Reliability-Based Robust Geotechnical Design of Retaining Walls

C. Hsein Juang[1], Zhifeng Liu[2], and H. Sez Atamturktur[2]

[1]Department of Civil Engineering, Clemson University, Clemson, SC 29634-0911, <hsein@clemson.edu>

[2]Department of Civil Engineering, Clemson University, Clemson, SC 29634-0911

ABSTRACT: Geotechnical design often involves high, hard-to-control parameter uncertainties, which result in high variability in the system response. The variability in system response, which is typically addressed by satisfying a minimum safety measure in the form of a factor of safety or reliability index, forces the geotechnical designer to compromise between safety and efficiency (i.e., cost). When robustness of the geotechnical design against such system response variability is not evaluated during the design process, the tradeoff between over-design for safety and under-design for cost-savings is exacerbated. This paper introduces a novel design approach, Reliability-based Robust Design Optimization that considers explicitly the reliability, robustness, and cost. This design methodology is demonstrated with the design of a cantilever retaining wall. System reliability index is used as the performance measure and the tradeoff among the computed reliability index, the variance of the reliability index (as a measure of the robustness), and the cost are investigated. The results show that for some designs (with reliability index between 3 and 3.65), no tradeoff exists between the reliability index and its variance; hence, the design with the greatest reliability index also has the highest robustness (smallest variance) for a given cost. For other designs, a tradeoff relationship exists between the reliability index and its variance for a given cost.

INTRODUCTION

Uncertainties are often divided into two groups: aleatoric and epistemic uncertainties. Aleatoric uncertainty is intrinsic uncertainty, which cannot be reduced with further knowledge and is usually described as random variable. Epistemic uncertainty is due to incomplete knowledge and may be reduced by acquiring more knowledge. In geotechnical engineering, inherent variability in soil parameters (such as cohesion factor c, friction angle ϕ and unit weight γ, etc.) and loading are common aleatoric uncertainties. On the other hand, uncertainty due to inaccurate model, small sample size, and measurement error are examples of epistemic uncertainties.

Two approaches are usually taken by geotechnical engineers to deal with uncertainties: factor of safety (FS)-based approach and reliability-based approach. In a FS-based approach, known as the *deterministic* approach, uncertainty is not considered explicitly; rather, it uses a FS value to account for all uncertainties. Reliability-based approach considers uncertainties *explicitly* in the design, and for simplicity, it is often implemented as partial factors design such as LRFD (Load and Resistance Factor Design), where safety is satisfied by applying load and resistance factors to nominal value of load and resistance, respectively. Load factors and resistance factors dictated by design code are generally calibrated based on reliability models, targeted on getting a design with certain reliability.

Accuracy of the assessed system reliability hinges on the accuracy of the statistical characterization of parameters uncertainty. The statistics (mean and standard deviation) of uncertain soil parameters are often imprecise or uncertain due to small sample size, measurement error, or human error. This leads to the question of how robust the assessed reliability (and the reliability-based design) is with this uncertainty of sample statistics. Without a quantitative measure of this effect, the designer may have to choose an overly conservative design to satisfy the reliability requirement. The proposed methodology: Reliability Based Robust Design Optimization (RRDO) will shed some light on this question by quantitatively considering robustness of the assessed reliability.

Robust Design, a widely used tool in quality engineering (Taguchi 1986; Phadke 1989), aims to make the product of a process insensitive to (or *robust* against) "hard-to-control" parameters (called "noise factors"), by carefully adjusting "easy-to-control" parameters (called "design parameters"). Reported applications of robust design concept in geotechnical engineering are scarce. In this paper, robust design is first introduced with emphasis on Robust Design Optimization using Non-dominated Sorting Genetic Algorithm, NSGA-II (Deb et al. 2002), which is a fast multi-objective optimization method. Then, a RRDO framework is proposed, considering reliability, robustness, and cost as objectives in the optimization. This framework and procedure are demonstrated through robust design of a retaining wall.

