THE LOGIC BOOK

Sixth Edition

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Chapter

7

PREDICATE LOGIC: SYNTAX AND SYMBOLIZATION

Section 7.1 discusses the aspects of English syntax that cannot be captured by *SL* but are mirrored in *PL*. In Section 7.2 we present the formal syntax of *PL*. In Section 7.3 we symbolize a wide range of English sentences in *PL*. In Section 7.4 we explore a variety of issues bearing on how we symbolize sentences in *PL*. In Section 7.5 we present *PLE*, an extension of *PL* that includes identity and functors.

7.1 PREDICATES, SINGULAR TERMS, AND QUANTITY EXPRESSIONS OF ENGLISH

As we noted in Chapter 2, the syntax of English (and every natural language) is much more complicated than is the syntax of *SL*. *SL* is a language for sentential logic and uses sentence letters to symbolize whole sentences of English. Consequently, subsentential components of English sentences have no counterparts in *SL*. Among the subsentential components of English sentences are singular terms, predicates, and quantity expressions. Consider the following fairly simple argument:

> None of David's friends supports Republicans. Sarah supports Breitlow and Breitlow is a Republican. So Sarah is no friend of David's.

This is a valid argument. Although we can symbolize this argument in *SL* using the following symbolization key:

N: None of David's friends supports Republicans.
S: Sarah supports Breitlow
B: Breitlow is a Republican
F: Sarah is a friend of David's

the resulting argument is *not* valid in *SL*:

$$\begin{array}{l} \text{N} \\ \underline{\text{S \& B}} \\ \sim \text{F} \end{array}$$

The problem is that the atomic sentences of *SL* are logically independent of each other—the truth-value of an atomic sentence on a truth-value assignment has no bearing on the truth-value of other atomic sentences on that truth-value assignment. But the English sentences we are symbolizing *are not* logically independent of each other. If the sentences symbolized by 'S' and 'B' are true, then it is also true that Sarah supports a Republican. And if that is so, and the first premise of our English argument is true, it follows that Sarah is not a friend of David's. That is, it is the interconnections between the predicates 'supports Republicans', 'supports Breitlow', 'is a Republican', and 'is a friend of David's', the singular terms 'Sarah', 'David', and 'Breitlow', and the quantity expression 'None' that make the English language argument valid.

The sentence

> Each citizen will vote or will not vote

provides another illustration of the limitations of *SL*. This sentence is not presenting two alternatives, that every citizen votes or that no citizen votes. Rather it is expressing the logical truth that the predicate 'will vote or will not vote' is true of each citizen. This generalization about citizens applies to each citizen individually, not to citizens as a collective group. For example, it applies to Cynthia (presuming she is a citizen) and says of Cynthia that she will either vote or not vote. This claim about Cynthia, or any other specified citizen, can readily be symbolized as a truth-functional truth of *SL*. Where 'C' abbreviates 'Cynthia will vote', '$C \vee \sim C$' says of Cynthia what the general claim says of each citizen. But there is, barring heroic measures, no symbolization of the general claim in *SL* that is truth-functionally true.[1]

[1]Since there are presumably only finitely many citizens, we could construct a very long iterated conjunction with as many conjuncts of the sort '$C \vee \sim C$' as there are citizens. But even such heroic measures fail when the items about which we wish to talk (for example, the positive integers) constitute an infinite, and not just an exceedingly large, set.

What we need is a symbolic language whose syntax does mirror the use of singular terms, predicates, and quantity expressions in English. *PL* is such a language. However, before introducing *PL* we will explore how singular terms, predicates, and quantity expressions function in English. A **singular term** is any word or phrase that designates or purports to designate (or denote or refer to) some one thing. Singular terms are of three sorts: proper names, definite descriptions, and pronouns used in place of proper names or definite descriptions, that is, pronouns that make pronominal cross-reference to proper names or definite descriptions. Examples of proper names include 'George Washington', 'Marie Curie', 'Sir Arthur Conon Doyle', 'Rhoda', and 'Henry'. Generally speaking, proper names are attached to the things they name by simple convention. On the other hand, definite descriptions—for example, 'the discoverer of radium', 'the person Henry is talking to', 'Mary's best friend', and 'James' only brother'—pick out or purport to pick out a thing by providing a unique description of that thing.[2] A definite description is a description that, by its grammatical structure, describes or specifies at most one thing. Thus 'James' only brother' is a definite description whereas 'James' brother' is not—the latter could accurately apply to more than one person because James may have many brothers, whereas the former can apply to at most one. Pronouns that bear pronominal cross-reference to singular terms refer to the things those singular terms designate.

The sentence 'If Sue has read Darwin's works, she's no creationist' contains three singular terms, the proper name 'Sue', the definite description 'Darwin's works', and the pronoun 'she'. Each of these singular terms does refer to something. 'Sue' refers to Sue because, by convention, it is her name. 'Darwin's works' refers to the works of Darwin because it is a definite description of those works. And the pronoun 'she' refers to Sue because in this sentence 'she' is going proxy for 'Sue' (it bears pronominal cross-reference to 'Sue' and hence refers to the same entity as does 'Sue').

Predicates, such as 'supports Republicans', can be thought of as incomplete sentences that contain gaps or holes such that when those gaps are filled with singular terms the result is a complete sentence. However, writing predicates as we have just done does not visibly display the gap into which a singular term can be placed, and it is not suitable for displaying predicates containing more than one gap or hole. We could display predicates by indicating the gaps with underscores, as in

___ supports Republicans
___ is located between ___ and ___

[2]As these examples illustrate, definite descriptions can themselves contain singular terms. But we are here concerned only with singular terms that do not occur as constituents of other singular terms. For example, we here take 'The Roman general who defeated Pompey invaded both Gaul and Germany' to contain just three singular terms: 'The Roman general who defeated Pompey', 'Gaul', and 'Germany'. In Section 7.5 we shall introduce techniques that allow us to recognize and symbolize singular terms that are themselves constituents of singular terms—including 'Pompey' as it occurs in 'The Roman general who defeated Pompey'.

But for reasons that will emerge, it is useful to use the lowercase letters 'w', 'x', 'y', and 'z' (called 'variables' for reasons to be subsequently explained) to mark the places in predicates where singular terms can be placed. Using this convention we can specify the two predicates displayed above as 'x supports Republicans' and 'x is located between y and z'. A predicate with one gap is a one-place predicate, a predicate containing two gaps is a two-place predicate, and in general a predicate containing **n** gaps is an **n**-place predicate.

One way of generating a predicate is to start with a complete sentence of English containing one or more singular terms and delete one or more of those terms. And one way of generating a sentence from a predicate is to fill all the holes that are marked by variables with singular terms.

Because the gaps in predicates that are marked by variables can be filled with referring expressions—proper names, definite descriptions, and some uses of pronouns—we will say that these gaps are 'referential positions' and that expressions occurring in these positions, including variables, '**occur in referential position**'. But not all expressions that occur in referential positions do refer. Consider

If you play with fire you are likely to get burned.

