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Spontaneous Dimensional Reduction in Quantum Gravity

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Abstract

Hints from a number of different approaches to quantum gravity point to a phenomenon of “spontaneous dimensional reduction” to two space-time dimensions near the Planck scale. I examine the physical meaning of the term “dimension” in this context, summarize the evidence for dimensional reduction, and discuss possible physical explanations.

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The world of everyday experience has three dimensions of space and one of time. The idea that this could change at small scales dates back a century [1–3], and the notion of “extra dimensions” is now commonplace in string theory and brane cosmology. Recent research, though, hints at the opposite behavior, a *reduction* in the number of dimensions near the Planck scale. If true, this could require drastic revisions of our conception of spacetime, and might point toward new directions in quantum gravity.

Dimension as an observable

To even pose the question of dimensional reduction, we must think carefully about the term “dimension.” In general relativity, spacetime is modeled as a smooth manifold, and dimension is unambiguous. Kaluza-Klein theory uses higher-dimensional manifolds, again with no real ambiguity. But quantum gravity is harder. Already in quantum mechanics, path integrals are dominated by nowhere-smooth paths with fractal dimensions [4], and smooth one-dimensional paths appear only semiclassically. If smooth large-scale spacetime is emergent, as many approaches to quantum gravity suggest, then dimension, too, must be emergent.

What, then, do we mean by dimension? What we need are “dimensional estimators,” quantum observables with a simple dependence on dimension that can be generalized to situations in which the dimension is unclear. Many possibilities exist, and different choices need not always agree; dimension may depend on the precise question being asked.

Some dimensional estimators arise directly from geometry. The simplest is the volume V of a geodesic ball of radius r . If, at some scale, $V \sim r^d$, then d is a measure of dimension at that scale. The more general Hausdorff dimension d_H counts the number of balls needed to cover a region as a function of r .

Other estimators come from random walks or diffusion processes. On a smooth manifold, the probability for a random walk from x to reach x' in time s is given by a heat kernel,*

$$K(x, x'; s) \sim (4\pi s)^{-d/2} e^{-\sigma(x, x')/2s} (1 + \mathcal{O}(s)) \quad (1)$$

The dependence on the dimension d is universal, so $K(x, x'; s)$ determines an effective dimension whenever a random walk can be defined. In fact, there are two such dimensions: the “walk dimension” d_W , determined from mean distance as a function of time, and the spectral dimension d_S , determined from the “return probability” $K(x, x; s) \sim (4\pi s)^{-d_S/2}$.

These estimators assume a Riemannian (positive definite) metric, and their application to spacetimes requires analytic continuation, which might distort the physics [5]. There are, however, several Lorentzian alternatives. The “causal spectral dimension” [6] exploits the relationship between two random walkers, each moving forward in time. The Myrheim-Meyer dimension [7, 8] is built from volumes of causal diamonds. We can also examine the behavior of geodesics [9, 10]: in certain spacetimes such as regions of Kasner space, although the manifold is four-dimensional, typical geodesics probe fewer dimensions [11].

* $\sigma(x, x')$ is half the squared geodesic distance between x and x' .

We can augment these geometric probes with dimension-dependent physical phenomena. This is an old idea: a century ago, Ehrenfest used such features as stability of orbits to single out three-dimensional space [12]. In thermodynamics, free energy grows with temperature as $F/VT \sim T^{d-1}$, determining a “thermodynamic dimension” $d = d_T$; dimensional reduction in quantum gravity was first observed as a scale-dependent change in this quantity [13]. Other thermodynamic dimensions can be built from such objects as the equation of state parameter and specific heat [5, 14, 15].

A final estimator comes from the fact that operators in quantum field theory acquire anomalous scaling dimensions, which change with energy under renormalization group flow. In field theoretical approaches like asymptotic safety [16], these scaling dimensions determine a spacetime dimension. In particular, two-point functions—propagators and Greens functions—have a characteristic dimensional dependence: a massless Greens function $G(x, x')$ in d dimensions goes as $\sigma(x, x')^{-(d-2)/2}$ (or $\ln \sigma(x, x')$ in two dimensions). Quantum corrections then make d a scale-dependent effective dimension.

Dimensional reduction

With these estimators in hand, we can ask what quantum gravity says about spacetime dimension. This is easier said than done: while we have promising research programs, we are still far from a complete quantum theory of gravity. We may hope, though, that if a phenomenon like dimensional reduction occurs in enough different settings, it may point to a fundamental feature of the physics.

Scale dependence of dimension was originally noticed in anisotropic cosmologies [11], but its broader significance was not appreciated. In quantum gravity, the phenomenon first appeared in string theory, where a thermodynamic dimension was found to unexpectedly drop to $d_T = 2$ at high temperatures, leading Atick and Witten to postulate “a lattice theory with a $(1 + 1)$ -dimensional field theory on each lattice site” [13]. But it was only with the computation of spectral dimension in causal dynamical triangulations [17] that the idea really took hold.