OVERVIEW OF ROBUST DESIGN OPTIMIZATION

Deterministic and Reliability-based Optimization

Traditional optimization, which is a deterministic approach, aims at finding a set of design variables **d** (**d** is a vector of design parameters) to minimize the performance function $f(\mathbf{d})$ subjected to a set of constraints functions **g**:

$$\begin{aligned}&\text{Minimize: } f(\mathbf{d})\\&\text{Subject to: } g_i(\mathbf{d}) \le 0 \quad i = 1,\ldots,n\end{aligned} \tag{1}$$

In the presence of uncertainties, optimum design obtained using the deterministic optimization may be infeasible for two reasons: (1) the objective function may deviate from the desired optimum value, and (2) the constraints may be violated. To address the first concern, robustness concept may be introduced; and to address the second concern, reliability-based constraints may be introduced. The reliability-based optimization is discussed first, followed by robust design optimization. Reliability-based optimization can be formulated as (Schueller and Jensen 2008):

$$\begin{aligned}&\text{Minimize: } f(\mathbf{d})\\&\text{Subject to: } [1-\Pr(g_i(\mathbf{d},\mathbf{z})\le 0)]\le \alpha_i \text{ or } \beta_i \ge \beta_{Ti} \quad i=1,...,n\end{aligned} \tag{2}$$

where $f(\mathbf{d})$ is the performance function [often expressed in terms of displacement, stress, or safety measure], $\mathbf{z}$ is uncertain variables vector (noise factors), α_i is the maximum allowable failure probability, β_i is the reliability index, and β_{T_i} is the threshold (or minimum required) reliability index for the i^{th} constraint.

Robust Design Optimization (RDO)

As noted previously, the objective function may deviate from desired optimum value. Thus, the robustness requirement should be considered in the performance function. With the recent advancements in the computational power, robust design through optimization is gaining popularity (Beyer and Sendhoff 2007). The Robust Design Optimization (RDO) is a new design concept that considers the robustness requirement in the performance function. RDO can be treated deterministically or probabilistically. In deterministic RDO (Lewis 2002), min-max optimization is performed by searching for **d** to minimize *worst-case performance* due to uncertainty. Beyer and Sendhoff (2007) concluded that deterministic RDO usually results in overly conservative design, which is undesirable. In the probabilistic RDO, first two statistical moments of the performance function are minimized simultaneously:

$$\begin{aligned}&\text{Minimize:} [\mu_f(\mathbf{d},\mathbf{z}), \sigma_f(\mathbf{d},\mathbf{z})]\\&\quad \text{where} \quad \mu_f(\mathbf{d},\mathbf{z}) = \int f(\mathbf{d},\mathbf{z})\cdot p(\mathbf{z})\cdot d\mathbf{z}\\&\qquad\qquad \sigma_f{}^2(\mathbf{d},\mathbf{z}) = \int (f(\mathbf{d},\mathbf{z})-\mu_f(\mathbf{d},\mathbf{z}))^2\cdot p(\mathbf{z})\cdot d\mathbf{z}\\&\text{Subject to:} \, g_i(\mathbf{d},\mathbf{z})\le 0 \quad i=1,...,n\end{aligned} \tag{3}$$

where $p(\mathbf{z})$ is joint probability distribution function of $\mathbf{z}$; μ_f and σ_f are the mean and standard deviation of f, which are the measures of "performance" and "robustness" respectively. Both μ_f and σ_f are minimized simultaneously in RDO.

The concept and significance of robustness are illustrated with Fig. 1. As shown in Fig. 1, three designs result in different system responses (such as displacement). Design A, which has a smaller mean response value than Design B, is not necessarily a better design, since it has a very large variance. On the other hand, minimizing σ_f alone may not be adequate. As also shown in Fig. 1, Design C, which has a smaller variance than Design B, is not necessarily a better design, since its mean response value is much larger. When μ_f and σ_f are to be minimized simultaneously, the process becomes a multi-objective optimization problem.

Statistical Moment Estimation

Statistical moments such as the mean and variance can be computed using Monte Carlo simulation, Taylor expansion, or numerical integration methods (Deodatis and Shinozuka 1988; Doltsinis and Kang 2005; Huang and Du 2007). In this paper, Taylor expansion method is used. With this method, the expansion of the performance function with respect to noise factors is integrated to determine the statistical moments, as shown below:

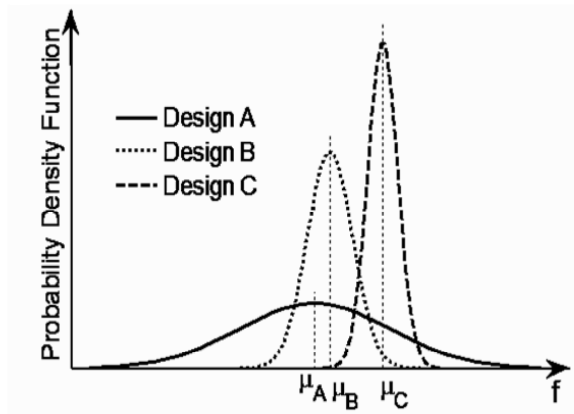

FIG 1. Comparison of three designs

$$y = f(\mathbf{d},\mathbf{z}) = f(\mathbf{d},\boldsymbol{\mu}_\mathbf{z}) + \sum_{i=1}^{m}\left(\left.\frac{\partial f}{\partial z_i}\right|_{\mu_{z_i}}\right)(z_i - \mu_{z_i})$$