In most contexts this sentence is used to comment about what is likely to happen to one, anyone, who plays with fire. Hence it would be a mistake to ask whom 'you', in either occurrence in the sentence, refers to. The sentence is a warning to all persons but does not refer to any particular person. Similarly, though 'Nobody' is a pronoun and does occur in referential position in

Nobody knows where Tom is

it does not make reference to anyone. There is no one whose name is 'nobody', nor does 'nobody' describe someone. We shall shortly discuss at length the use of pronouns that occur in referential position but do not in fact refer to anyone or anything.

It is not the case that all the singular terms of natural languages do refer or denote some one thing. For example, the only singular term in

Sherlock Holmes was a great detective

does not refer to a nineteenth-century English detective named 'Sherlock Holmes' who lived at 221B Baker Street, because there was no such detective. There are also definite descriptions that occur in referential positions but do not refer. Two examples are 'the present prime minister of the United States' and 'the largest prime number'. There is no prime minister of the United States and there is no largest prime number. This is a matter of some importance because by stipulation all of the individual constants of *PL*, the analogues of proper names and definite descriptions of English, do refer. Various strategies have been advanced for dealing with singular terms that do not refer, and we will explore one of them later in this chapter. But for the present we stipulate that all the singular terms we use in examples and exercises should be taken to refer.

Of course, what a singular term refers to is often context dependent. In its most familiar use 'George Washington' refers to the first president of the United States. But the U.S. Navy has an aircraft carrier named after the first president and so there are contexts in which 'George Washington' refers to a ship, not a man. Similarly, at a cocktail party where there is only one Henry, 'the person Henry is talking to' may refer to different persons at different times. Hereafter, when we use a sentence of English as an example or in an exercise set, we are assuming that sentence is being used in a context such that it is clear who or what the singular terms in that sentence refer to. We also note that when we are working with a group of sentences, the context that is assumed must be the same for all the sentences in the group. That is, we assume that a singular term that occurs several times in the piece of English discourse under discussion designates the same thing in each of its occurrences.

Predicates may contain multiple singular terms; in generating a predicate from a sentence containing multiple singular terms we may, but need not, delete all the singular terms. For example, 'New York City is north of Philadelphia' contains two singular terms, 'New York City' and 'Philadelphia', and we can obtain three distinct predicates by deleting one or both of these singular terms:

x is north of Philadelphia
New York City is north of x
x is north of y

As far as grammar is concerned, any singular term can be used to replace a variable in a predicate. Hence among the sentences we can generate from the two-place predicate 'x is north of y' and the singular terms 'Minneapolis', 'Chicago', and '3' are

Minneapolis is north of Chicago.
Chicago is north of Minneapolis.
Chicago is north of Chicago.
Chicago is north of 3.

The semantics we will adopt will assign a truth-value to each of these sentences.[3]

Given a stock of predicates, singular terms, and the sentential connectives '. . . and . . .', '. . . or . . .', 'if . . . then . . .', '. . . if and only if . . .', and 'it is not the case that . . .', we can generate a wide variety of sentences of English. For example, from these sentential connectives, the singular terms 'Henry', 'Sue', 'Rita', and 'Michael', and the predicates

x is easygoing
x likes y
x is taller than y

[3]It may be suggested that the fourth sentence is neither true nor false, as it "makes no sense". Numbers do not have location, so 3 is not located anywhere. But the semantics we will provide in Chapter 8 does allow for such sentences and counts them as false. Precisely because numbers do not have location it is false that any given number is spatially related to anything.

we can generate the following sentences

Michael is easygoing.
Sue is easygoing.
Michael is taller than Sue and Sue is taller than Henry.
Sue likes Henry and Michael likes Rita.
If Rita likes Henry, then Rita is taller than Henry.
Michael is easygoing if and only if it is not the case that Rita is easygoing.

If we allow the use of quantity expressions as well as singular terms to generate sentences from predicates, that is expressions such as 'everything', 'something', 'nothing', 'everyone', someone', and 'no one', we can also generate the following sentences from these same predicates:

Everyone is easygoing.
No one is easygoing.
Someone is easygoing.
Michael likes everyone.
Someone likes Sue.
No one is taller than her- or himself.

Note that the quantity expressions we used to generate these sentences, 'everyone', 'no one', and 'someone', all occur in referential positions, in positions where singular terms can occur. But these and other quantity expressions *are not* singular terms. 'No one' obviously does not refer to anyone. And neither does 'someone'. Consider 'Someone will win tonight's lottery'. We can all agree that this is true—there will be a winner. Suppose the winner turns out to be Henry Jacobson. It is not the case that when we asserted, in the morning, that someone would win the lottery we were referring to Henry Jacobson. All 'Someone will win tonight's lottery' asserts is that there is a person, identity presumably unknown, who will win tonight's lottery.

Similarly, 'Everyone' in 'Everyone is easygoing' does not refer to the totality of people or the set of all people, for it is individuals, not collections or sets of individuals, that are claimed to be easygoing. Rather, the force of 'Everyone is easygoing' is just that 'is easygoing' can be truly predicated of each and every individual.

7.1 EXERCISES

1. Identify the singular terms in the following sentences, and then specify all the predicates that can be obtained from each sentence by deleting one or more singular terms.

a. The president is a Democrat.
*b. The speaker of the house is a Republican
c. Sarah attends Smith College.

*d. Bob flunked out of U Mass.
e. Charles and Rita are siblings.
*f. 2 is greater than 1 and less than 4.

7.2 THE FORMAL SYNTAX OF *PL*

The language *PL* is far more powerful than the language *SL* because it includes constituents whose functions largely mirror the functions of **n**-place predicates, singular terms, and quantity terms ('every', 'some', 'no', . . .) in English.[4] In this section we present the formal syntax of *PL*, which is somewhat complicated (though not as complicated as the syntax of English).

The vocabulary of *PL* consists of:

Sentence Letters: The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts	A, B, C, . . . , Z, $A_1, B_1, C_1, \ldots, Z_1, \ldots$
Predicates: The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts, followed by one or more primes	$A', B', C', \ldots, Z'$, $A_1', B_1', C_1', \ldots, Z_1', \ldots$
Individual terms:	
Individual constants: The lowercase Roman letters 'a' through 'v', with or without positive-integer subscripts	a, b, c, . . . , v, $a_1, b_1, c_1, \ldots, v_1, \ldots$
Individual variables: The lowercase Roman letters 'w' through 'z', with or without positive-integer subscripts	w, x, y, z, $w_1, x_1, y_1, z_1, \ldots$
Truth-functional connectives:	$\sim, \&, \vee, \supset, \equiv$
Quantifier symbols:	$\forall, \exists$
Punctuation marks:	( )

The sentence letters of *PL* are just the sentence letters of *SL*. This makes every sentence of *SL* a sentence of *PL*. Officially, that a predicate of *PL* is an **n**-place predicate is indicated by the presence of **n** primes. So 'A'' is a one-place predicate and 'B''' is a two-place predicate of *PL*. But we also adopt the informal convention of allowing the omission of primes when the context makes it clear whether the predicate in question is a 1-place, 2-place, 3-place, . . . predicate, as when we write an **n**-place predicate with **n** distinct variables

[4] *PL* does not mirror all the subsentential relations present in English and other natural languages. Consequently, there are, as one might expect, English language arguments that are deductively valid but whose symbolizations in *PL* are not valid, English sentences that are logically true but whose symbolizations in *PL* do not reflect this, and so on. To deal with natural language discourse that cannot be adequately represented in *PL*, even more powerful formal systems are available—for example, tense logic and modal logic. A discussion of these systems is beyond the scope of this text.

following the predicate. So we will often write 'Ax' (or 'Ay' or 'Az' . . .) rather than 'A′', 'Bxy' rather than 'B″', and so on.