Causal dynamical triangulations is a discrete, nonperturbative path integral approach to quantum gravity in which spacetimes are approximated as piecewise flat simplicial complexes (higher dimensional “geodesic domes”) with a prescribed direction of time. The model has been remarkably successful, yielding not only a de Sitter ground state but also a correct spectrum of quantum volume fluctuations [18]. Lattice quantum gravity is notoriously bad at reproducing four-dimensional spacetime, though, and one must check the dimension carefully. Using spectral dimension as their estimator, Ambjørn et al. found that causal dynamical triangulations has a phase in which spacetime is, indeed, four-dimensional at large scales, but—much to their surprise—appears two-dimensional at small scales.

At about the same time, Percacci and Perini pointed out that if the renormalization group flow for quantum gravity has a high energy (ultraviolet) fixed point—the hope of the asymptotic safety program—the scaling dimensions of fields at the fixed point are necessarily those of a two-dimensional field theory [19]. The same dimensional reduction

was confirmed for the spectral and walk dimensions [20].

These results sparked a broad investigation of small-scale dimension, with surprisingly consistent results. Geometric and spectral dimension seem to fall to two at small scales for loop quantum gravity [21, 22] and spin foams [23]. For models with a minimum length, geometric [24] and spectral [25] dimension do the same. For noncommutative Snyder space, thermodynamic dimensions show the same short-distance reduction [15]; for other noncommutative geometry models, the short-distance spectral dimension also decreases, although in a manner that depends on the choice of Laplacian [26, 27]. Models based on a generalized uncertainty principle show a reduction in thermodynamic dimension, now to $d_T = 2.5$ [14]. The short-distance approximation of the Wheeler-DeWitt equation is dominated by spacetimes in which typical geodesics probe only two dimensions [9, 10]. In causal set theory, the naive spectral dimension increases at small scales [6], but with a better choice of Laplacian it falls to $d_S = 2$ [28]. The Myrheim-Meyer dimension also falls to approximately two for small causal sets [29]. For proposed renormalizable modifications of general relativity, including Hořava-Lifshitz gravity [30] and higher curvature models [31], a generalized spectral dimension again falls to $d_S = 2$.

Extracting the physics

By itself, no single one of these results is very convincing. Together, though, they show a clear pattern. Like spontaneous symmetry breaking in field theory, dimensional reduction at short distances and high energies needs no explicit external mechanism; it is indeed “spontaneous dimensional reduction.”

But while this convergence suggests that the phenomenon is real, the underlying physical mechanism remains obscure. The answer may require a full quantum theory of gravity, but two interesting suggestions have been made:

1. **Scale invariance:** For general relativity to be “asymptotic safe” [16], its renormalization group flow must have an ultraviolet fixed point. Such a fixed point is, by definition, scale-invariant, and this invariance is enough to guarantee effective two-dimensional behavior for any theory that includes general relativity [32]. While the details are technical, the underlying reason is simple: only in two dimensions is Newton’s constant G_N dimensionless and thus scale-free.

This is not completely satisfying, since it fails to explain *why* such a fixed point should exist. But perhaps the argument should be reversed: if we define the theory at high energy, it is natural to postulate a high degree of symmetry, which is then broken at lower energies.

2. **Asymptotic silence:** Near a spacelike singularity, classical general relativity exhibits “asymptotic silence”: light cones shrink to lines, and nearby points become causally disconnected [33]. This, in turn, leads to BKL behavior [34], in which the metric is locally Kasner with axes of anisotropy that vary chaotically in space

and time. In such a spacetime, each point has a “preferred” spatial direction, and geodesics effectively see only $1 + 1$ dimensions [9, 10].

While this behavior was discovered in cosmology, asymptotically silent spacetimes dominate the short-distance Wheeler-DeWitt equation, and quantum fluctuations near the Planck scale may lead to short-distance asymptotic silence everywhere [35]. If so, the small scale metric will have two length scales,

$$ds^2 = \ell_{\parallel}^2 g_{\mu\nu} dx^\mu dx^\nu + \ell_{\perp}^2 h_{ij} dx^i dx^j \quad (2)$$

and the Einstein-Hilbert action will become very nearly that of a two-dimensional conformal field theory for the transverse metric h_{ij} [36]. Such reductions have been studied in different contexts [37–39], and could give a physical explanation for the fixed point in asymptotic safety.

It is too soon to tell whether either, or perhaps both, of these ideas will explain spontaneous dimensional reduction. But it seems fair to say that the phenomenon offers an exciting new glimpse into the decades-old problem of quantize gravity.

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