Half-integral Spin from Quantum Gravity

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Summary

For a certain class of three-manifolds, the angular momentum of an asymptotically flat quantum gravitational field can have half-integral values. In the absence of a full theory of quantum gravity, this result relies on a set of apparently natural assumptions governing the kinematics of such a theory. A key feature is that state vectors are in general invariant only under asymptotically trivial diffeomorphisms that can be continuously deformed to the identity. Angular momentum is associated with diffeomorphisms that look asymptotically like rotations; and the question of whether half-integral values occur depends on whether the diffeomorphism associated with a 2π rotation is itself deformable to the identity.

Objects that change under a 2π rotation entered physics via quantum mechanics; and unlike field with integral spin, they have no classical counterparts. Nonetheless, to a flat space physicist, fields with integral and half-integral spin appear formally on an equal footing, in that their components transform under a change of basis as finite dimensional representations of the (covering group of the) Lorentz group. That objects with half-integral spin play no classical role appears simply as a manifestation of the connection between spin and statistics: classical fields do not represent a large number limit for fermions.

But in curved space, the different in behavior between the two types of fields is accompanied by a more uncomfortable formal distinction. Integral spin fields -- tensors -- are geometrical objects [1] whose components transform under a change of basis as representations of the general linear group GL_4. Half-integral spin fields -- spinors -- are in contrast defined only when a metric is present and have components only along orthonormal frames; their components thus belong to no representation of GL_4 (in fact GL_4 has no finite dimensional spinor representations).

Thus, while supersymmetry [2], in which bosons and fermions are treated as equal, seems natural from a flat space point of view, the older geometrodynamics [3] program, which accords a primary role to bosons seemed more appropriate in the context of general relativity. Within geometrodynamics, there seemed no clear way to construct objects with half integral spin, although Finkelstein and Misner [4] suggested that nontrivial spacetime topology might permit state vectors which change sign under rotations.*

Although the actual mechanism differs from the possibilities discussed in [4], it is remarkable that states with half integral spin do in fact arise from the quantization of integral spin fields having a gauge degree of freedom [6], and in particular from quantum gravity [7]. In the absence of a well defined quantum theory of nonlinear fields, this statement relies on a set of assumptions governing the character of such a theory, and we will list these explicitly below. First, an intuitive indication of how half-integral spin states can arise: When a gauge degree of freedom is present, the classical configuration space (the space on which a Lagrangian function is defined) is, in general, larger than the space of physically distinct fields. The image of a field under a 2π rotation need only be a gauge related field, and the effect on configuration space of a 2π rotation will be a gauge transformation $R(2\pi)$. As a result, even at the classical level an action of the rotation group (or of the Lorentz group) on configuration space will not in general be single valued. In the corresponding quantum theory, if state vectors are regarded as functions on the classical configuration space, the induced representation of O_3 on the quantum space of states will be single valued only if ψ is invariant under the gauge transformation $R(2\pi)$.

Let us now relate the ideas sketched above to quantum gravity. Let M denote a 3-manifold that can be compactified by adding a single point (I_0) at infinity. Let $\mathcal{M}_M$ be the set of smooth, positive definite, asymptotically

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*A related suggestion there that twists in the light cone structure could lead to half-integral spin was considered by Williams and by Williams and Zvengrowski [5] who, however, incorrectly identified the angular momentum operator by treating metric components, in effect, as scalars.

flat 3-matrics g_{ab} on M.* Then the various $\mathcal{M}_M$ comprise the classical configuration space, and diffeomorphisms of M represent the gauge freedom of the classical theory. Denote by $\mathcal{H}_M$ a vector space of functionals Ψ: $\mathcal{M}_M \to \mathbb{C}$. Our first (of four) assumptions asserts that a Schrödinger representation exists and that manifolds which arise classically are also present in the quantum theory.

I. There is a representation of quantum gravity in which the state space includes a nonvanishing subspace $\mathcal{H}_M$ for every M that occurs classically as a Cauchy surface of an asymptotically flat vacuum spacetime.

