The Constant of Gravity in General Relativity

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ABSTRACT

It is shown that the coupling constant in general relativity is arbitrary and does not enter the dynamical equations governing the motion of bodies. By considering the Universe plus one body we are able to show that gravity is necessarily attractive, independent of the magnitude or sign of the coupling constant, and that the 'constant of gravity' that enters the equations of motion is given by the large scale distribution of matter in motion in the Universe.

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1. Introduction

In the general theory of relativity the metric tensor g_{ij} is given by the field equations

$$R_{ij} = -\kappa(T_{ij} - \frac{1}{2} g_{ij} T) \quad (1)$$

where T_{ij} is the energy momentum tensor of the sources and κ a coupling constant. The value of κ is determined from the condition that the weak field approximation reduces to Newton's theory giving $\kappa = 8\pi G$, where G is the Newtonian constant of gravity and units are such that $c = 1$.

This situation is far from satisfactory for it would require that identical kinematic distributions of matter in motion would yield different dynamical consequence for different choices of κ; indeed, if κ were of the opposite sign, gravity would be repulsive. Apparently, the value of κ has to be introduced from outside the theory and determined empirically. In this essay I will demonstrate that this is not the case; κ is arbitrary both in magnitude and in sign and scales out of the theory, the 'constant of gravity' G is in fact given by the distribution of matter in motion, gravity is necessarily attractive, and identical kinematic distributions give identical dynamical consequences. Indeed, a very simple observation shows that this must be the case, for under the scale transformation $g_{ij} \to \lambda g_{ij}$ (λ constant) the field equations are unchanged except that $\kappa \to \kappa/\lambda^{1/2}$ and the geodesic equations are invariant depending only on ratios of g_{ij}. Such a transformation changes κ but not the dynamics, which are therefore independent of κ.

The standard procedure of comparing the weak field limit considers the solution for a localised source in an otherwise empty Universe, demanding that far away from the source $g_{ij} \to \eta_{ij}$ the Minkowski tensor. But we have no experience of such a situation; our experience is of localised sources in a full Universe and it is the solution of this problem that should be compared to the Newtonian physics of our experience. As I will show below, when this is done it emerges that "G" is in fact determined by the rest of the Universe and κ is arbitrary.

2. Cosmological Solutions

I shall confine my attention to homogeneous isotropic cosmological models, in which case the metric can be expressed as *

$$ds^2 = C^2(\sigma)\left\{ d\sigma^2 - \left[\frac{d\rho^2 + \rho^2 d\Omega^2}{(1 + k\rho^2/R^2)^2} \right] \right\} \tag{2}$$

where $k = 0, +1, -1$, and R is an arbitrary constant. The fundamental particles of the substratum are at rest, and the σ time scale is measured by the round trip travel time of a light signal between such particles. Once a unit for σ is chosen, the constant number density of particles, n_0, is an empirical quantity (alternatively the unit could be chosen so that $n_0 = 1$).

The conformal factor $C(\sigma)$ is given by the field equations which reduce to

* H.P. Robertson, Proc.Nat.Acad. of Sciences, U.S.A., Vol 15, p.822, 1929
A.G.Walker, Proceedings of London Math.Soc. Vol 42, p.90, 1936.
L.Infeld & A.Schild, Phys.Rev. Vol 68, p.250, 1945

$$\frac{\dot{C}^2}{C} + \frac{kC}{R^2} = \frac{\kappa n_o m_o}{3} \tag{3}$$

where m_o is a (conventional) mass of the fundamental particles. Defining $c(\sigma)$ such that $C(\sigma) = \kappa\, n_o m_o\, c(\sigma)/12$ we have

k	0	+1	-1
$c(\sigma)$	$2R^2(1-\cos\sigma/R)$		$2R^2(\cosh\sigma/R-1)$

3. The Universe plus local source

We now consider the effect of adding a local source density n_1 on top of the cosmological distribution. Expanding the metric about the cosmological solution, setting $g_{oo} = C^2(\sigma,\rho)$ then in the weak field, local, slow motion approximation, the R_{oo} equation reduces to

$$R_{oo} = \frac{1}{C}\left(2\, C_{oo} + \eta_{oo}\Box C\right) - \frac{4}{C^2}\left(C_o C_o - \frac{1}{4}\eta_{oo} C^{2} C_a\right) = \frac{1}{2}\frac{\kappa n\, m_o}{C} \tag{4}$$

