The Sign and Magnitude of the Constant of Gravity in General Relativity

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Abstract

The magnitude and sign of the constant of gravity in the general theory of relativity are usually considered to be undetermined by the theory and are chosen so that in the weak field limit the theory reduces to Newtonian theory and is therefore in agreement with empirical results about the world. It is here emphasised that the weak field limit is really the Universe plus a localised source, and an analysis of this problem demonstrates that in general relativity the constant of gravity is necessarily positive and is determined by the large scale distribution of matter in motion. The significance of this result is highlighted by comparison with a scalar theory where gravity is shown to be necessarily repulsive. The relationship between these results and the usual interpretation of general relativity is discussed and the differences are resolved by an analysis of the interpretation of the equations governing possible cosmological models.

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The magnitude and sign of the constant of gravity in the general theory of relativity are usually considered to be undetermined by the theory and are chosen so that in the weak field limit the theory reduces to Newtonian theory and is therefore in agreement with empirical results about the world. It is here emphasised that the weak field limit is really the Universe plus a localised source, and an analysis of this problem demonstrates that in general relativity the constant of gravity is necessarily positive and is determined by the large scale distribution of matter in motion. The significance of this result is highlighted by comparison with a scalar theory where gravity is shown to be necessarily repulsive. The relationship between these results and the usual interpretation of general relativity is discussed and the differences are resolved by an analysis of the interpretation of the equations governing possible cosmological models.

1. Introduction

The constant of gravity is usually considered to be an arbitrary constant of nature whose magnitude and sign is to be determined by experiment and observation. Thus in general relativity with field equations

$$R_{ij} - \frac{1}{2} g_{ij}R = \kappa T_{ij}, \quad ds^2 = g_{ij}dx^i dx^j, \quad \delta \int ds = 0 \tag{1}$$

it is customary to derive the 'weak field' limit

$$ds^2 = (1 - 2\phi)dt^2 - (1 + 2\phi)(dx^2 + dy^2 + dz^2) \tag{2}$$

$$\nabla^2\phi = -\frac{\kappa\rho}{2}, \quad \frac{d^2x^i}{dt^2} = \frac{\partial\phi}{\partial x^i}$$

and by comparing this with Newtonian theory to deduce that $\kappa = -8\pi G$. The sign of κ is chosen to make gravity an attractive force, the magnitude (in terms of say atomic units) is chosen to give quantitative agreement between theory and observation.

If a peverse theorist wished to choose κ of opposite sign and of a different magnitude the criticisms levelled against him would be that his theory did not agree with our empirical knowledge of the world, but there would be no logical inconsistency in such a choice.

In the above standard derivation of the magnitude and sign of the gravitational constant, no account is taken of the rest of the matter in the Universe. This is sometimes considered a weakness of general relativity since it appears that the theory does not incorporate the philosophical principles of Leibniz and Mach that local physics is determined by the large scale distribution of matter in motion. In particular, if the kinematics of the rest of the Universe were unchanged but the mean density was different, the magnitude and sign of the constant of gravity would be unchanged, local physics is therefore independent of the large scale distribution of matter in motion.

The standard derivation can be criticised on several grounds. The world of our experience is not of a local source in an otherwise empty Universe, on the contrary it is of the effect of local sources embedded in a cosmological distribution of matter, and it is this problem that should be compared with our empirical knowledge of the world. Moreover the standard weak field analysis needs the boundary conditions $g_{ij} \to \eta_{ij}$, the Minkowski metric at large distances from the source. What justification is there for imposing this boundary condition? The answer must be that the cosmological metric is approximately Minkowskian if we consider times and distances that are small compared to typical cosmological values, so even in the standard treatment the large scale properties of the Universe enter implicitly through the imposition of boundary conditions. This suggests that we should analyse the Universe plus a local source and then compare this to the world of our experience, when this is done it emerges that gravity is necessarily attractive and the magnitude of the constant of gravity is determined by the large scale distribution of matter in motion in accord with the philosophical principles of Leibniz and Mach. It should be emphasised that this is a consequence of General Relativity in its standard form, what is at issue is the question of interpretation of results of the theory. The coupling constant κ in equations (1) is shown to cancel out of the results, its place is taken by cosmological factors.

2. A simple model

Before considering the problem in General Relativity it is worth looking at a very simple theory where

$$ds^2 = \phi(dt^2 - dx^2 - dy^2 - dz^2), \quad \phi(x) = \sum \kappa \int mG(x, y)ds_y, \quad \delta \int ds = 0 \tag{2.1}$$

This is essentially the Einstein-Nordstrøm theory where

$$C^{i}_{jk\ell} = 0, \qquad R = \kappa T, \qquad \delta \int ds = 0 \tag{2.2}$$

with $C^{i}_{jk\ell}$ the Weyl tensor and R and curvature scalar, except that the second of equations (2.1) is in integral form with $G(x, y)$ the retarded Greens function.

