# What drives the time evolution of the spacetime geometry?

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**Abstract**

I show that in a general, dynamic spacetime, the rate of change of gravitational momentum is related to the difference between the number of bulk and boundary degrees of freedom. All static spacetimes maintain holographic equipartition; i.e., in these spacetimes, the number of degrees of freedom in the boundary is equal to the number of degrees of freedom in the bulk. It is the departure from holographic equipartition that drives the time evolution of the spacetime. This result, which is equivalent to Einstein's equations, provides an elegant, holographic, description of spacetime dynamics.[1]

The *mathematical* answer to the question in the title is given by Einstein's equation $G^a_b = 8\pi T^a_b$, which determine the metric in terms of the matter source. But what does this equation mean *physically*? I will show that one can provide an elegant, holographic answer in terms of an alternative equation:

$$\int_{\mathcal{V}} \frac{d^3x}{8\pi} h_{ab} \pounds_\xi p^{ab} = \frac{1}{2} k_B T_{\rm avg} (N_{\rm bulk} - N_{\rm sur}) \tag{1}$$

Here, $h_{ab}$ is the induced metric on the $t =$ constant surface, $p^{ab}$ is its conjugate momentum and $\xi^a = N u^a$ is the proper-time evolution vector corresponding to observers moving with four-velocity $u_a = -N\nabla_a t$ which is the normal to the $t =$ constant surface. The $N_{\rm sur}$ and $N_{\rm bulk}$ are the degrees of freedom in the surface and bulk of a 3-dimensional region $\mathcal{V}$ and $T_{\rm avg}$ is the average Davies-Unruh temperature [1] of the boundary. The left hand side is the time rate of change of gravitational momentum which is driven by the departure from holographic equipartition, indicated by a non-zero value for $(N_{\rm bulk} - N_{\rm sur})$. The time evolution will cease when $N_{\rm sur} = N_{\rm bulk}$ and, in fact, all static geometries obey this condition of holographic equipartition. The validity of Eq. (1) for all observers (i.e., foliations) ensures the validity of Einstein's equations; thus, Eq. (1) carries the same physical content as the gravitational field equations. In short, *holographic equipartition dictates the evolution of spacetime geometry.*

I will now describe how this result arises [2]. Several recent investigations suggest that the gravitational field equations have the same status as the equations of elasticity

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or fluid mechanics (for reviews, see e.g., [3]). This connection becomes most apparent when we use $f^{ab} \equiv \sqrt{-g}g^{ab}$ as the dynamical variable (instead of the usual $g_{ab}$) and the corresponding canonical momenta $N^c_{ab}$ defined by:

$$N^a_{bc} = -\Gamma^a_{bc} + \frac{1}{2}(\Gamma^d_{bd}\delta^a_c + \Gamma^d_{cd}\delta^a_b) \tag{2}$$

The variations of these *dynamical* variables ($f^{ab}\delta N^c_{ab}$, $N^c_{ab}\delta f^{ab}$) and the variations of the *thermodynamic* variables ($S\delta T, T\delta S$) have an one-to-one correspondence [4] when evaluated on the null surfaces.[2]

It also turns out that a very similar combination $f^{ab}\pounds_v N^i_{ab}$ occurs in the expression for the conserved current associated with a vector field $v^a$. If we decompose the derivative $\nabla_k v_j$ of any vector field $v^j$ into the symmetric and anti-symmetric parts by $\nabla^{(j}v^{k)} \equiv S^{jk}$ and $\nabla^{[j}v^{k]} \equiv J^{jk}$, then the anti-symmetric part $J^{lm}$ leads to a conserved current $J^i \equiv \nabla_k J^{ik}$; in other words, from every vector field $v^k$ in the spacetime, we can obtain a conserved current, quite trivially. To find a more useful form for this current, we proceed as follows: From the Lie derivative of the connection $\pounds_v \Gamma^a_{bc} = \nabla_b \nabla_c v^a + R^a{}_{cmb} v^m$, one can obtain, on using Eq. (2), the relation: $g^{bc}\pounds_v N^a_{bc} = \nabla_b J^{ab} - 2R^a_b v^b$. This gives us the explicit form of the conserved current

$$J^a[v] = \nabla_b J^{ab}[v] = 2R^a_b v^b + g^{ij}\pounds_v N^a_{ij} \tag{3}$$

In fact, this *is* the standard Noether current associated with $v^a$ which we have derived *without ever mentioning the action principle for gravity or its diffeomorphism invariance!*.

