g \right]
\end{align*}
\]

And so:
\[
\dot{\hat{K}}^S(t) = \frac{1}{\bar{i}(t)} \cdot \left[ \hat{K}^S(t)^\psi \cdot \frac{D}{\psi} - \hat{c}(t) \right] - \left[ \frac{\partial \bar{i}(t)}{\partial \hat{K}^S(t)} \cdot \dot{\hat{K}}^S(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1 - \psi} \cdot g \right] \cdot \dot{\hat{K}}^S(t)
\]

\[
= \frac{\hat{K}^S(t)^\psi \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1 - \psi} \cdot g \cdot \hat{K}^S(t) - \hat{c}(t)}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{K}^S(t)} \cdot \dot{\hat{K}}^S(t)}
\]

It follows that:

\[
\frac{\dot{\hat{K}}^S(t)}{\hat{K}^S(t)} = \frac{\hat{K}^S(t)^{\psi - 1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1 - \psi} \cdot g - \frac{\hat{c}(t)}{\hat{K}^S(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{K}^S(t)} \cdot \dot{\hat{K}}^S(t)}
\]

5.6. Transition Path

To show that \(\bar{i}(t)\) increases as \(\hat{K}^S(t)\) increases on the transition path, I first derive the following relationship between the growth rates in \(\bar{i}(t)\), \(K^S(t)\) and \(g\) that must hold:

\[
g \bar{i}(t) = \phi(t) \left[ g^{K^S(t)} + g \right]
\]

where:

\[
\phi(t) = \frac{1 - \bar{i}(t)}{\bar{i}(t) - \bar{i}(t)^2} \cdot y(t) + 1 \quad \text{and} \quad y(t) = \left[ \frac{a^H(\bar{i}(t), t)}{a^H(\bar{i}(t), t) - a^S(\bar{i}(t), t)} \right]
\]

and \(\phi(t)\) has the three properties: \(\phi(t) \geq 0; \phi(t) \leq 1; \text{ and } \lim_{\bar{i}(t) \to 1} \phi(t) = 0\). The expression in (68) implies that the growth rate in the equilibrium cut-off, \(g \bar{i}(t)\), is proportional to the sum of the growth rate in the s-capital stock \(g^{K^S(t)}\) and the productivity of \(K^S(t)\), \(\frac{\psi}{1 - \psi} \cdot g\) where the constant of proportionality \(\phi(t)\) is time dependent with the three features set out above. I now derive this expression for \(g \bar{i}(t)\) and these three properties of \(\phi(t)\).

**Deriving \(g \bar{i}(t)\)**

To derive the expression for \(g \bar{i}(t)\), note that the equilibrium condition, \(A(\bar{i}(t), t) = B(\bar{i}(t), t)\) can be re-arranged as:

36
(70) \[ a^H(\tilde{i}(t), t) \cdot \tilde{i}(t) = [1 - \tilde{i}(t)] \cdot a^S(\tilde{i}(t), t) \cdot \frac{K^S(t)}{L^H} \]

Taking time derivatives of (70) implies a further condition that must hold in any \( t \):

(71) \[ \dot{\tilde{i}}(t) \cdot \frac{a^H_1(\tilde{i}(t), t)}{a^H(\tilde{i}(t), t)} + \dot{\tilde{i}}(t) \cdot \frac{\dot{a}^H(\tilde{i}(t), t)}{a^H(\tilde{i}(t), t)} + \dot{\tilde{i}}(t) \cdot \frac{\dot{a}^S(\tilde{i}(t), t)}{a^S(\tilde{i}(t), t)} + a^S(\tilde{i}(t), t) + \dot{K}^S(t) \]

and substituting in the expression for the growth rates implies that:

(72) \[ g^{\tilde{i}}(t) \left[ \frac{a^H_1(\tilde{i}(t), t)}{a^H(\tilde{i}(t), t)} - \frac{a^S_1(\tilde{i}(t), t)}{a^S(\tilde{i}(t), t)} \right] + \frac{1}{[1 - \tilde{i}(t)]} = g + g^K^S(t) \]

and so:

(73) \[ g^{\tilde{i}}(t) = \phi(t) \left[ g^K^S(t) + g \right] \]

where:

(74) \[ \phi(t) = \left[ \frac{a^H_1(\tilde{i}(t), t)}{a^H(\tilde{i}(t), t)} - \frac{a^S_1(\tilde{i}(t), t)}{a^S(\tilde{i}(t), t)} \right] + \frac{1}{[1 - \tilde{i}(t)]}^{-1} = \frac{1 - \tilde{i}(t)}{[\tilde{i}(t) - \tilde{i}(t)^2] \cdot y(t) + 1} \]

**Deriving the Three Properties of \( \phi(t) \)**

First to see that \( \phi(t) \geq 0 \) note that the weakly positive slope of \( A(i, t) \) implies \( \forall i, t \):\(^{24}\)

(75) \[ A_1(i, t) \geq 0 \]

which, given the definition of \( A(i, t) \), can be re-written as:

(76) \[ \frac{a^H(i, t) \cdot a^S(i, t) - a^S_1(i, t) \cdot a^H(i, t)}{[a^S(i, t)]^2} \geq 0 \]

Since the denominator of the expression in (76) is always positive, this implies...

\(^{24}\)This is Assumptions 2 and 3.
that the following must hold:

\[ a_1^H(i, t) \cdot a_S(i, t) \geq a_1^S(i, t) \cdot a_H(i, t) \]

\[ \frac{a_1^H(i, t)}{a_H(i, t)} \geq \frac{a_1^S(i, t)}{a_S(i, t)} \]  

(77)

Therefore so long as the \( A(i, t) \) schedule is weakly positively sloped, \( y(t) \) remains weakly positive, the denominator in the expression for \( \phi(t) \) in (74) remains positive, and so \( \phi(t) \) also remains weakly positive. To see the second property, that \( \phi(t) \leq 1 \), consider the following proof by contradiction. Assume instead that \( \phi(t) > 1 \). The expression for \( \phi(t) \) in (74) then implies:

\[ \bar{i}(t) \cdot \left[ \frac{a_1^H(\bar{i}(t), t)}{a_H(\bar{i}(t), t)} - \frac{a_1^S(\bar{i}(t), t)}{a_S(\bar{i}(t), t)} \right] + \frac{1}{1 - \bar{i}(t)} < 1 \]

\[ \bar{i}(t) \cdot [1 - \bar{i}(t)] \cdot y(t) < 1 - \bar{i}(t) - 1 \]

\[ \bar{i}(t) \cdot y(t) - \bar{i}(t)^2 \cdot y(t) < -\bar{i}(t) \]  

(78)

But since \( \bar{i}(t) \in [0, 1] \) this condition cannot hold. If \( \bar{i}(t) = 0 \), then the result is a contradiction since (78) requires that \( 0 < 0 \). Similarly if \( \bar{i}(t) > 0 \) then (78) requires:

\[ \frac{1}{1 - \bar{i}(t)} < 0 \]

(79)

which is not possible since \( \bar{i}(t) \in [0, 1] \) and \( y(t) \geq 0 \). The third property, that \( \lim_{\bar{i}(t) \to 1} \phi(t) = 0 \), follows from the fact that:

\[ \lim_{\bar{i}(t) \to 1} \left[ \frac{1}{1 - \bar{i}(t)} \right] = \infty \]

(80)

and that:

\[ \lim_{\bar{i}(t) \to 1} \left[ \bar{i}(t) \cdot \left[ \frac{a_1^H(\bar{i}(t), t)}{a_H(\bar{i}(t), t)} - \frac{a_1^S(\bar{i}(t), t)}{a_S(\bar{i}(t), t)} \right] \right] \geq 0 \]

(81)

And so \( \lim_{\bar{i}(t) \to 1} \phi(t) = 0 \), given (74), (80) and (81).