An **expression** of *PL* is a sequence of not necessarily distinct elements of the vocabulary of *PL*. All of the following are expressions of *PL*:

Lab
(∀∃xy
(∀w)(∃y)Fwy
((a ⊃ B)

In each case every character in the expression is an element of the vocabulary of *PL*. But the following are not expressions of *PL*:

{AbcB
(A ⊃ π)
(∀x/W

Each of these expressions contains a character that is not part of the vocabulary of *PL*. These are, respectively, '{', 'π' (the Greek letter pi), and '/'.

In what follows we will use the boldface capital letters '**P**', '**Q**', '**R**', and '**S**' as metavariables ranging over expressions of *PL*. We will also use boldface '**a**' as a metavariable ranging over individual constants of *PL* and boldface '**x**' as a metavariable ranging over individual variables of *PL*.

> *Quantifier of PL:* An expression of *PL* of the form (∀**x**) or (∃**x**). An expression of the first form is a **universal** quantifier, and one of the second form is an **existential** quantifier.

We will say that a quantifier *contains* a variable. Thus '(∀y)' and '(∃y)' both contain the variable 'y' (and are 'y-quantifiers'); '(∀z)' and '(∃z)' both contain the variable 'z' (and are 'z-quantifiers').

> *Atomic formulas of PL*: Every expression of *PL* that is either a sentence letter of *PL* or an **n**-place predicate of *PL* followed by **n** individual terms of *PL*.

We are now able to recursively define 'formula of *PL*':

1. Every atomic formula of *PL* is a formula of *PL*.
2. If **P** is a formula of *PL*, so is ~ **P**.
3. If **P** and **Q** are formulas of *PL*, so are (**P** & **Q**), (**P** ∨ **Q**), (**P** ⊃ **Q**), and (**P** ≡ **Q**).
4. If **P** is a formula of *PL* that contains at least one occurrence of **x** and no **x**-quantifier, then (∀**x**)**P** and (∃**x**)**P** are both formulas of *PL*.

5. Nothing is a formula of *PL* unless it can be formed by repeated applications of clauses 1–4.

Lastly, we specify the **logical operators** of *PL*:

An expression of *PL* that is either a quantifier or a truth-functional connective is a *logical operator* of *PL*.

It will emerge that not every formula of *PL* is a sentence of *PL*. However, before we can specify which formulas of *PL* are sentences of *PL* we need to define the terms 'subformula' and 'main logical operator':

1. Every formula is a subformula of itself.
2. If **P** is an atomic formula of *PL*, then **P** contains no logical operator, and hence no main logical operator, and **P** has no immediate subformula.
3. If **P** is a formula of *PL* of the form ~ **Q**, then the tilde ('~') that precedes **Q** is the main logical operator of **P**, and **Q** is the immediate subformula of **P**.
4. If **P** is a formula of *PL* of the form (**Q** & **R**), (**Q** $\vee$ **R**), (**Q** $\supset$ **R**), or (**Q** $\equiv$ **R**), then the binary connective between **Q** and **R** is the main logical operator of **P**, and **Q** and **R** are the immediate subformulas of **P**.
5. If **P** is a formula of *PL* of the form ($\forall$**x**)**Q** or of the form ($\exists$**x**)**Q**, then the quantifier that occurs before **Q** is the main logical operator of **P**, and **Q** is the immediate subformula of **P**.

The *subformulas* of a formula **P** of *PL* are

- **P** itself,
- The immediate subformulas of **P**,
- The subformulas of **P**'s immediate subformulas.

We can classify formulas of *PL* (and later sentences) by their main logical operator. Atomic formulas have no main logical operator. Quantified formulas have a quantifier as their main logical operator. Formulas whose main logical operator is a sentential connective are truth-functional formulas. Below we display several formulas and all of their subformulas. Remember that every formula is a subformula of itself.

Formula	Main Logical Operator	Formula Type
1. Rabz	None	Atomic
2. ~ (Rabz & Hxy)	~	Truth-functional
(Rabz & Hxy)	&	Truth-functional
Rabz	None	Atomic
Hxy	None	Atomic

3. (Hab ≡ (∀z)(Fz ⊃ Gza))	≡	Truth-functional
Hab	None	Atomic
(∀z)(Fz ⊃ Gza)	(∀x)	Quantified
(Fz ⊃ Gza)	⊃	Truth-functional
Fz	None	Atomic
Gza	None	Atomic
4. (∀y)(Hay ∨ (Fy ⊃ Gya))	(∀y)	Quantified
(Hay ∨ (Fy ⊃ Gya))	∨	Truth-functional
Hay	None	Atomic
(Fy ⊃ Gya)	⊃	Truth-functional
Fy	None	Atomic
Gya	None	Atomic

Quantifiers interpret the variables that fall within their **scope**.

Scope of a quantifier: The scope of a quantifier in a formula **P** of *PL* is the quantifier itself and the subformula **Q** that immediately follows the quantifier.

In other words, the scope of a quantifier is all of the formula of which the quantifier is the main logical operator, including the quantifier itself. Some examples will be helpful here. In the formula '(∃y)(Fyz & Gzy)' the subformula that immediately follows the quantifier '(∃y)' is '(Fyz & Gzy)' and accordingly the scope of that quantifier is all of '(∃y)(Fyz & Gzy)', and all of the variables in that formula, including the 'y' following '∃', fall within the scope of that quantifier. But in the formula 'Hx ⊃ (∀y)Fxy' the formula immediately following the quantifier '(∀y)' is 'Fxy' and the scope of that quantifier is therefore all of '(∀y)Fxy'. The first occurrence of 'x' (in 'Hx') does not fall within the scope of '(∀y)'. Similarly in '(∃w)(Gwa ⊃ Fa) ≡ Hw' the scope of '(∃w)' does not include the whole formula, for the formula that immediately follows that quantifier is 'Gwa ⊃ Fa'. Hence the first and second occurrences of 'w' in '((∃w)(Gwa ⊃ Fa) ≡ Hw)' fall within the scope of '(∃w)' but the third, in 'Hw', does not.

The final concepts we need to introduce before we define 'sentence of *PL*' are those of **free** and **bound** variables.

Bound variable: An occurrence of a variable **x** in a formula **P** of *PL* that is within the scope of an **x**-quantifier.
Free variable: An occurrence of a variable **x** in a formula **P** of *PL* that is not bound.

At long last we are ready to formally introduce the notion of a **sentence of *PL*:**

Sentence of PL: A formula **P** of *PL* is a sentence of *PL* if and only if no occurrence of a variable in **P** is free.

The formula '(Hx ⊃ (∀y)Fxy)' is not a sentence of *PL* because it contains a free variable. In fact, both occurrences of 'x' in this formula are free.