With $n = n_o + n_1$, this gives

$$C(\sigma,\rho) = C(\sigma) - \frac{\kappa\, N m_o}{8\pi\rho}\,, \quad N = \int n_1 dV \tag{5}$$

and the metric becomes

$$ds^2 = \left(\frac{\kappa\, n_o\, m_o}{12}\right)^2 c^2(\sigma)\left[\left(1 - \frac{3N}{\pi\, n_o c(\sigma)\rho}\right)d\sigma^2 - \left(\frac{1+3N}{\pi n_o c(\sigma)\rho}\right)\left(d\rho^2+\rho^2 d\Omega^2\right)\right] \tag{6}$$

While the above solution is adequate for our purposes, we can obtain an exact solution by embedding the Schwarzschild solution into the Cosmological solution as done by Einstein and Strauss (1945).* The analysis is somewhat lengthy but eventually yields the metric for $\rho \leq \rho_o$

$$ds^2 = \left[\frac{\kappa n_o m_o}{12}\right]^2 \left|\frac{b^2}{a^2}(z_\sigma^2 - a^2\rho^2 y_\sigma^2)d\sigma^2 - a^4 y^2(d\rho^2 + \rho^2 d\Omega^2)\right| \quad (7)$$

where

$$a = 1 + \frac{\rho_o^3}{(1+k\rho^2/R^2)^3 \rho y}, \quad b = 1 - \frac{\rho_o^3}{(1+k\rho^2/R^2)^3 \rho y} \quad (8)$$

and y and z are given by

$$z_\rho^2 = \frac{2a^6}{b^2}\rho y_\rho (y + \rho y_\rho/2), \quad z_\rho z_\sigma = \frac{a^6}{b^2}\rho y_\sigma (y + \rho y_\rho) \quad (9)$$

subject to the boundary conditions

$$a^2 y = \frac{c(\sigma)}{(1+k\rho^2/R^2)}, \quad \frac{b^2}{a^2} z_\sigma^2 - a^4\rho^2 y_\sigma^2 = c^2(\sigma), \text{ at } \rho = \rho_o . \quad (10)$$

The subscripts ρ and σ denote differentiation with respect to these variables. The solutions for y and z are independent of κ, so too are the geodesic equations.

In the local, weak field, slow motion limit, equations (9) can be solved by successive perturbations, for $\rho_o^2 << \sigma^2$ the solution for $\rho << \rho_o$ reduces to equation (7), with $N = 4\pi n_o \rho_o^3/3$. These are

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* Einstein A and Straus, E.G, (1945), Reviews of Modern Physics, Vol 17, p.120 and Volume 18, p.148.

just the conditions in which we expect Newtonian physics to be valid, for example for the sun $\rho_o^2/\sigma^2 \sim 10^{-16}$, $\rho_o \sim 10^8$ A.U.

4. Geodesic Equation The equation of motion of a test particle in the metric (7) is given by

$$\frac{d}{d\sigma}\left[c(\sigma)\frac{d\rho^i}{d\sigma}\right] = -\frac{3N}{2\pi n_o}\frac{\rho^i}{\rho^3} \tag{11}$$

The gravitational force exerted by the local matter is attractive, and the coupling constant κ does not enter; since $c(\sigma)$ is slowly varying there is an effective constant of gravity $G^* \simeq 3/2\pi n_o m_o c(\sigma)$ given in terms of the cosmological solution.