$$E(y) = f(\mathbf{d},\boldsymbol{\mu}_\mathbf{z})$$

$$Var(y) = E\left[\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\left.\frac{\partial f}{\partial z_i}\right|_{\mu_{z_i}}\right)\cdot\left(\left.\frac{\partial f}{\partial z_j}\right|_{\mu_{z_j}}\right)\cdot(z_i - \mu_{z_i})\cdot(z_j - \mu_{z_j})\right] = \nabla^{\mathrm{T}}\cdot\mathbf{C}\cdot\nabla \tag{4}$$

Where: $\boldsymbol{\mu}_\mathbf{z}$ – mean value vector of $\mathbf{z}$

E, Var – expectancy and variance operator, respectively

m – number of noise factors

∇ – gradient vector evaluated at mean value

$\mathbf{C}$ – covariance matrix, $[C]_{ij} = E[(z_i - \mu_{Z_i})(z_j - \mu_{Z_j})]$.

Numerical integration may be implemented with point estimate method (PEM). Rosenblueth (1975) first suggested that continuous integration could be approximated by evaluating the weighted sum of functions at finite sampling points. This method is further developed by Zhao and Ono (2000) for improved accuracy.

Multi-Objective Optimization using NSGA-II

Multi-objective optimization can be solved with many optimization methods, such as goal programming, compromise programming (Chen et al. 1999), and the weighted sum method. Marler (2003) provides a survey of multi-objective optimization. In this paper, Non-dominated Sorting Genetic Algorithm version II (NSGA-II), developed by Deb et al. (2002), is adopted. A multi-objective optimization problem may be formulated as follows:

$$\text{Minimize: } \mathbf{F}(\mathbf{d}) = \left[f_1(\mathbf{d})\, f_2(\mathbf{d}) \; \ldots f_l(\mathbf{d}) \right] \quad (5)$$
$$\text{Subject to: } g_i(\mathbf{d}) \le 0 \;\; i = 1,\ldots,n$$

Unlike single objective optimization, which aims to yield a single best design, in multi-objective optimization, typically no single best design that is superior to all other designs in all objectives can be found. In Fig. 2, such single best design is referred to as the *utopia* point, which is unattainable in most cases since the objectives may conflict with each other. However, there generally exists a set of designs in the solution space (i.e., set $\{\mathbf{F}(\mathbf{d}) \mid g_i(\mathbf{d}) \le 0 \text{ for all i}\}$), which are superior to all other designs; but within this set, no design is superior to another in all criteria. These designs constitute a Pareto optimum set or Pareto-Front (Gencturk and Elnashai, 2011), as shown in Fig. 2. In the Pareto-Front, no improvement can be made with respect to one objective without worsening the other objective.

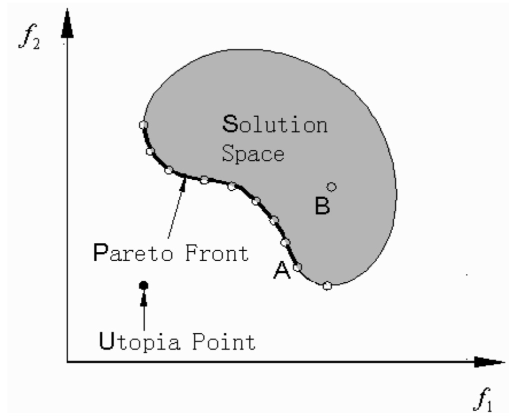

FIG 2Illustration of Pareto Front

The Pareto-Front can also be viewed as a set of designs that are *not dominated* by any design in the solution space. Here, the "domination" relation is defined as follows: design B is *dominated* by design A if B is inferior or equal to A in *every* objective measure, with one exception that B is not dominated by A if B is equal to A in *all* objective measures. Though Pareto Front can be constructed by comparing all designs and then selecting all non-dominated designs, it is not efficient to do so. Optimization methods discussed earlier can be used instead, which are much more efficient than the enumeration scheme.