While Eq. (3) associates a conserved current (and charge) with *any* vector field, those related to the vector, describing the time evolution, are of special interest. The vector $\xi^a = Nu^a$ measures the proper-time lapse corresponding to the normal $u_a = -N\nabla_a t$ of the $t =$ constant surfaces in any spacetime. (In static spacetimes, for example, we can choose $\xi^a$ to be the Killing vector.) An elementary calculation shows [2] that the Noether charge associated with $\xi^a$ has a simple, nice form with direct thermodynamic meaning. We get:

$$u_a J^a(\xi) = 2D_\alpha(Na^\alpha) \tag{4}$$

where $a^i \equiv u^j \nabla_j u^i$ is the acceleration and $D_\alpha$ is the covariant derivative on the $t =$ constant surface. The acceleration $a_i$ has the explicit form $Na_i = Nu^l \nabla_l u_i = h^j_i \nabla_j N$. Integrating Eq. (4) over $\sqrt{h}d^3x$ to obtain the total Noether charge, we find that the flux of the acceleration is essentially the total Noether charge contained inside a volume. Noting that we have set $16\pi G = 1$ and adding the correct proportionality constant (now with $G = L_P^2$!), we get:

$$\int_{\mathcal{V}} \sqrt{h}\, d^3x\ u_a J^a[\xi] = \int_{\mathcal{V}} d\Sigma_a J^a[\xi] = \int_{\partial\mathcal{V}} \frac{\sqrt{\sigma}\, d^2x}{8\pi L_P^2}(Nr_\alpha a^\alpha) \tag{5}$$

This result is valid for any region $\mathcal{V}$ in any spacetime.

[2] I use the $(-+++)$ signature and units with $c = 1, \hbar = 1, k_B = 1, 16\pi G = 1$, so that Einstein's equations reduce to $2G_{ab} = T_{ab}$. The Latin letters run through 0-3 while the Greek letters run through 1-3.

Let us now choose the boundary to be a $N(t, \mathbf{x}) =$ constant surface within the $t =$ constant surface. In the above expression, $r_\alpha$ is then the normal to the $N(t, \mathbf{x}) =$ constant surface within the $t =$ constant surface. Therefore, one can write $r_\alpha \propto D_\alpha N$ or $r_i \propto h_i^j \nabla_j N$ where $h_j^i = \delta_j^i + u^i u_j$ is the projection tensor to the $t =$ constant surface. Since $Na_i = h_i^j \nabla_j N$, it follows that $r_i$ and $a_i$ are in the same direction even in the most general (non-static, $N_\alpha \neq 0$) case. This leads to $N r_\alpha a^\alpha = Na = (h^{ij} \nabla_i N \nabla_j N)^{1/2}$. So, if we choose the boundary to be a surface with $N =$ constant (which is a generalization of the notion of an equipotential surface), we can interpret $T_{\text{loc}} = N r_\alpha a^\alpha / 2\pi = Na/2\pi$ as the (Tolman redshifted) local Davies-Unruh temperature [1] of the observers with four-velocity $u_a = -N \delta_a^0$. These observers, who are moving normal to the $t =$constant hypersurfaces will have the acceleration $a$ with respect to the local freely falling observers. The local vacuum of the freely falling frame will appear to be a thermal state with temperature $T_{\text{loc}} = Na/2\pi$ to these observers. So we can write:

$$2 \int_{\mathcal{V}} \sqrt{h}\, d^3x\ u_a J^a[\xi] = \int_{\partial \mathcal{V}} \frac{\sqrt{\sigma}\, d^2x}{L_P^2} \left( \frac{1}{2} T_{\text{loc}} \right) \tag{6}$$

Thus, (twice) the Noether charge contained in a $N =$ constant surface is equal to the equipartition energy of the surface when we attribute one degree of freedom to each cell of Planck area $L_P^2$. Another, equivalent interpretation emerges, if we think of $s = \sqrt{\sigma}/4L_P^2$ as the analogue of the entropy density. Then we get, directly from Eq. (5), the result:

$$\int_{\mathcal{V}} \sqrt{h}\ d^3x\ u_a J^a[\xi] = \int_{\partial \mathcal{V}} d^2x\ Ts \tag{7}$$

which is the heat (enthalpy) density ($TS/A$) of the boundary surface. Thus, the Noether charge for the time-development vector, contained in a region of space bounded by an $N(t, \mathbf{x}) =$ constant surface, is equal to the surface heat content. This delightfully simple interpretation is valid in *the most general* context without any assumptions like static nature, existence of Killing vectors, asymptotic behaviour, etc.