The first occurrence of 'x' does not fall within the scope of any quantifier and is therefore free, and the second occurrence of 'x', while falling within the scope of a quantifier, does not fall within the scope of an x-quantifier and is therefore free. The formula '$(\forall z)Gz \supset \sim Hz$' is not a sentence of *PL* because the third occurrence of 'z' does not fall within the scope of a z-quantifier. The scope of '$(\forall z)$' is limited to the subformula of which it is the main logical operator—that is, to '$(\forall z)Gz$'.

Earlier we considered the following four formulas of *PL*:

Rabz

$\sim$ (Rabz & Hxy)

$(Hab \equiv (\forall z)(Fz \supset Ga))$

$(\forall y)(Hay \vee (Fy \supset Gya))$

The first formula is not a sentence of *PL* because it contains 'z' as a free variable. We can construct a sentence from this formula by prefacing it with a z-quantifier; both '$(\exists z)Rabz$' and '$(\forall z)Rabz$' are sentences of *PL*. The second formula in our list is not a sentence of *PL* because 'z', 'x', and 'y' all occur free in that formula. This formula can be converted to a sentence by prefacing it with three distinct quantifiers, as in '$(\exists z)(\exists x)(\exists y) \sim (Rabz \& Hxy)$'. The third formula in our list is a sentence of *PL*. The only variable in '$(Hab \equiv (\forall z)(Fz \supset Ga))$' is 'z' and its two occurrences both fall within the scope of the quantifier '$(\forall z)$'. The fourth formula is also a sentence of *PL*. The formula of which '$(\forall y)$' is the main logical operator is the entire formula, and hence all four occurrences of 'y' fall within the scope of that quantifier.

There are formulas of *PL* that cannot be transformed into sentences of *PL* by adding quantifiers within whose scope the entire original formula falls. Consider '$(Fy \supset (\exists y)Gy)$'. The first occurrence of 'y' in this formula is free as it does not fall within the scope of any quantifier. The result of attaching a universal quantifier to this entire formula is '$(\forall y)(Fy \supset (\exists y)Gy)$'. But this expression is neither a formula nor a sentence of *SL*. The only way quantifiers become attached to formulas is in accordance with the fourth clause of the recursive definition of 'formula of *PL*', which is

4. If **P** is a formula of *PL* that contains at least one occurrence of **x** and no **x**-quantifier, then $(\forall \mathbf{x})\mathbf{P}$ and $(\exists \mathbf{x})\mathbf{P}$ are both formulas of *PL*.

This clause does not allow attaching '$(\forall y)$' to '$(Fy \supset (\exists y)Gy)$' because the latter already contains a y-quantifier, '$(\exists y)$'.

We have been omitting the primes that, by the formal requirements of *PL*, are parts of the predicates of *PL*, and we will continue to do so. We will also frequently omit the outermost parentheses of a formula of *PL*, as we did with *SL*. In our usage outermost parentheses are a pair of left and right parentheses that are added, as a pair, when a binary connective is inserted between two formulas of *PL*.

The omission of outermost parentheses should cause no confusion. Note, however, that when outermost parentheses are customarily dropped, it is not safe to assume that every sentence that begins with a quantifier is a quantified sentence. Consider

$$(\forall x)(Fx \supset Ga)$$

and

$$(\forall x)Fx \supset Ga$$

Both begin with quantifiers, but only the first is a quantified sentence. The scope of the x-quantifier in this sentence is the whole formula. The second sentence is a truth-functional compound; the scope of the x-quantifier is just '$(\forall x)Fx$'. It turns out that these two sentences say very different things.

To make complicated formulas of *PL* easier to read, we also allow the use of square brackets, '[' and ']', in place of the parentheses required by clause 3 of the recursive definition of 'formula of *PL*'. (But we will not allow square brackets in place of parentheses in quantifiers.) So, instead of

$$\sim (\forall y)((\exists z)Fzy \supset (\exists x)Gxy)$$

we can write

$$\sim(\forall y)[(\exists z)Fzy \supset (\exists x)Gxy]$$

In later chapters we shall require one further syntactic concept, that of a **substitution instance** of a quantified sentence. We use the notation $\mathbf{P}(\mathbf{a}/\mathbf{x})$ to specify the formula of *PL* that is like **P** except that it contains the individual constant **a** wherever **P** contains the individual variable **x**. Thus if **P** is

$$(Fza \vee \sim Gz)$$

then $\mathbf{P}(c/z)$ is

$$(Fca \vee \sim Gc)$$

Substitution instance of **P**: If **P** is a sentence of *PL* of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$, and **a** is an individual constant, then $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is a substitution instance of **P**. The constant **a** is the **instantiating constant**.

For example, 'Ga', 'Gb', and 'Gc' are all substitution instances of '$(\exists z)Gz$'. And 'Fab', 'Fbb', and 'Fcb' are all substitution instances of '$(\forall z)Fzb$'. 'Fab' is the result of substituting 'a' for 'z' in 'Fzb', 'Fbb' is the result of substituting 'b' for 'z' in 'Fzb', and 'Fcb' is the result of substituting 'c' for 'z' in 'Fzb'. In forming a substitution instance of a quantified sentence, we drop the initial quantifier and replace all remaining occurrences of the now free variable with some one constant. Thus '$(\exists y)Hay$' and '$(\exists y)Hgy$' are both substitution instances of '$(\forall x)(\exists y)Hxy$', but 'Hab' is not. (In forming substitution instances *only* the initial quantifier is dropped, and every occurrence of the variable that becomes free when that quantifier is dropped is replaced by the *same* constant.) All the following are substitution instances of '$(\exists w)[Fw \supset (\forall y)(\sim Dwy \equiv Ry)]$':

$$Fd \supset (\forall y)(\sim Ddy \equiv Ry)$$
$$Fa \supset (\forall y)(\sim Day \equiv Ry)$$
$$Fn \supset (\forall y)(\sim Dny \equiv Ry)$$

but

$$Fd \supset (\forall y)(\sim Dny \equiv Ry)$$

is not—for here we have used one constant to replace the first occurrence of 'w' and a different constant to replace the second occurrence of 'w'. Again, in generating substitution instances, each occurrence of the variable being replaced must be replaced by the same individual constant.

Only quantified sentences have substitution instances, and the substitution instances are formed by dropping the initial quantifier. Thus '$\sim Fa$' is *not* a substitution instance of '$\sim (\forall x)Fx$'. '$\sim (\forall x)Fx$' is a negation, not a quantified sentence, and hence has no substitution instances. '$(\forall x)Fxb$' is not a substitution instance of '$(\forall x)(\forall y)Fxy$' because, while the latter is a quantified sentence, only the initial quantifier can be dropped in forming substitution instances, and here the initial quantifier is '$(\forall x)$', not '$(\forall y)$'.

7.2E EXERCISES

1. Determine, for each of the following, whether it is a formula of *PL*, and if it is, whether it is a sentence of *PL*. If it is not a formula, explain why not. If it is a formula but not a sentence, explain why it is not a sentence. Then, if it is a formula of *PL*, list all of its components and identify the main logical operator, if any, of each by circling it, and for each subformula indicate whether it is an atomic, truth-functionally compound, or quantified formula. We here allow the deletion of outer parentheses and the use of square brackets in place of parentheses around binary compounds.