RELIABILITY-BASED ROBUST DESIGN OPTIMIZATION

As discussed previously, model error or uncertainty in sample statistics makes reliability uncertain. Model error may be considered as a random variable in reliability analysis; so only sample statistic effect on the reliability will be studied in this paper. Several approaches are available for evaluating the effect of insufficient sample size. Ditlevsen and Madsen (1996) model samples statistics with random variables. Most (2010) use bootstrap method to calculate confidence interval of reliability. In this paper, sample statistics is considered probabilistically with assumed mean and variation.

Thus, RDO is further modified, as formulated in Equation (3), into Reliability-based Robust Design Optimization (RRDO). In the RRDO, the performance function is specifically expressed in terms of reliability index, and the robustness is measured with the variance (or standard deviation) of the reliability index. Furthermore, the cost of a given design [C(d)] is added as the third objectives, in addition to reliability [in terms of $\mu_\beta(\mathbf{d},\mathbf{z})$] and robustness [in terms of $\sigma_\beta(\mathbf{d},\mathbf{z})$]. Thus, the RRDO may be formulated as:

$$\begin{aligned} &\text{Minimize:} [-\mu_\beta(\mathbf{d},\mathbf{z}), \sigma_\beta(\mathbf{d},\mathbf{z}), \mathrm{C}(\mathbf{d})] \\ &\text{Subject to:} \; g_i(\mathbf{d},\mathbf{z}) \le 0 \quad i = 1,...,n \end{aligned} \tag{6}$$

In reliability-based optimization, target reliability must be specified prior to decision-making. In the proposed RRDO approach, however, no preference needs to be specified before optimization. A Pareto-Front that reflects the tradeoff relationship between cost, reliability and robustness will be acquired after optimization. All designs in the Pareto-Front are safe designs (i.e., constraint is satisfied), while they are equally good in one or more aspect. With the Pareto Front, geotechnical engineers have a broader view about the relation between reliability, robustness and cost. What safety and robustness levels can be anticipated with an increase or decrease in the cost can be readily known from Pareto Front. Compared to a fixed design obtained from reliability-based optimization, geotechnical engineer may feel more comfortable to choose a design from the Pareto-Front based on the cost and safety requirements.

The tradeoff between the mean reliability index and the cost is easily understood. However, the relationship between the mean reliability index and its variance is not as obvious for some problem. Thus, understanding such tradeoff relationship can be beneficial in achieving a safe and efficient design.

EXAMPLE: RETAINING WALL DESIGN

A retaining wall design is used as an example to demonstrate the RRDO framework. The required height above ground surface is H_w = 3.66 m (12ft); the backfill is cohesionless soil (ϕ = 35°, $c = 0$); and the design parameters are x_1, x_2, x_3, and x_4, as shown in Fig. 3. Proportional ranges for these design variables are: $0.5H_w \le x_1 \le H_w$, $0 \le x_2 \le 0.5H_w$, $(H_w/12) \le x_3 \le (H_w/10)$, and $(H_w/12) \le x_4 \le (H_w/10)$. For this example, a horizontal backfill (λ= 0) and a vertical wall face (α = 90°) are assumed, although cases with $\lambda \neq 0$ and $\alpha \neq 90°$ can be considered. Random variables considered are friction angle (ϕ), wall-backfill interface friction angle (δ), and soil unit weight (γ). These noise factors are assumed to follow Normal distribution for simplicity (although other distribution types may be used). Symbolically, their means and standard deviations (sample

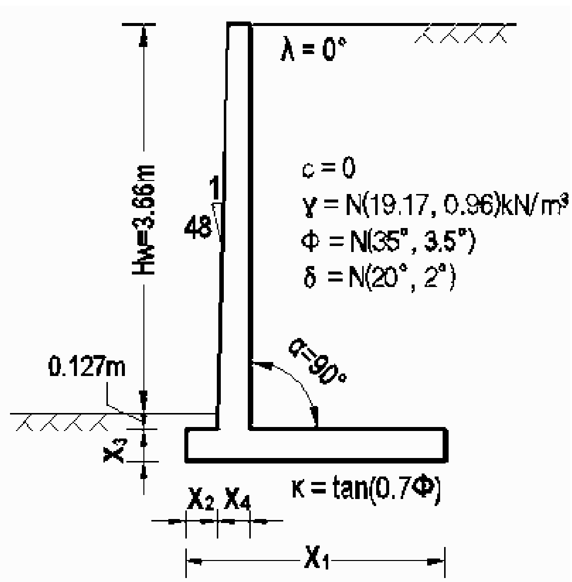

FIG 3. Illustration of the example retaining wall

statistics) are denoted as $\overline{X_i}$ and S_i respectively (where i = 1, 2, 3 for ϕ, δ, and γ, respectively). For this example, the values of $\overline{X_i}$ and S_i are listed in Table 1. The parameters δ and ϕ are assumed to be correlated with a correlation coefficient of 0.8. The coefficient of friction for sliding at the base is assumed herein to be $\kappa = \tan(0.7\phi)$, although this friction angle between the base and the soil beneath it may also be treated as a random variable if so desired. The purpose of robust design is to ensure the system response (in this example, system reliability) of a selected design is less sensitive to, or robust against, the variation of sample statistics.