\( \bar{i} \) During Transition
The proof that $\frac{d\bar{i}(t)}{d\hat{K}^S(t)} \geq 0$ during the conjectured transition in Figure 5 now follows from two relationships. The first is implied by the fact that if $\hat{K}^S(t)$ is increasing, (21) implies:

$$g^{K^S}(t) > g^\bar{i}(t) + \frac{\psi}{1-\psi} \cdot g$$

where $g^{K^S}(t)$ is the growth rate in the stock of s-capital and $g^\bar{i}(t)$ is again the growth rate in the equilibrium cut-off $\bar{i}(t)$. The second relationship is that set out in (68). Combining these two sets of relationships implies that if $\hat{K}^S(t)$ is increasing then:

$$\frac{1-\phi(t)}{\phi(t)} \cdot g^\bar{i}(t) > \frac{1}{1-\psi} \cdot g$$

And so:

$$g^\bar{i}(t) > \frac{\phi(t)}{(1-\psi) \cdot (1-\phi(t))} \cdot g$$

Given the properties of $\phi(t)$ derived before, and that $g > 0$, this implies that if $\hat{K}^S(t)$ is increasing then:

$$g^\bar{i}(t) > 0$$

i.e. $\bar{i}(t)$ is also increasing. (82) also implies $\lim_{\bar{i}(t) \to 1} g^\bar{i}(t) = 0$ given the properties of $\phi(t)$.

### 5.7. Simulation of the Dynamic Model

To make the simulation tractable, I need to explicitly define the absolute productivity schedules for the factors – the $a^C(i, t)$, $a^H(i, t)$, and $a^S(i, t)$ schedules. To do this, I make the following assumptions:
where $a^C$, $a^H$, $b$, and $a^S(0,0)$ are positive constants. The first three expressions in (86) are absolute productivity schedules. They are the simplest possible schedules that generate a relative productivity schedule, $A(i,t)$, with the properties set out in Assumption 2. Intuitively, they imply that the absolute productivity of s-capital is diminishing in $i$, at a constant rate $b$, and the productivity of the other factors is constant across the task-spectrum. The final expression is a growth process. It is the simplest possible process that ensures Assumption 3 is maintained. This is shown in the Appendix. It implies that, again, the productivity of s-capital in producing the numeraire good grows at rate $g$. (86) allows for an explicit solution to be derived for $\tilde{i}(t)$. From the $A(i,t)$ and $B(i,t)$ schedules it follows that in any $t$:

\begin{equation}
\frac{a^H(i,t) \cdot L^H}{(a^S(0,t) - b \cdot \tilde{i}(t)) \cdot (1 - \tilde{i}(t)) \cdot a^S(0,t)^{\psi}} = \hat{K}^S(t)
\end{equation}

and so:

\begin{equation}
\left[ b \cdot a^S(0,t)^{\psi} \right] \cdot \tilde{i}(t)^2 - \left[ a^S(0,t)^{\psi} + b \cdot a^S(0,t)^{\psi} \right] \cdot \tilde{i}(t) + \left[ a^S(0,t)^{\psi} - \frac{a^H(i,t) \cdot L^H}{\hat{K}^S(t)} \right] = 0
\end{equation}

Given the quadratic form of (88), $\tilde{i}(t)$ can be found in a straightforward way.\(^{25}\)

Figure 6 shows the results for one particular parameterisation of the dynamic model. This is solved using the relaxation algorithm from Trimborn et al. (2008).\(^{26}\)

\(^{25}\)Note that the quadratic produces two solutions for $\tilde{i}(t)$ – in each case, the larger can be discarded as it takes place outside $i \in [0,1]$.

\(^{26}\)Here $a^C = 1$, $a^H = 40$, $a^S(0,0) = 10$, $y = 0.05$, $\rho = 0.02$, $\psi = 0.5$, $b = 1$, $D = 1$, $L^H = 50$. Note that the transition path of $r^{S*}(t)$ now does depend upon the normalisation – although the steady-state $r^{S*}$ does not depend upon it. This simulation is not robust to all parameterisations.
6. References


Figure 1: The Relative Productivity Schedule

Figure 2: The Zero-Profit Schedule
Figure 3: Market Equilibrium

Figure 4: A New Market Equilibrium
Figure 5: Dynamic Equilibrium

\[ \hat{K}^*(t) = 0 \]

\[ \hat{c}(t) = 0 \]

\[ \hat{K}^*(t) \text{ and } \hat{r}^* = 1 \]

Figure 6: Simulation of a Dynamic Equilibrium