The second assumption concerns the meaning of the "momentum constraint" [8]. In classical relativity, the momentum π^{ab} conjugate to g_{ab} is constructed from the extrinsic curvature of M (when M is regarded as an embedded hypersurface) and it satisfies the equation

$$\nabla_b \pi^{ab} = 0 \ . \tag{1}$$

Replacing π^{ab} by $\frac{1}{i}\frac{\delta}{\delta g_{ab}}$, and contracting with a vector field ξ^a that vanishes at infinity, one finds for any state vector ψ that

$$\int dx \ D_a \xi_b \frac{\delta}{\delta g_{ab}(x)} \psi(g) = \frac{d}{d\lambda} \psi(\chi_\lambda g) = 0 \ , \tag{2}$$

where χ_λ is a family of diffeomorphisms for which the path $\lambda \to \chi_\lambda(P)$ is tangent at P to ξ^a. Equation (2) means that ψ is invariant under diffeomorphisms that are asymptotically trivial (approach the identity at infinity) and which can be joined to the identity by a path χ_λ of such diffeomorphisms; this is the content of the next assumption.

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*We can take this to mean that there is a flat metric δ_{ab} defined near I_o on M and that, to within boosted Schwarzschild terms, $h_{ab} := g_{ab} - \delta_{ab}$ is $O(r^{-2})$ and $\nabla_a h_{bc}$ is $O(r^{-3})$ where r is a radial coordinate and ∇_a the covariant derivative associated with δ_{ab}.

II. For each M, there is a set of D of asymptotically trivial diffeomorphisms which includes all diffeomorphisms of compact support and is such that for all χ in D_o, the component of the identity in D, $\psi(g) = \psi(\chi g)$.

If $\mathcal{C} := \mathcal{M}/D_o$ is the space of equivalence classes $[g_{ab}]$, assumption (II) is the statement that ψ is a function on $\mathcal{C}$. There is a precise analogue of (II) in Yang-Mills theory. Eq. (1) is replaced by the Gauss constraint $\nabla_a \underset{\sim}{E}^a = 0$, and the corresponding quantum constraint is the statement that state vectors ψ are invariant under gauge transformations in G_o, the component of the identity in the space G of asymptotically trivial gauge transformations. Gauge transformations not in G_o, however, do not leave invariant the states of Yang-Mills quantum theory. Transformations that are asymptotically finite are associated with the charge operator, while transformations in G but not in G_o change the phase of ψ by a multiple of the Yang-Mills angle θ [9]. In gravity, diffeomorphisms that are finite at infinity are associated with momentum operators; and a change of states under diffeomorphisms in D but not in D_o is the key to half-integral spin. In particular, if a vector field ξ^a is the generator of an asymptotic symmetry, the associated conserved quantity (the ADM momentum or angular momentum associated with ξ^a) is

$$P_\xi\, \psi(g) = \frac{1}{i}\frac{d}{d\lambda}\, \psi(\chi_\lambda g)\Big|_{\lambda=0},$$

where χ_λ is the family of diffeomorphisms generated by ξ^a. (II implies that P_ξ depends only on the asymptotic behavior of ξ^a.) Thus, to require invariance under diffeomorphisms that are finite at infinity would be to rule out states of nonzero momentum and angular momentum. To preserve generality, we therefore assume

III. If $[g_{ab}]$ and $[g'_{ab}]$ are in $\mathcal{C}_M$ and $[g_{ab}] \neq [g'_{ab}]$, there is some ψ in $\mathcal{H}_M$ for which $\psi(g_{ab}) \neq \psi(g'_{ab})$.

Finally, we assume as a prerequisite for a reasonable theory that one be able to define the angular momentum operators corresponding to a rotational subgroup of the symmetry group at spatial infinity. Let ${\phi_\alpha}^a$, $\alpha = 1 - 3$, be generators of asymptotic rotations which satisfy the commutation relations

$$[\phi_\alpha, \phi_\beta] = -\varepsilon_{\alpha\beta\gamma}\, \phi_\gamma$$

in a neighborhood of spatial infinity (I_o); let $R_\alpha(\theta)$ be the family of diffeomorphisms generated by ϕ_α.