TABLE 1. Random variables for retaining wall design

Noise factors (random variables)	$\overline{X}$	$\text{cov} = S/\overline{X}$
ϕ (º)	35	0.1
δ (º)	20	0.1
γ (kN/m^3)	19.17	0.05

For geotechnical design, the design parameters are to be selected so that the design satisfies three stability requirements: (1) sliding failure, (2) overturning failure, and (3) bearing capacity failure. For this example, the lateral earth pressure is computed using Coulomb theory, in which the active earth pressure coefficient is expressed as (Coulomb, 1776; Perloff and Baron, 1976):

$$K_a = \left[\frac{\sin(\alpha+\phi)/\sin(\alpha)}{\sqrt{\sin(\alpha-\delta)} + \sqrt{\sin(\phi+\delta)\sin(\phi-\lambda)/\sin(\alpha+\lambda)}} \right]^2 \tag{7}$$

Reliability analysis can be performed to compute the reliability index for each of the three failure modes. In this paper, the system reliability, an overall reliability derived from reliability indexes against the three failure modes, is used as the performance measure. Assuming that these failure modes are completely uncorrelated, the system failure probability and reliability index can be computed as follows (Ditlevsen, 1979):

$$\Pr_{sys} = 1 - \prod_{i=1}^{l} [1 - \Phi(\beta_i)] \tag{8}$$

$$\beta_{sys} = \Phi^{-1} \left\{ \prod_{i=1}^{l} [1 - \Phi(\beta_i)] \right\} \tag{9}$$

To apply the RRDO methodology, Eq. (6) is implemented with three objectives with respect to the system reliability index [$\mu_\beta(\mathbf{d},\mathbf{z})$], the standard deviation of the system reliability index [$\sigma_\beta(\mathbf{d},\mathbf{z})$], and the cost [$\mathrm{C}(\mathbf{d})$]. It is noted that the standard deviation [$\sigma_\beta(\mathbf{d},\mathbf{z})$] is used herein as a measure of robustness (smaller standard deviation implying higher robustness). As in most optimization problem, area of the retaining wall is used to represent the cost. Thus, the robust design of this retaining wall can be set up as a three-objective optimization problem, and RRDO is formulated as:

$$\begin{aligned}
&\text{Find}: \mathbf{d} = [x_1\ x_2\ x_3\ x_4] \\
&\text{To minimize: } [-\mu_\beta(\mathbf{d},\mathbf{z}), \sigma_\beta(\mathbf{d},\mathbf{z}), \text{Area}(\mathbf{d})] \\
&\text{Subject to: } \mu_\beta > 2.6;\ e < x_1/6 \\
&\qquad 0.5H_w \le x_1 \le H_w;\ 0 \le x_2 \le 0.5x_1 \\
&\qquad H_w/12 \le x_3 \le H_w/10;\ H_w/12 \le x_4 \le H_w/10
\end{aligned} \tag{10}$$

where e is the eccentricity. Note that the constraint $e < x_1/6$ is required to prevent tension at the base, and the constraint of system reliability $\mu_\beta > 2.6$ is chosen to satisfy serviceability limit-state requirement (Phoon et al., 1995). The search for optimal design parameters is carried out with NSGA-II algorithm (Deb et al., 2002). Each design parameter is assigned a discrete integer value (Cai and Thierauf, 1996).

For each design, the reliability index for each stability component (sliding, overturning or bearing capacity failure mode) is computed, and then the system reliability index β_{sys} is determined. Recall that there are three random variables, namely, δ, ϕ, and γ, as shown in Table 1. Uncertainty in sample statistics ($\overline{X}$ and S^2), due to insufficient sample size, measurement error or human error, is described probabilistically as shown in Equation (11) with z_i (i =1 to 3 for $\overline{X}$, $i+3$ =4 to 6 for S^2). Mean values of the sample statistics are set as those listed in Table 1, while the coefficients of variation are assumed to be 0.1 and 0.3 for $\bar{X}$ and S^2 respectively without losing generality, since S^2 contains much greater uncertainty than $\bar{X}$. Thus,

$$\begin{aligned}
z_i &= N\left(\overline{X}_i, (0.1\overline{X}_i)^2\right); \qquad \text{for } i = 1,2,3 \\
z_{i+3} &= N\left(S_i^2, (0.3S_i^2)^2\right); \qquad \text{for } i+3 = 4,5,6
\end{aligned} \tag{11}$$