For $c(\sigma)$ increasing the test particle spirals inwards, to see this let $d\tau = d\sigma/c(\sigma)$ and the geodesic equation becomes

$$\frac{d^2\rho^i}{d\tau^2} = -\frac{c(\sigma)\ 3N\rho^i}{2\pi n_o \rho^3} \tag{12}$$

and for an almost circular orbit this has $\rho \propto 1/c(\sigma)$. In (σ,ρ) coordinates the Universe is at rest and the test particle spirals inwards. If we use the test particle orbit as a clock, then the orbit is constant and the Universe is expanding - these are two alternative ways of describing the same relative behaviour. If we therefore scale all lengths and times by $c(\sigma)$ so that

$$c(\sigma)\ d\sigma = A\ dt, \quad c(\sigma)d\rho = Ad\ell, \quad c(\sigma)\rho = A\ell \tag{13}$$

the orbit equation reduces to

$$\frac{d^2\ell^i}{dt^2} = -\frac{3\,N}{2\pi A n_o}\frac{\ell^i}{\ell^3} = -\frac{GM\ell^i}{\ell^3} \text{ if } G \equiv \frac{3}{2\pi A n_o m_o} \quad . \tag{14}$$

In the gravitational time t, the cosmological metric becomes

$$ds^2 = \left(\frac{\kappa n_o m_o}{12}\right)^2 A^2 \left(dt^2 - \frac{c^2(\sigma)}{A^2}\frac{(d\rho^2+\rho^2 d\Omega^2)}{(1+k\rho^2/R^2)^2}\right) \tag{15}$$

5. The constant of gravity G The constant of gravity G defined by the last of equations (14) can be expressed solely in terms of measures in t time. The number density is n_o in the σ time scale, since $c(\sigma)d\sigma = Adt$, the number density in t time is $n(t) = A^3 n_o/c^3(\sigma)$. If we now define $R(t) = c(\sigma)\,A$ then on using equation (3) we find

$$n(t) = -\frac{A^3 n_o}{2}\left(\frac{\ddot{c}}{c^3} - \frac{\dot{c}^2}{c^4}\right) = -\frac{A n_o}{2}\frac{\ddot{R}}{R} \tag{16}$$

and hence

$$G = \frac{-3}{4\pi n(t) m_o}\left(\frac{\ddot{R}}{R}\right) = \frac{3\,q\,H^3}{4\pi\rho(t)} \tag{17}$$

where H is the Hubble constant, q the deceleration parameter, and the line element is

$$ds^2 = \left(\frac{\kappa\, m_o n_o A}{12}\right)^2 \left[dt^2 - R^2(t)\left(\frac{d\rho^2+\rho^2 d\Omega^2}{(1+k\rho^2/R^2)^2}\right)\right] \tag{18}$$

Conclusions

By considering the Universe plus one body problem in cosmic coordinates, we see that the coupling constant κ scales out of the problem and that gravity is necessarily attractive. The "constant of gravity" that enters the Newtonian limit is $3/2\pi n_o m_o c(\sigma)$. By transferring to gravitational (or proper) time t we see that G is determined by the distribution of matter in motion through the equation

$$G = \frac{3}{4\pi\rho(t)} q H^2$$

This equation should be interpreted as determining the G that enters the weak field limit from the large scale distribution of matter in the Universe, not as imposing restrictions on the possible motion with G a given constant brought in from outside the theory.

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Name:	Ian Walter Roxburgh
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Present position:	Professor of Applied Mathematics, Queen Mary College, University of London - since 1967.
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Board of the Solar Physics Section European Physical Society, 1975-

Advisory Board, Ministry of Defence, 1973-

Vice Chairman, Fundamental Physics Panel, European Space Agency, 1970-73

Astrophysics Working Group, European Space Agency, 1973-75

Chairman, Mathematical Sciences Coordinating Committee, Queen Mary College, 1971-73

Steering Committee, Conference of Professors of Applied Mathematics, 1971-73

Council, Royal Astronomical Society, 1968-71.

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