For example, a correct answer for the expression '$\sim (\exists z)Fz \ \& \ Hz$' would be Formula but not a sentence. The 'z' in 'Hz' is free.

$\sim (\exists z)Fz \ \& \ Hz$	Truth-functional
$\sim (\exists z)Fz$	Truth-functional
Hz	Atomic
$(\exists z)Fz$	Quantified
Fz	Atomic

a. Ba & Hc
*b. (∃x)(Fx & Gx)
c. ~ (∀y)Fya
*d. Fz ⊃ (∀z)Fz
e. (∃a)Ga
*f. Hxx ≡ (∃w)Fw
g. (∀x)(∀y) ~ Hxy
*h. (∃y) ~ Hyy & Ga
i. (∀y) ~ Fy ≡ ~ (∃w)Fw
*j. (∀x)Faa
k. (∃z)(Fz & ~ Baz)
*l. (∃x)[Fx & (∀x)(Fx ⊃ Gx)]
m. (∃x)Fx ∨ ~ (∃x)Fx
*n. ~ (∀x)(Gx ≡ Fx)
o. (∃x)(∃y)Lxx
*p. (∀x)[(∃y)Fyx ⊃ (∃y)Fxy]
q. Fa ⊃ (∃x)Fx
*r. Fa ≡ (∀x)Fa
s. ~ Fw ⊃ ~ (∃w)Gww

2. Indicate, for each of the following sentences of *PL*, whether it is an atomic sentence, a truth-functionally compound sentence, or a quantified sentence. Circle the main logical operator, if any.

a. (∀x) (Fx ⊃ Ga)
*b. (∀x) ~ (Fx ⊃ Ga)
c. ~ (∀x)(Fx ⊃ Ga)
*d. (∃w)Raw ∨ (∃w)Rwa
e. ~ (∃x)Hx
*f. Habc
g. (∀x)(Fx ≡ (∃w)Gw)
*h. (∀x)Fx ≡ (∃w)Gw
i. (∃w)(Pw ⊃ (∀y)(Hy ≡ ~ Kyw))
*j. ~ (∃w)(Jw ∨ Nw) ∨ (∃w)(Mw ∨ Lw)
k. ~ [(∃w)(Jw ∨ Nw) ∨ (∃w)(Mw ∨ Lw)]
*l. Da
m. (∀z)Gza ⊃ (∃z)Fz
*n. ~ (∃x)(Fx & ~ Gxa)
o. (∃z) ~ Hza
*p. (∀w)(~ Hw ⊃ (∃y)Gwy)
q. (∀x) ~ Fx ≡ (∀z) ~ Hza

3. Give a substitution instance of each of the following sentences in which ‘a’ is the instantiating term.

a. (∀w)(Mww & Fw)
*b. (∃y)(Mby ⊃ Mya)
c. (∃z) ~ (Cz ~ Cz)
*d. (∀x)[(Laa & Lab) ⊃ Lax]
e. (∃z)[Fz & ~ Gb) ⊃ (Bzb ∨ Bbz)]
*f. (∃w)[Fw & (∀y)(Cyw ⊃ Cwa)]
g. (∀y)[~ (∃z)Nyz ≡ (∀w)(Mww & Nyw)]
*h. (∀y)[(Fy & Hy) ⊃ [(∃z)(Fz & Gz) ⊃ Gy]]

i. $(\exists x)(Fxb \equiv Gbx)$
*j. $(\forall x)(\forall y)[(\exists z)Hzx \supset (\exists z)Hzy]$
k. $(\forall x) \sim (\exists y)(Hxy \ \& \ Hyx)$
*l. $(\forall z)[Fz \supset (\exists w)(\sim Fw \ \& \ Gwaz)]$
m. $(\forall w)(\forall y)[(Hwy \ \& \ Hyw) \supset (\exists z)Gzw]$
*n. $(\exists z)(\exists w)(\exists y)[(Fzwy \equiv Fwzy) \equiv Fyzw]$

4. Which of the following examples are substitution instances of the sentence '$(\exists w)(\forall y)(Rwy \supset Byy)$'?

a. $(\forall y)Ray \supset Byy$
*b. $(\forall y)(Ray \supset Byy)$
c. $(\forall y)(Rwy \supset Byy)$
*d. $(\forall y)(Rcy \supset Byy)$
e. $(\forall y)(Ryy \supset Byy)$
*f. $(\exists y)(Ray \supset Byy)$
g. $(Ray \supset Byy)$
*h. $(\forall y)(Ray \supset Baa)$

7.3 INTRODUCTION TO SYMBOLIZATION

Recall the sentences about Michael and his co-workers that we discussed in Section 7.1:

Michael is easygoing.
Sue is easygoing.
Michael is taller than Sue and Sue is taller than Henry.
Sue likes Henry and Michael likes Rita.
If Rita likes Henry, then Rita is taller than Henry.
Michael is easygoing if and only if it is not the case that Rita is easygoing.

We can now symbolize these sentences in *PL*. Here, as will frequently be the case throughout the rest of this chapter, we will use a symbolization key. A symbolization key specifies the **universe of discourse** ('UD' for short) or set of things we are talking about. Every UD is a *nonempty* set. A symbolization key also gives the English readings of the predicates of *PL* we will use in our symbolizations and assigns members of the UD to the individual constants we will use. Our symbolization keys will also assign truth-values to any sentence letters of *PL* that we will use in our symbolizations. We will specify the set we are using as the UD either as we do below, by listing the members inside curly brackets, or by using a description of the set, for example 'The set of positive integers'. In symbolizing our sentences about Michael, Sue, Henry, and Rita we will use the following symbolization key:

UD: The set {Michael, Sue, Henry, Rita}
Ex: x is easygoing
Txy: x is taller than y

Lxy:	x likes y
m:	Michael
s:	Sue
h:	Henry
r:	Rita

The English sentences we are symbolizing and our symbolizations of them in *PL* are as follows:

Michael is easygoing.	Em
Sue is easygoing.	Es
Michael is taller than Sue and Sue is taller than Henry.	Tms & Tsh
Sue likes Henry and Michael likes Rita.	Lsh & Lmr
If Rita likes Henry, then Rita is taller than Henry.	Lrh ⊃ Trh
Michael is easygoing if and only if it is not the case that Rita is easygoing.	Em ≡ ~ Er

In constructing our symbolization key we selected predicate letters and individual constants that may help us remember what English predicates and singular terms they symbolize. We will follow this practice throughout this chapter, but we note that a strong mnemonic connection between the predicates and singular terms of *PL* and the expressions of English they symbolize is not always possible.

We can also use symbolization keys to provide English readings of sentences of *PL*. For example, using our current symbolization key we can read

Lrh ≡ (Lhr & ~ Lhs)

as

Rita likes Henry if and only if Henry likes Rita and does not like Sue.

Using the same symbolization key we can also provide English readings for the following sentences of *PL*:

Lhr & ~ Lrh
Lrh ⊃ Lrm
Trh & ~ Trs
Tsh ⊃ Lhs
(Lmh ∨ Lms) ⊃ (Lmh & Lms)

In English these become, respectively,

Henry likes Rita and Rita does not like Henry.