Because of the uncertainty in sample statistics, the system reliability is uncertain; thus, it is necessary to compute μ_β and σ_β. Recall that in RRDO framework, μ_β addresses the reliability requirement and σ_β addresses the robustness requirement. In this study, Taylor expansion method (Eq. 4) is used to calculate μ_β and σ_β. Here, the gradient $\partial\beta_{sys}/\partial z_i$ is computed with central differential method. To this end, a suitable step size is required. Convergence test show that a step size of $\sigma_z/8$ will yield an accurate determination of the gradient $\partial\beta_{sys}/\partial z_i$.

Figure 4(a) shows the Pareto-Front obtained with the RRDO optimization with respect to the three objectives, $\mu_\beta(\mathbf{d},\mathbf{z}), \sigma_\beta(\mathbf{d},\mathbf{z})$, and $\text{Area}(\mathbf{d})$. Fig. 4(b) shows a projection of this front on the $\mu_\beta - \text{Area}$ plane. The results show that when μ_β is in the range 2.6 to 3 (Fig. 4b), there is a clear tradeoff relationship between μ_β and σ_β. For a given *Area* (cost) value, a set of competing designs exist, as each point on the Pareto-Front is not dominated by all other points. Thus, a tradeoff decision between greater reliability (higher μ_β) and higher robustness (lower σ_β) must be made in the design. When μ_β is in the range of approximately 3 to 3.65 (Fig. 4b), no tradeoff relationship exist between μ_β and σ_β, as for a given *Area* value, there exists one design that dominates all other designs. Therefore, the plot of μ_β versus *Area* is reduced to a line; in other words, for a given *Area* value, a design with the highest μ_β value (implying greatest reliability) also has a smallest σ_β (implying highest robustness). When μ_β> 3.65, again, a set of competing designs exist, and thus the tradeoff between μ_β and σ_β exists, albeit not as obvious. When the tradeoff decision between μ_β (reliability) and *Area* (cost) must be made, the derived Pareto-Front is a valuable tool to aid in the decision-making.

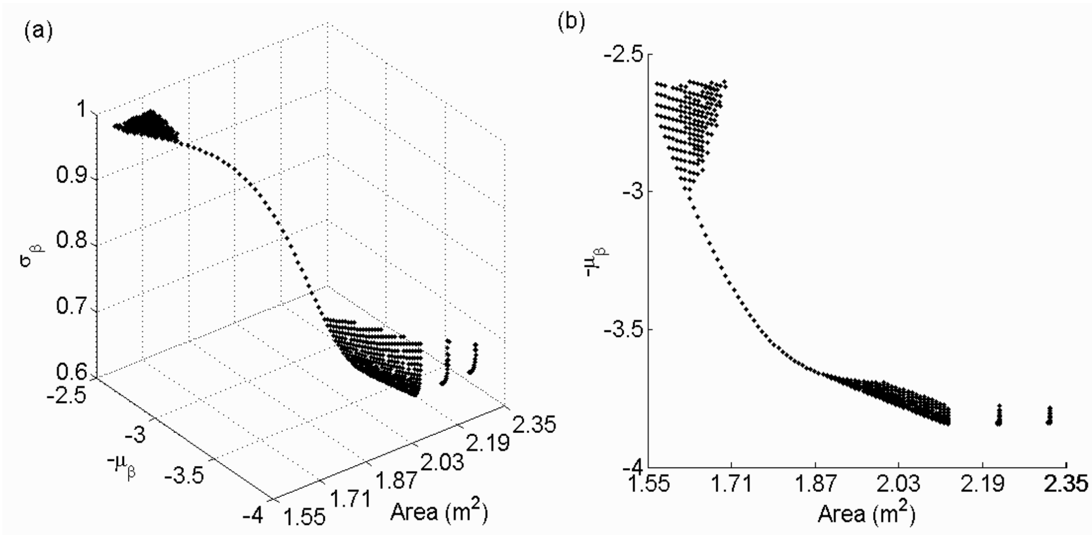

FIG 4. Pareto Front obtained using the RRDO

Fig. 5 further shows the tradeoff between μ_β and σ_β for different ranges of Area (cost). When μ_β is in the range 2.6 to 3 (right part of Fig. 5a), there is an obvious tradeoff relationship. If reliability is of greater concern, a design with higher μ_β should be chosen, which will result in a greater σ_β (or lower robustness). On the other hand, if a higher robustness is preferred, a design with smaller σ_β may be chosen.