If we solve this theory for a cosmological distribution plus a local source of mass M, the solution is of the form

$$\phi = \sum \frac{\kappa m}{r} = \kappa \frac{M_u}{R_u} + \frac{\kappa M}{r} \tag{2.3}$$

where $\kappa M_u/R_u$ is the sum over the cosmological distribution and is the 'potential' of the Universe, that is the sum over the back light cone of individual $\kappa m/r$.

The geodesic equation for the motion of a test particle in the weak field limit is then

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\frac{1}{\phi}\frac{\partial\phi}{\partial x^i} \approx \frac{1}{2}\frac{R_u}{M_u}\frac{M}{r^3}x^i \tag{2.4}$$

The coupling constant κ does not enter the equation for a geodesic since it cancels from top and bottom of equation (2.4). In the geodesic equation the 'constant of gravity' is $\frac{1}{2}R_u/M_u$, i.e. the sum over the cosmological distribution. Moreover the resulting weak field equation of motion gives a repulsive force, bound orbits are therefore not possible in this theory.

This result should be contrasted with the result obtained by following the normal weak field analysis for a localised source in an otherwise empty universe. Following this procedure we would arrive at the result

$$\phi = 1 + \frac{\kappa M}{r}, \qquad \frac{d^2x^i}{dt^2} = \frac{1}{2}\kappa \frac{M}{r^3}x^i \tag{2.5}$$

and by choosing $\kappa = -2G$ we would obtain agreement with Newtonian theory, in particular we could make gravity an attractive force. Clearly the normal procedure is incorrect.

3. The Universe plus one body problem in general relativity

For simplicity I shall confine my attention to homogeneous isotropic cosmological models, the metric can then be expressed in the form[+]

$$ds^2 = C^2(\tau)\,[d\tau^2 - \frac{dx^2 + dy^2 + dz^2}{(1 + kr^2/R^2)^2}] \tag{3.1}$$

where $k = 0, +1, -1$, and R is an arbitrary constant. In these coordinates the fundamental particles of the substratum are at rest (r = constant), and τ is measured by the round trip travel time of a light signal between these particles. The usual Robertson-Walker metric is derived from this by the transformation

$$t = \int C(\tau)d\tau, \qquad R(t) = C(\tau) \tag{3.2}$$

The number density of particles n_o is a constant in these coordinates and the field equations of general relativity give the one condition

$$\frac{\dot{C}^2}{C} + \frac{4kC}{R^2} = \frac{\kappa n_o m_o}{3} = \frac{\kappa \rho_o}{3} \tag{3.3}$$

where $\rho_o = n_o m_o$ and m_o is the mass of the particles. This equation is readily solved to yield

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[+] L. Infeld and A. Schild, Physical Review, Vol.68, p.250, 1945.

$$\frac{12C(\tau)}{\kappa n_o m_o} = c(\tau) = \tau^2 \qquad k = 0$$

$$= R^2 \sin^2(\tau/R) \qquad k = +1 \qquad (3.4)$$

$$= R^2 \sinh^2(\tau/R) \qquad k = -1$$

We notice that $C(\tau)$ is proportional to κ and is always positive.

We now consider the effect of adding a local source density $\rho_1 = n_1 m_o$, at $r = 0$. If we use the weak field approximation we expand the metric $g_{ij} = C_{ij} + h_{ij}$ where C_{ij} is the cosmological metric given by equations (3.1) and (3.4), and impose the boundary condition $g_{ij} \to C_{ij}$ as $r \to \infty$. After a little manipulation we find that g_{oo} is given by the equation $R_{oo} = \kappa(T_{oo} - \frac{1}{3} g_{oo} T)$ which is

$$2\ddot{g}_{oo} + \eta^{ij} g_{oo,ij} - \frac{4\dot{g}_{oo}\dot{g}_{oo}}{g_{oo}} + \frac{\eta^{ij} g_{oo,i} g_{oo,j}}{g_{oo}} = \frac{1}{2}\kappa(\rho_o + \rho_1) \qquad (3.5)$$

where $C_{ij} = c^2(\tau)\eta_{ij}$. In the weak field slow motion approximation this gives

$$ds^2 = \left(\frac{\kappa\rho_o}{12}\right)^2 c^2(\tau) \left[\left(1 - \frac{3M}{\pi\rho_o c(\tau) r}\right) d\tau^2 - \left(1 + \frac{3M}{\pi\rho_o c(\tau) r}\right) \frac{(dx^2+dy^2+dz^2)}{(1+kr^2/R^2)^2}\right] \qquad (3.6)$$

where $M = \int \rho_1 dV$ is the mass of the local source.

While this weak field analysis is adequate for our purposes it is possible to produce an exact solution by embedding a local source in a spherical hole in the cosmological model as was first done by Einstein and Strauss.[+] The analysis is somewhat tedious and lengthy and will be

[+]A. Einstein and E.G. Strauss, Review of Modern Physics, Vol.17, p.120, 1945.

published elsewhere, but in the weak field slow motion approximation we again obtain the result given in equation (3.6).