If Rita likes Henry, then Rita likes Michael.

Rita is taller than Henry and Rita is not taller than Sue.

If Sue is taller than Henry, then Henry likes Sue.

If Michael likes Henry or Michael likes Sue, then Michael likes Henry and Michael likes Sue.

We can, of course, improve on the English. For example, the last sentence of *PL* can be more colloquially read as

If Michael likes either Henry or Sue he likes both of them.

Earlier we gave several examples of English sentences having quantity expressions in positions that singular terms can also occupy. Among these were

Everyone is easygoing.
No one is easygoing.

We can symbolize such sentences in *PL* without using quantifiers provided the discourse within which such sentences occur is about a finite, and for practical purposes, a reasonably small number, of things or individuals. For example, suppose we are again talking about only Michael, Sue, Henry, and Rita. Given this context, 'Everyone is easygoing' is equivalent to 'Michael, Sue, Henry, and Rita are easygoing' and this claim can be symbolized as an iterated conjunction:

$$(\mathrm{Em} \ \& \ \mathrm{Es}) \ \& \ (\mathrm{Eh} \ \& \ \mathrm{Er})$$

And 'No one is easygoing' is in this context equivalent to 'Neither Michael nor Sue nor Henry nor Rita is easygoing' and can be symbolized as the negation of an iterated disjunction:

$$\sim [(\mathrm{Em} \vee \mathrm{Es}) \vee (\mathrm{Eh} \vee \mathrm{Er})]$$

But these techniques are impractical when the number of things or individuals we are talking about is even moderately large. And we cannot, even in principle, symbolize quantity claims about an infinite number of things, say the positive integers, by using iterated conjunctions and disjunctions. For these purposes we do need the quantifiers of *PL*.

Suppose substantially more than four people work in Michael's office and we want to symbolize sentences about this larger group of individuals. We will use the following symbolization key to do so.

UD:	The set of people who work in Michael's office
Lxy:	x likes y
Rxy:	x respects y
m:	Michael
r:	Rita
h:	Henry

The first sentence we will symbolize is

Everyone likes Michael.

In symbolizing English sentences in *PL* it will often be useful to first paraphrase those sentences. Our paraphrases will be analogous to those we used in Chapter 2. We will use the terms 'each' and 'there is a(n)' followed by a variable to specify where quantifiers will occur in *PL*, and we will underline all expressions that are counterparts to the logical operators of *PL*. As 'Everyone likes Michael' is a claim about everyone in the UD, we will paraphrase it as:

<u>Each x is such that</u> x likes Michael.

Using 'Lxy' to symbolize 'x likes y' and using 'm' to designate Michael, our symbolization is

(∀x)Lxm

This sentence says that 'Lxm' is true of each thing in the UD, that is, each thing in the UD likes Michael. Our symbolization key includes

Lxy: x likes y

That we chose, arbitrarily, to use 'x' and 'y' in assigning an English reading to this 2-place predicate *does not mean* that whenever we use this predicate it must be followed by 'x' and then 'y'. A predicate of *PL* can be followed by *any* combination of the appropriate number of variables and individual constants. More generally, in symbolization keys, variables are used to mark the gaps in **n**-place predicates, not to specify what variables are to be used in symbolizations containing those predicates. We also could have used any variable in our paraphrase and any variable in our symbolization. For example, '(∀y)Lym', '(∀z)Lzm', and '(∀w)Lwm' are all correct symbolizations of 'Everyone likes Michael'.

Having symbolized 'Everyone likes Michael' it is easy to symbolize 'Michael likes everyone'. An appropriate paraphrase is

<u>Each x is such that</u> Michael likes x.

Our symbolization is '(∀x)Lmx'. Note that since Michael is part of the UD, it follows both from 'Everyone likes Michael' and from 'Michael likes everyone' that Michael likes Michael, that is, that Michael likes himself.

The sentence

Someone likes Michael and someone does not like Michael can be paraphrased and symbolized as a conjunction. Our paraphrase is

> <u>There is a y such that</u> y likes Michael <u>and</u> <u>there is a y such that</u> <u>it is not the case that</u> y likes Michael.

Our paraphrase is readily symbolized as

$$(\exists y)Lym \ \& \ (\exists y) \sim Lym$$

Again, there is no requirement that the variables we use in symbolization keys also be used in corresponding positions in symbolizations based on those symbolization keys. And there is no requirement that the variable we use in one subformula of a sentence of *PL* also be used in other subformulas unless those variables are being interpreted by the same quantifier. So we could equally correctly have symbolized 'Someone likes Michael and someone does not like Michael' as

$$(\exists x)Lxm \ \& \ (\exists z) \sim Lzm$$

Note that '$(\exists y)(Lym \ \& \sim Lym)$' says something very different from '$(\exists y)Lym \ \& \ (\exists y) \sim Lym$'. The former sentence says that there is someone who both likes Michael and does not like Michael.

The sentence

> Everyone who likes Michael also respects him

is readily paraphrased and symbolized as follows:

> <u>Each x is such that</u> (<u>if</u> x likes Michael <u>then</u> x respects Michael).
>
> $(\forall w)(Lwm \supset Rwm)$

And 'Someone likes and respects Michael' can be paraphrased and symbolized as

> <u>There is a y such that</u> (y likes Michael <u>and</u> y respects Michael).
>
> $(\exists y)(Lym \ \& \ Rym)$

It is important to understand why the main logical operator of the immediate subformula of '$(\forall w)(Lwm \supset Rwm)$' is a '$\supset$' while that of '$(\exists x)(Lxm \ \& \ Rxm)$' is an '&'. '$(\forall w)(Lwm \ \& \ Rwm)$' and '$(\forall w)(Lwm \supset Rwm)$' say quite different things. The former says that each member of the UD both likes and respects Michael. The latter attributes 'respects Michael' *only* to those members of the UD who do like Michael. When the UD is heterogeneous and we want to attribute some property to members of the UD that are of a particular sort, the most common way of doing so is to use a universally quantified sentence whose immediate subformula is a material conditional, that is, to use a sentence of the form $(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$. Such a sentence does not say that every member of the UD is of the sort **P**, nor does it say that every member of the UD is of the sort **Q**. Rather, it says of those members of the UD that *are* of the sort **P** that they are *also* of the sort **Q**. So, while a sentence of the form $(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ does say of every member of the UD that it is of the sort $\mathbf{P} \supset \mathbf{Q}$, when a member of the UD is not of the sort **P** this comes to naught.

On the other hand, when we do want to say that one or more members of the UD are of the sort **P** and of the sort **Q** our symbolization will be a sentence of the form $(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$. Note that '$\supset$' is not appropriate in this case, for $(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ says that there is at least one member of the UD such that if it is of the sort **P** then it is also of the sort **Q**. If a member of the UD is not of the sort **P**, then trivially it is such that *if* it is of the sort **P** (which it is not) *then* it is also of the sort **Q**. This is a much weaker claim than the claim made by a sentence of the form $(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$.