When μ_β is in the range 3 to 3.65, which includes left part of Fig. 5a, left part of Fig. 5b, and right part of Fig. 5c, there is no tradeoff relationship for a given Area (cost): higher μ_β will result in a smaller σ_β (or greater robustness). In such case, the design with the highest μ_β should be chosen if the cost is not a concern. Otherwise, there is a tradeoff decision to be made between reliability (μ_β) and cost (Area). When μ_β> 3.65, which includes lower left part of Fig. 5c, and all of Figs. 5d and 5e, the Pareto-Front consists of many μ_β - σ_β tradeoff relationships in each Area range. Thus, the tradeoff between reliability and robustness must be made for a specific Area (cost).

CONCLUDING REMARKS

A new design methodology, Reliability-based Robust Design Optimization (RRDO) is proposed, which considers reliability, robustness of reliability, and cost as the design objectives. To demonstrate this methodology, RRDO is applied to the design of a cantilever retaining wall. The results are presented in terms of a Pareto-Front, which depicts acceptable designs with three competing objectives, the reliability index (system response), the robustness of reliability index, and the cost. In the example presented herein, Pareto-Front proved itself an effective design aid. The Robust Geotechnical Design methodology, implemented herein as RRDO, is novel and has the potential to improve simultaneously reliability and cost by reducing the sensitivity of the design to the hard-to-control noise factors (i.e., soil parameters and construction factors). The proposed RRDO approach has the potential to improve a wide range of geotechnical design problems.

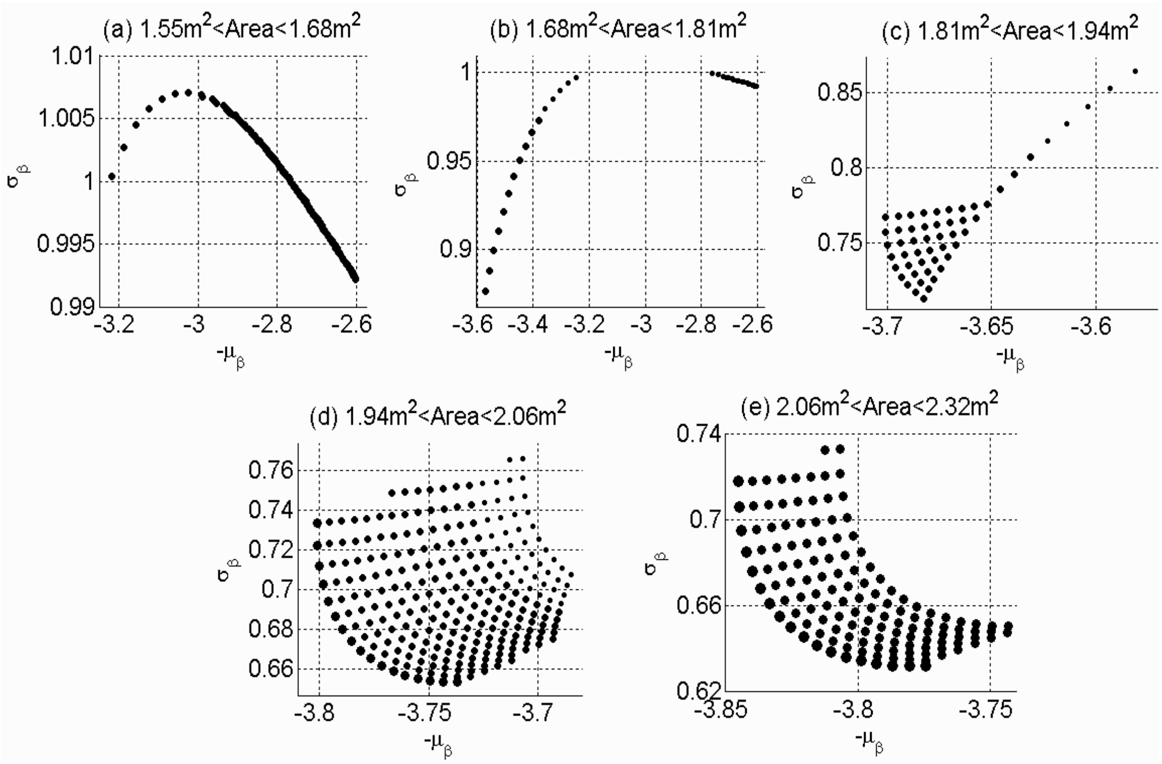

FIG 5. Tradeoff relationship between μ_β and σ_β for different ranges of area

ACKNOWLEDGMENTS

The study on which this paper is based was supported in part by National Science Foundation through Grant CMMI-1200117 and the Glenn Department of Civil Engineering, Clemson University. The results and opinions expressed in this paper do not necessarily reflect the view and policies of the National Science Foundation. The first author wishes to thank Professor Robert D. Holtz for his advices during Purdue years and throughout the first author's professional career.