Incidentally, the factor 2 on the left hand side of Eq. (6), also solves a puzzle familiar to general relativists. The integral on the right of Eq. (6) gives $(1/2)TA = 2TS$ if we take (for the sake of illustration) $T=$ constant on the boundary and $S = A/4$. Therefore, the Noether charge $Q$ is just the heat content (enthalpy) $Q = TS$, which is also clear from Eq. (7). Thus, the Noether charge is *half* of the thermal, equipartition energy of the surface $(1/2)TA = 2TS$ if we attribute $(1/2)T$ per surface degree of freedom. In the case of the Schwarzschild geometry, for example, the thermal, equipartition energy of the surface is just the total mass $M = 2TS$. *But what the Noether charge measures is the heat content (enthalpy)* $E - F = TS$ *which is precisely* $(M/2)$. This leads to a well-known "problem" when one tries to define the total mass of a spacetime (which asymptotically tends to the Schwarzschild limit) using the so-called Komar integral. In this context, $\xi^a$ will become the standard timelike Killing vector and the Noether potential will be the Komar potential. The integral one performs with the Killing vector $\xi^a$ is identical to the computation of the Noether charge above and one gets $(M/2)$. In classical relativity, this was considered very puzzling because, in classical general relativity, we (at best!) only have a notion of energy but no notion of heat content ($TS$), free energy ($F = E - TS$), etc. The thermodynamic perspective —

which requires $\hbar$ to define the Davies-Unruh temperature $k_B T = (\hbar/c)(\kappa/2\pi))$ from an acceleration $\kappa$ — tells us that the Noether charge is the heat content (enthalpy) $TS$ and *not* the energy $2TS$, and that the result *must* be $M/2$ for consistency. In short, classical general relativity can only interpret $M$ physically (as energy), while the thermodynamic considerations allow us to *also* interpret $M/2$ physically as the heat content $TS$. *This is yet another case of thermodynamic considerations throwing light on some puzzling features of classical general relativity.*

Let us next consider the main theme, viz. the dynamics of spacetime. To do this, we take the dot product of the Noether current $J^a[\xi]$ (given in Eq. (3) with $v^a = \xi^a$) with $u_a$, use Eq. (4), introduce the gravitational dynamics through $R_{ab} = (8\pi L_P^2)\mathscr{T}_{ab}$ (where $\mathscr{T}_{ab} \equiv T_{ab} - (1/2)g_{ab}T$) and integrate the result over a 3-dimensional region $\mathcal{R}$ with the measure $\sqrt{h}\, d^3x$. This leads to:

$$\int_{\mathcal{R}} \frac{d^3x}{8\pi L_P^2} \sqrt{h}\, u_a g^{ij} \pounds_\xi N^a_{ij} = \int_{\partial\mathcal{R}} \frac{d^2x\,\sqrt{\sigma}}{L_P^2} \left(\frac{N a_\alpha r^\alpha}{4\pi}\right) - \int_{\mathcal{R}} d^3x\, N\sqrt{h}\, (2u^a u^b \mathscr{T}_{ab}) \quad (8)$$

where $r_\alpha$ is the normal to the boundary of the 3-dimensional region. We now choose the boundary to be a $N(t, \mathbf{x})$= constant surface within the $t$ = constant surface. As before, we can then interpret $T_{\rm loc} = N a_\alpha r^\alpha/2\pi = Na/2\pi$ as the Tolman redshifted Davies-Unruh temperature. Further, in the second term, we identify $2N\mathscr{T}_{ab}u^a u^b = (\rho + 3p)N$ as the Komar energy density. So Eq. (8) becomes:

$$\frac{1}{8\pi L_P^2} \int_{\mathcal{R}} d^3x \sqrt{h}\, u_a g^{ij} \pounds_\xi N^a_{ij} = \int_{\partial\mathcal{R}} \frac{d^2x\,\sqrt{\sigma}}{L_P^2} \left(\frac{1}{2} k_B T_{\rm loc}\right) - \int_{\mathcal{R}} d^3x\, \sqrt{h}\, \rho_{\rm Komar} \quad (9)$$