For the reasons just given, many of our symbolizations of English sentences will be either of the form $(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ or of the form $(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$ (where the variable **x** occurs in both **P** and **Q**). Sentences of the form $(\forall \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$ as well as those of the form $(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ are far less common as symbolizations of English sentences. Sentences of the form $(\forall \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$ are very strong. They say each thing in the UD is both of the sort **P** and of the sort **Q**. On the other hand, sentences of the form $(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ are extremely weak. On truth-functional grounds such sentences are equivalent to sentences of the form $(\exists \mathbf{x})(\sim \mathbf{P} \vee \mathbf{Q})$, which means all they say is that there is at least one thing that either is not of the sort **P** or is of the sort **Q**. The moral in both cases is that when we find we have constructed a symbolization that is of the form $(\forall \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$ or of the form $(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$ it is a good idea to double-check to make sure our symbolization is correct.

The quantity terms 'any' and 'anyone' are often appropriately symbolized by universal quantifiers. Such is the case in the sentence

Anyone who respects Michael also respects Rita.

Our paraphrase and symbolization are

<u>Each x is such that</u> (<u>if</u> x respects Michael <u>then</u> x respects Rita)

$(\forall x)(Rxm \supset Rxr)$

But some uses of 'any' *can* be symbolized by an existential quantifier. Consider 'If anyone respects Rita, Henry does'. This sentence is a material conditional and the consequent says that a specific person, Henry, respects Rita. This sentence can be paraphrased in two different ways:

<u>If</u> <u>there is an x such that</u> x respects Rita <u>then</u> Henry respects Rita,
<u>Each x is such that</u> (<u>if</u> x respects Rita <u>then</u> Henry respects Rita).

These can be symbolized, respectively, as '$(\exists x)Rxr \supset Rhr$' and '$(\forall x)(Rxr \supset Rhr)$'. The first sentence of *PL* is a material conditional, while the second is a universally quantified sentence whose immediate component is a material conditional. These sentences are equivalent, as are our paraphrases. If it is true that if there is a person that respects Rita, then Henry does, it is also true of each person that if that person respects Rita (which means that at least one person respects Rita) then Henry does, and vice versa. But neither of the following is a correct symbolization of 'If anyone respects Rita, Henry does':

$$(\forall x)Rxr \supset Rhr$$
$$(\exists x)(Rxr \supset Rhr)$$

The first of these sentences of *PL* says that if each person x is such that x respects Rita then Henry respects Rita, that is, that if *everyone* respects Rita then Henry does. This is not news, for Henry, being one of 'everyone', of course respects Rita if everyone does. That is, the sentence is logically true.

The second of these sentences of *PL* is an existentially quantified sentence whose immediate subformula, '$Rxr \supset Rhr$', is a material conditional. As pointed out above, a sentence of a form such as '$(\exists x)(Rxr \supset Rhr)$' is equivalent to '$(\exists x)(\sim Rxr \sim Rhr)$', which says that there is someone such that either that person does not respect Rita or Henry respects Rita. This is also a logical truth—and not surprisingly, because it is equivalent to the symbolization '$(\forall x)Rxr \supset Rhr$' that we discussed in the previous paragraph (later in this section we will explain why the equivalence holds). Not all sentences of the form $(\exists \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$, where $\mathbf{P}$ contains $\mathbf{x}$ but $\mathbf{Q}$ does not, are logically true but, as we have noted, no such sentence makes a very strong claim.

In English there are a fair number of different ways we can say that everything of this sort is also of that sort. For example, if we are talking about the people in Michael's office, all of the following sentences can be used to make the same claim:

Everyone who respects Henry also respects Rita.
Each person who respects Henry also respects Rita.
All those who respect Henry also respect Rita.
Anyone who respects Henry also respects Rita.
Those who respect Henry also respect Rita.
A person who respects Henry also respects Rita.

All of these can appropriately be paraphrased and symbolized as follows:

Each x is such that (if x respects Henry then x respects Rita).

$$(\forall x)(Rxh \supset Rxr)$$

Examples of English sentences that are appropriately symbolized as existentially quantified sentences are

There is someone who respects Henry and Rita,
Someone respects Henry and Rita,

and

At least one person respects Henry and Rita

These can all be paraphrased and symbolized as

<u>There is an x such that</u> (x respects Henry <u>and</u> x respects Rita)

$(\exists x)\,(Rxh \ \&\ Rxr)$

It is at least arguable that there are some uses of 'some' in English where 'some' means 'at least two'. We here note that the existential quantifier of *PL* always means 'there is at least one'. In Section 7.5 we will introduce an expansion of *PL*, *PLE,* and in that language we will be able to adequately symbolize such expressions as 'there are at least two' and 'there are exactly two' and thus accommodate those uses of 'some' in English where 'some' means' 'at least two'.

We next symbolize some sentences about the animals in the Saint Louis Zoo.[5]

- The dolphins want to swim with us.
- The jaguars prance on tree limbs.
- The grizzlies are discontent when forced to dine without wine.
- The alligators sup in sullen silence and the polar bears sunbathe without swim suits.
- The gorillas stare mutely but intently as the rhinos dance divinely.
- The great horned owls see and know all but say nothing.
- Neither the tigers nor the zebras ever change their stripes.

Our symbolization key will be

UD:	The set consisting of animals in the Saint Louis Zoo
Az:	z is an alligator
Bz:	z is a grizzly bear
Cz:	z sometimes changes its stripes
Dz:	z is a dolphin
Ez:	z sees everything
Fz:	z is discontent when forced to dine without wine
Gz:	z is a gorilla

[5]Some readers will recognize the influence of Simon and Garfunkel's whimsical song *At the Zoo.*

Iz:	z stares intently
Jz:	z is a jaguar
Kz:	z knows everything
Mz:	z stares mutely
Nz:	z says nothing
Oz:	z is a great horned owl
Pz:	z is a polar bear
Rz:	z is a rhinoceros
Sz:	z sups in sullen silence
Tz:	z is a tiger
Uz:	z prances upon tree limbs
Vz:	z dances divinely
Wz:	z wants to swim with us
Xz:	z sunbathes without a swim suit
Zz:	z is a zebra

As is to be expected with this large a symbolization key, not all of the predicate letters we have selected are mnemonic reminders of what they symbolize. Our first three symbolizations are straightforward:

- The dolphins want to swim with us.

  $(\forall x)(Dx \supset Wx)$

- The jaguars prance upon tree limbs.

  $(\forall y)(Jy \supset Uy)$

We can paraphrase our third example

- The grizzlies are discontent when forced to dine without wine. <u>Each x is such that</u> if x is a grizzly <u>then</u> x is discontent when forced to dine without wine and symbolize it as

  $(\forall w)(Bw \supset Fw)$

Our next three examples can be paraphrased and symbolized as conjunctions.

- The alligators sup in sullen silence and the polar bears sunbathe without swimsuits.

  <u>Each w is such that</u> if w is an alligator <u>then</u> x sups in sullen silence <u>and</u> <u>each x is such that</u> if x is a polar bear then x sunbathes without a swimsuit.

  $(\forall w)(Aw \supset Sw)\ \&\ (\forall x)(Px \supset Xx)$

Our current example can also (and equivalently) be paraphrased and symbolized as

Each w is such that [(if w is an alligator then w sups in sullen silence) and (if w is a polar bear then w sunbathes without a swimsuit)]

$(\forall w)[(Aw \supset Sw) \ \& \ (Pw \supset Xw)]$

- The gorillas stare mutely but intently; moreover, the rhinos dance divinely.