REFERENCES

Beyer, H., and Sendhoff, B. (2007). "Robust optimization - A comprehensive survey." *Comput. Methods Appl. Mech. Eng.,* 196(33-34), 3190-3218.

Cai, J., and Thierauf, G. (1996). "Evolution strategies for solving discrete optimization problems." *Adv. Eng. Software,* 25(2-3), 177-183.

Chen, W., Wiecek, M. M., and Zhang, J. (1999). "Quality utility - a compromise programming approach to robust design." *J Mech Des, Trans ASME,* 121(2), 179-187.

Coulomb, C. A. (1776). Essai sur une application des regles des maximis et minimis a quelquels problemesde statique relatifs, a la architecture. *Mem. Acad. Roy. Div. Sav.,* Vol. 7, 343–387.

Deb, K., Pratap, A., Agarwal, S., and Meyarivan, T. (2002). "A fast and elitist multiobjective genetic algorithm: NSGA-II." *IEEE Transactions on Evolutionary Computation,* 6(2), 182-97.

Deodatis, G., and Shinozuka, M. (1988). "Stochastic FEM analysis of nonlinear dynamic problems." *Probabilistic Methods in Civil Engineering, Proceedings of the 5th ASCE Specialty Conference,* ASCE, Blacksburg, VA, 152-155.

Ditlevsen, O., (1979). "Narrow reliability bounds for structural systems" *Journal of Structural Mechanics, ASCE* 7(4), 453-472.

Ditlevsen, O., and Madsen, H. (1996). *Structural reliability methods*. Wiley, New York.

Doltsinis, I., and Kang, Z. (2004). "Robust design of structures using optimization methods." *Comput. Methods Appl. Mech. Eng.,* 193(23-26), 2221-37.

Rosenblueth, E., (1975). “Point estimates for probability moments.” *Proc. Natl. Acad. Sci.* U.S.A., Vol. 72, No. 10, pp. 3812-3814.

Gencturk, B., and Elnashai, A. S. (2011). "Multi-objective optimal seismic design of buildings using advanced engineering materials." *MAE Center Report No. 11-01*, Mid-America Earthquake Center, Univ. of Illinois at Urbana-Champaign, IL.

Huang, B., and Du, X. (2007). "Analytical robustness assessment for robust design." *Structural and Multidisciplinary Optimization,* 34(2), 123-137.

Lewis, A. (2002). “Robust regularization” *Technical Report*, Simon Fraser University, Vacouver, Canada.

Marler, R. T., and Arora, J. S. (2004). "Survey of multi-objective optimization methods for engineering." *Structural and Multidisciplinary Optimization,* 26(6), 369-95.

Most, T., and Knabe, T. (2010). “Reliability analysis of bearing failure problem considering uncertain stochastic parameters.” *Computers and Geotechnics*, 37(3), 299-310.

Park, G., Lee, T., Lee, K. H., and Hwang, K. (2006). "Robust design: An overview." *AIAA J.,* 44(1), 181-191.

Perloff, W. H., Baron, W. (1976). *Soil mechanics: principles and applications*, Ronald Press Co., New York.

Phadke, M. S., (1989). *Quality engineering using robust design*, Prentice Hall, Englewood Cliffs, New Jersey.

Phoon, K. K., Kulhawy, F. H., and Grigoriu, M. D. (1995). “Reliability-based design of foundations for transmission line structures.” *Technical Report TR-105000*, Electric Power Research Institute, Palo Alto, California.

Schueller, G. I., and Jensen, H. A. (2008). "Computational methods in optimization considering uncertainties - An overview." *Comput. Methods Appl. Mech. Eng.,* 198(1), 2-13.

Taguchi, G., *Introduction to Quality Engineering: Designing Quality into Products and Processes, Quality Resources*, White Plains, NY, 1986.

Zhao, Y., and Ono, T. (2000). "New point estimates for probability moments." *J. Eng. Mech.,* 126(4), 433-436.