IV. An SO(3) subgroup of the symmetry group at spatial infinity (of M) acts on $\mathcal{H}_M$ by the (possibly double valued) representation

$$\hat{R}_\alpha(\theta)\psi(g) = \psi(R_\alpha(\theta)g) \ .$$

In other words, the angular momentum associated with ϕ_α is

$$\hat{J}_\alpha\psi(g) = \frac{1}{i}\frac{d}{d\theta}\, \psi[R_\alpha(\theta)g] \ .$$

These four assumptions together imply the existence of state vectors having half-integral angular momentum. In particular (writing $R(2\pi) \equiv R_\alpha(2\pi)$ for some α), if $R(2\pi)$ is not deformable to the identity on some manifold M, then (III) implies $\hat{R}(2\pi)\psi \neq \psi$ for some ψ in $\mathcal{H}_M$, whence $\psi' := \hat{R}(2\pi)\psi - \psi$ is an eigenstate of $\hat{R}(2\pi)$ with eigenvalue -1: ψ' is a nonvanishing superposition of states having half-integral angular momentum.

When M is topologically Euclidean, $R(2\pi)$ is in D_o, for we can define rotational vector fields ϕ_α on the whole of M and thereby make $R(2\pi)$ the identity. In the general case, $R(2\pi)$ is in D_o only if one can communicate a rotation by 2π at infinity to the whole interior of the space. Surprisingly, a recent theorem [10] in differential topology in effect characterizes all 3-manifolds for which $R(2\pi)$ is in D_o; and it is easy to find manifolds M which occur classically as spacelike hypersurfaces of asymptotically flat vacuum spacetimes, and for which $R(2\pi)$ is <u>not</u> in D_o. In particular, if H is a subgroup of SU(2), the manifold of cosets SU(2)/H (called an elliptic space) occurs classically as a compact hypersurface and, with one point removed, as an asymptotically flat hypersurface of a vacuum spacetime. Choosing H to be, for example, the quaternion group then implies $R(2\pi) \notin D_0$ on $M = SU(2)/H - I_0$. (This M can be constructed by removing from $\mathbb{R}^3$ a solid cube and then identifying opposite faces of its boundary after a 90° rotation.) Then (I) and (II) imply the existence of state vectors ψ with half-integral angular momentum.

Some further aspects of the work deserve at least a brief mention. In the Schrödinger representation, there is an additional constraint associated with the Hamiltonian density of the theory which vanishes classically; a common way of treating this last constraint is to change to a representation in which the trace, π, of the extrinsic curvature and the conformal 3-metrics (the equivalence classes of 3-metrics that differ only by a conformal factor that becomes 1 at infinity) are independent variables.

The momentum constraint again has the meaning that state vectors are invariant under diffeomorphisms of M, and the space $\tilde{\mathcal{E}}$ of conformal metrics modulo diffeomorphisms in D_0 has the same number of disconnected components as $\mathcal{E}$ had. The criterion for the occurrence of half-integral spin is thus unchanged.

We have not considered here the question of a spin-statistics relation. The prototype construction of spin-1/2 from integer spin objects is the quantum mechanical system of an electric and magnetic charge where half integral spin arises from the angular momentum of their Coulomb fields. There an asymptotic interchange of two identical (electric charge, magnetic charge) systems changes the sign of the wave function precisely when it ought to--when the total angular momentum is half-integral [11]. In the Yang-Mills "spin from isospin" construction [6], asymptotic interchange of two isolated identical solitons can again be shown to change the sign of the state vector precisely when each has half-integral angular momentum [12]. In gravity, however, the connection between spin and statistics seems more subtle and may rely on understanding a dynamics that can accommodate the topology change needed to define "particle" creation.

Finally, it should be emphasized that any diffeomorphism on M not in the component of the identity acts nontrivially on the state space $\mathcal{H}_M$. Thus the group $D/D_o \simeq \pi_o(D)$ is a symmetry group acting on states associated with a manifold M. In the geometrodynamics picture of a nontrivial topology as a particle (an extended object of size on the order of the Planck length), D/D_o represents an internal symmetry group, and its irreducible representations will determine the possible particle multiplets associated with the particular manifold M.

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