This result, again, has a remarkable physical meaning. If the spacetime is static and we choose the foliation such that $\xi^a$ is the Killing vector, then $\pounds_\xi N^a_{ij} = 0$ and the left hand side vanishes. The equality of two terms on the right hand side can be thought of as representing the *holographic equipartition* [5] if we define the bulk and surface degrees of freedom along the following lines: We count the number of surface degrees of freedom by allotting one 'bit' for each Planck area:

$$N_{\rm sur} \equiv \frac{A}{L_P^2} = \int_{\partial\mathcal{R}} \frac{\sqrt{\sigma}\, d^2x}{L_P^2} \quad (10)$$

We next define an *average* temperature $T_{\rm avg}$ of the boundary surface $\partial\mathcal{R}$ by:

$$T_{\rm avg} \equiv \frac{1}{A} \int_{\partial\mathcal{R}} \sqrt{\sigma}\, d^2x\; T_{\rm loc} \quad (11)$$

Finally, we define the bulk degrees of freedom $N_{\rm bulk}$ by the following procedure: *If* the energy $E$ in the region $\mathcal{R}$ has reached equipartition at the average surface temperature $T_{\rm avg}$, *then* $|E| = (1/2)N_{\rm bulk}k_B T_{\rm avg}$; that is, we define the number of bulk degrees of freedom by:

$$N_{\rm bulk} \equiv \frac{|E|}{(1/2)k_B T_{\rm avg}} = \pm \frac{1}{(1/2)k_B T_{\rm avg}} \int_{\mathcal{R}} \sqrt{h}\; d^3x\; \rho_{\rm Komar} \quad (12)$$

where $E$ is the total Komar energy in the bulk region $\mathcal{R}$ contributing to gravity. (The $\pm$ is to ensure that $N_{\rm bulk}$ remains positive even when the Komar energy becomes negative.)

This is the relevant value of $N_{\rm bulk}$ if we assume equipartition holds for the energy $E$ in the bulk region with the average surface temperature. Our result in Eq. (9) then says that *comoving observers in any static spacetime* will indeed find:

$$N_{\rm sur} = N_{\rm bulk} \qquad \text{(Holographic equipartion)} \tag{13}$$

That is, the equipartition is holographic in all static spacetimes.

What is more, Eq. (9) *shows clearly that the discrepancy from holographic equipartition — resulting in a non-zero value for the right hand side — drives the dynamical evolution of the spacetime.* We can write Eq. (9) as:

$$\int \frac{d^3x}{8\pi L_P^2}\sqrt{h}u_a g^{ij}\pounds_\xi N^a_{ij} = \frac{1}{2}k_B T_{\rm avg}(N_{\rm sur} - N_{\rm bulk}) \tag{14}$$

Note that, even in a static spacetime, non-static observers will perceive a departure from holographic equipartition because Eq. (14) — while being generally covariant — is foliation dependent through the normal $u_i$. It is, of course, possible for the same spacetime to be described by two different sets of observers (i.e., foliations) such that the metric appears static for one while it is non-static for the other. (A simple example is de Sitter spacetime which is static in spherically symmetric coordinates while time dependent in FRW coordinates.) Unlike Einstein's equation $G^a_b = 8\pi T^a_b$, Eq. (14) clearly distinguishes observers who perceive the spacetime to be static (for which $N_{\rm sur} = N_{\rm bulk}$) from those who find it time dependent.

One can rewrite the left hand side of Eq. (14) by relating $u_a g^{ij}\pounds_\xi N^a_{ij}$ to more familiar constructs in the Hamiltonian formulation of relativity. A straightforward computation [2] shows that $\sqrt{h}u_a g^{ij}\pounds_\xi N^a_{ij}$ can be expressed as:

$$\sqrt{h}u_a g^{ij}\pounds_\xi N^a_{ij} = -h_{ab}\pounds_\xi p^{ab}; \qquad p^{ab} \equiv \sqrt{h}(Kh^{ab} - K^{ab}) \tag{15}$$

allowing us to rewrite Eq. (14) in the form of Eq. (1) presented at the beginning of the essay.

As I mentioned earlier, demanding the validity of Eq. (1) or Eq. (14) for all foliations is mathematically equivalent to Einstein's equations. While Eq. (14) is a classical equation, individual parts of it (like $T_{\rm avg}, N_{\rm sur}$) contain $\hbar$. This strengthens the idea that gravitational field equations have the same conceptual status as the equations of thermodynamics or fluid mechanics, with the Davies-Unruh temperature providing the link between microscopic and macroscopic descriptions of spacetime. It is remarkable that the dynamical evolution of the spacetime can be described in such an elegant, holographic language.

# References


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