Logic Worksheet

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At university, you will encounter many theorems and proofs. It is *logic* that underlies these proofs. In fact, it is logic that is the foundation of all of mathematics.

1 Propositions

A *proposition* is another word for a sentence or a statement, which is either true or false.

For example, "*25 is divisible by 5*" and "*every triangle has 4 sides*" are propositions. Clearly the first is true and the second is false.

We all use propositions in our daily lives, *e.g.* "*it is cold today*" or "*I am happy*". However, it is harder to establish whether such propositions are true or false. By contrast, such ambiguity and subjectivity are absent in mathematical propositions.

Just like a sentence, a statement needs to have a subject and some information about that subject, so, "*a green apple*" and "*walking fast*" are not sentences. Similarly, "12" and "$x+4$" are not propositions.

Example 1.1. *Write down*

i. a mathematical proposition that is always true

ii. a mathematical proposition that is always false

iii. an everyday sentence that is always true

iv. an everyday sentence that is always false

v. a non-proposition

Propositional logic, first developed by Aristotle, is the branch of mathematical logic concerned with the study of propositions and how they can combine with, and connect to, one another. In the next section we will see how two propositions can be combined. By the way, I will now use the words *proposition* and *statement* interchangeably.

2 Implications

Propositions are often labelled with a capital letter. For example, we can let P be the statement "*12 is an even number*"

In what follows, we will reserve the letter T and F to describe a proposition as either true or false.

Let P and Q be propositions. Consider the statement "______________" or "______________", denoted ____________.

P is called the *hypothesis* or *premise* and Q is called the *conclusion* or *consequence*.

The statement $P \implies Q$ is called an *implication*, or a *conditional* statement because $P \implies Q$ asserts that Q is true on the condition that P holds. Here is the truth table:

P	Q	$P \implies Q$
T	T	
T	F	
F	T	
F	F	

The last two lines might look quite odd. A way to understand these truth values is to interpret $P \implies Q$ as false when a promise has been broken (or when a lie has been told). Here's an example.

Example 2.1. *Consider the statement $P \implies Q$ where,*
P: "I am elected president"
Q: "I lower taxes."
To which line of the truth table above does each of these statements correspond?

i. I have been elected president and I have lowered taxes.

ii. I have been elected president but I don't lower taxes.

iii. I wasn't elected president, yet I lower taxes (e.g. by lobbying).

iv. I wasn't elected president. I don't lower the taxes.

In which of the following scenarios would you consider me a liar?

An interesting observation: *From a false statement, anything can be deduced!*

Remarks. *i. $P \implies Q$ can also be written backwards as ____________.*

ii. $P \implies Q$ can also be read "P __________ Q". Let's try to understand this phrase. Suppose $P \implies Q$ holds. When P holds, Q is guaranteed to follow. So P holds only if Q also holds.

iii. The statement $Q \implies P$ is called the ______________ of $P \implies Q$.

iv. If both $P \implies Q$ and $Q \implies P$ hold, then we can write ______________. This is read: "P ________________ Q". It means that P and Q say exactly the same thing, and are interchangable.

Example 2.2. *Fill in each blank by a statement that would make each implication true.*

i. *N is divisible by 8* $\Longrightarrow$

ii. *N is divisible by 8* $\Longleftarrow$

iii. *N is divisible by 8* $\Longleftrightarrow$

iv. *Sue has blue hair* $\Longrightarrow$

v. *Sue has blue hair* $\Longleftarrow$

vi. *Sue has blue hair* $\Longleftrightarrow$

vii. *A triangle has 4 sides* $\Longrightarrow$

Example 2.3. *Complete the missing entries in the following table with either a statement or an implication symbol so that each row is true. (Avoid trivial answers like* $P \Longrightarrow P$*.)*

	P	$\Longrightarrow$ or $\Longleftarrow$ or $\Longleftrightarrow$	Q
Example	$x > 1$	$\Longrightarrow$	$x > 0$
a)	x is an integer		$x = 2$
b)	$x = 3$		$x^2 = 9$
c)	x is a prime number	$\Longrightarrow$	
d)	$2 > 3$	$\Longleftarrow$	
e)		$\Longleftrightarrow$	This month is April.
f)		$\Longrightarrow$	I play the violin.

If you enjoy thinking about logic and would like to read more about it, I recommend this free textbook.

Book of Proof, Hammack R., 3rd ed. (2018), available for free at `http://www.people.vcu.edu/~rhammack/BookOfProof/`