The result (3.6) demonstrates that the metric scales with κ^2, and that the coefficient of $d\tau^2$ is of the form $(1 - \frac{\alpha}{r})$. This should be contrasted with the result of the scalar theory where we obtained $(1 + \frac{\alpha}{r})$. This change of sign is produced by the non linearity of equation (3.5), and it is this sign change that makes gravity attractive in general relativity but repulsive in the scalar theory.

4. Equation of motion in the weak field limit

The equation of motion of a test particle is given by the geodesics of the metric (3.6), in the weak field limit this gives

$$\frac{d}{d\tau}\left(c(\tau)\frac{dx^i}{d\tau}\right) = -\frac{3Mx^i}{2\pi\rho_o r^3} \tag{4.1}$$

The effect of the mass M is to produce an attractive force. In fact a test particle will spiral inwards, this is easily seen by transforming to a time scale σ such that $d\sigma = d\tau/c(\tau)$, equation (4.1) is then

$$\frac{d^2x^i}{d\tau^2} = -\frac{c(\tau)3Mx^i}{2\pi\rho_o r^3}$$

which for a circular orbit has $r \propto 1/c(\tau)$. In the τ time scale the Universe is at rest and an orbit spirals inwards; as Eddington pointed out long ago this is equivalent to saying that the orbit is stationary and the Universe expanding.

We have now reached our principal goal, equation (4.1) demonstrates that in the weak field limit of general relativity gravity is an attractive force, and that the 'constant of gravity' is given by ρ_o and $c(\tau)$, that is by the cosmological distribution of matter; the coupling constant κ does not appear in the final result.

5. Relation to the usual procedure

The time scale τ that has been used is one in which the cosmological substratum is at rest, it is related to the cosmological time of the Robertson-Walker form of the metric through the transformation

$$C(\tau)d\tau = Adt, \qquad R(t) = C(\tau)/A \tag{5.1}$$

where A is an arbitrary constant. If we now scale all lengths and times by $c(\sigma)$, so that $c(\tau)r = A\ell$, the orbit equation becomes

$$\frac{d^2\ell^i}{dt^2} = -\frac{3M\ell^i}{2\pi A\rho_o\ell^3} = -\frac{GM\ell^i}{\ell^3}, \qquad G = \frac{3}{2\pi A\rho_o} \tag{5.2}$$

and the cosmological metric becomes

$$ds^2 = A^2\left(\frac{\kappa\rho_o}{12}\right)^2 [dt^2 - R^2(t)\frac{(dx^2 + dy^2 + dz^2)}{(1 + kr^2/R^2)^2}] \tag{5.3}$$

In these Robertson-Walker coordinates the effect of a local source is attractive and the 'constant of gravity' is a constant given by the scaling factor A and the mean density of the Universe ρ_o as measured in τ units.

The density ρ_o is constant on the τ scale, in the t scale the density is therefore

$$\rho(t) \quad = \quad \frac{A^3 \rho_o}{c^3(\tau)} \quad = \quad \frac{\rho_o}{R^3(t)} \tag{5.4}$$

If we now differentiate the field equation (3.3) we obtain

$$\rho(t) \quad = \quad - \frac{A^3 \rho_o}{2} \left(\frac{\ddot{c}}{c^3} - \frac{\dot{c}^2}{c^4} \right) \quad = \quad - \frac{A \rho_o}{2} \frac{1}{R} \frac{d^2 R}{dt^2} \tag{5.5}$$

Equations (5.4) and (5.5) are recognisable as the field equations of general relativity in Robertson-Walker coordinates. With τ and t both positive, A is positive and equation (5.5) requires $\ddot{R}/R$ to be negative as $\rho(t)$ is positive. Turning now to equation (5.1) for the value of the constant of gravity we find

$$G \quad = \quad \frac{3qH^2}{4\pi\rho(t)}, \qquad q \quad = \quad - \frac{R\ddot{R}}{\dot{R}^2}, \qquad H \quad = \quad \frac{\dot{R}}{R} \tag{5.6}$$

which is again recognisable as a standard result of general relativity in Robertson-Walker coordinates.

In the usual procedure the first of equations (5.6) is interpreted as imposing constraints on the kinematics of the Universe given the value of G. Our analysis demonstrates that this equation should be interpreted as determining G given the kinematics. It should be noted that equation (5.5) which is just one of the standard Robertson-Walker results requires G as given by equation (5.6) to be positive. The theory would be inconsistent if G were negative.

Equation (5.6) also demonstrates that general relativity satisfies the Leibniz-Mach principle; if the kinematics is unchanged, but the average density is changed, then G is changed. Similarly changing the kinematics but keeping the matter density unchanged also changes the value of G.

We conclude that an analysis of the Universe plus a local source in general relativity shows that gravity is necessarily attractive and that the 'constant of gravity' is determined by the large scale distribution of matter in motion.