  Each z is such that [if z is a gorilla then (z stares mutely and z stares intently)] and each z is such that (if z is a rhino then z dances divinely)

  $(\forall z)[Gz \supset (Mz \ \& \ Iz)] \ \& \ (\forall z)(Rz \supset Vz)$

An alternative paraphrase and symbolization are equally appropriate:

Each x is such that ([if x is a gorilla then (x stares mutely and x stares intently)] and (if x is a rhino then x dances divinely))

$(\forall x)([Gx \supset (Mx \ \& \ Ix)] \ \& \ (Rx \supset Vx))$

Note that in our paraphrases we have used 'and' in place of both 'but' and 'moreover'.

- The great horned owls see and know all but say nothing.

  Each y is such that (if y is a great horned owl then [(y sees all and y knows all) and y says nothing])

  $(\forall y)(Oy \supset [(Ey \ \& \ Ky) \ \& \ Ny])$

- Neither the tigers nor the zebras ever change their stripes

can be paraphrased and symbolized in various ways, including as a conjunction of two universally quantified sentences and as a quantified sentence whose immediate component is a material conditional whose antecedent is a disjunction:

Each x is such that (if x is a tiger then it is not the case that x sometimes changes its stripes) and each y is such that (if y is a zebra then it is not the case that y sometimes changes its stripes).

$(\forall x)(Tx \supset \sim Cx) \ \& \ (\forall y)(Zy \supset \sim Cy)$

Each x is such that [if (x is a tiger or x is a zebra) then it is not the case that x sometimes changes its stripes]

$(\forall x)[(Tx \vee Zx) \supset \sim Cx]$

We specified that the sentences we have just symbolized are about a specific group of animals—those at the Saint Louis Zoo. This makes the use of 'the' ('The dolphins . . .', 'The grizzlies . . .') appropriate. In English, as we have already seen, we don't always use 'all', 'every', 'each' or other quantity terms when making universal claims. For example, 'Dolphins are good swimmers' is appropriately used to make a claim about all dolphins, everywhere. But in 'The dolphins want to swim with us' the use of 'the' indicates we are talking about some specific group of dolphins.

In our symbolization key we used 'z' in interpreting our one-place predicates. But in our symbolizations we sometimes used 'x', sometimes 'y', sometimes 'w', and sometimes 'z'. As we noted earlier, the variables we use in symbolization keys to interpret predicates need not be the variables we use in quantified sentences containing those predicates. What matters is that the variable we use in a quantifier *matches* the variables that the quantifier is intended to interpret. '$(\forall y)(Dy \supset Wy)$' and '$(\forall w)(Dw \supset Ww)$' are equally good symbolizations of 'The dolphins want to swim with us' but '$(\forall x)(Dx \supset Wy)$' is not a symbolization of that sentence at all. It is, in fact, not a sentence of *PL* because it contains a free variable, 'y'.

In symbolizing our sentences about zoo animals we constructed universally quantified sentences whose immediate components are truth-functional compounds, often material conditionals. Because these symbolizations are *universally* quantified sentences their immediate components—truth-functional compounds—are attributed to each and every member of the UD. We again note that the attribution is vacuous, comes to nothing, when the attribution is to a member of the UD that is *not* of the sort specified by the antecedent of the material conditional. So while '$(\forall w)(Jw \supset Uw)$' attributes '$Jw \supset Uw$' to all members of the UD, it attributes 'Uw' ('w prances upon tree branches') only to those members that 'Jw' ('w is a jaguar') is true of and says nothing of the non-jaguars.

We now augment our present symbolization key by adding the following two-place predicates to symbolize the sentences that follow:

Hxy:	x is heavier than y
Lxy:	x likes y

- Every animal likes every animal

  <u>Each x and each y are such that</u> (or <u>each pair x and y is such that</u>) x likes y.

  $(\forall x)(\forall y)Lxy$

- Every animal likes at least one animal.

  <u>Each x is such that</u> <u>there is a y such that</u> x likes y.

  $(\forall x)(\exists y)Lxy$

- There is an animal that likes all the animals.

  There is a z such that each w is such that z likes w.

  $(\exists z)(\forall w)Lzw$

- No animal is heavier than every animal.

  It is not the case that there is an x such that each y is such that x is heavier than y.

  $\sim (\exists x)(\forall y)Hxy$

- If an animal is heavier than another, then the second is not heavier than the first.

  Each x and each y are such that (if x is heavier than y then it is not the case that y is heavier than x).

  $(\forall x)(\forall y)(Hxy \supset \sim Hyx)$

- No animal is heavier than itself.

  It is not the case that there is an x such that x is heavier than x.

  $\sim (\exists x)Hxx$

- Every gorilla likes every rhinoceros.

  Each x is such that [if x is gorilla then each y is such that (if y is a rhinoceros then x likes y)].

  $(\forall x)[Gx \supset (\forall y)(Ry \supset Lxy)]$

'Every gorilla likes every rhinoceros' can also be equivalently paraphrased and symbolized as:

  Each x and each y are such that if [(x is a gorilla and y is a rhinoceros) then x likes y].

  $(\forall x)(\forall y)[(Gx \,\&\, Ry) \supset Lxy]$

- Every gorilla likes at least one rhinoceros.

Each x is such that [if x is a gorilla then there is a y such that (y is a rhinoceros and x likes y)].

$(\forall x)[Gx \supset (\exists y)(Ry \,\&\, Lxy)]$

- Every gorilla likes at least one rhinoceros and does not like at least one jaguar.

  Each z is such that (if z is a gorilla then [there is a w such that (w is a rhinoceros and z likes w) and there is a y such that (y is a jaguar and it is not the case that z likes y)]).

  $(\forall z)(Gz \supset [(\exists w)(Rw \,\&\, Lzw) \,\&\, (\exists y)(Jy \,\&\, \sim Lzy)])$

- The dolphins don't like the grizzly bears.

  Each y is such that [if y is a dolphin then each w is such that (if w is a grizzly bear then it is not the case that y likes w)].

  $(\forall y)[Dy \supset (\forall w)(Bw \supset \sim Lyw)]$

- Some tigers like all the jaguars but no tiger likes any grizzly bear.

  There is an x such that [x is a tiger and each y is such that (if y is a jaguar then x likes y)] and it is not the case that there is a w such that [w is a tiger and there is a z such that (z is a grizzly bear and w likes z)].

  $(\exists x)[Tx \,\&\, (\forall y)(Jy \supset Lxy)] \,\&\, \sim (\exists w)[Tw \,\&\, (\exists z)(Bz \,\&\, Lwz)]$

The right conjunct of our paraphrase can also be correctly symbolized as '$\sim (\exists w)(\exists z)[(Tw \,\&\, Bz) \,\&\, Lwz)]$'.

- Every dolphin is heavier than every great horned owl but no dolphin is heavier than any rhinoceros.

  Each x and each y are such that [if (x is a dolphin and y is a great horned owl) then x is heavier than y] and it is not the case that there is an x and there is a y such that [(x is a dolphin and y is an rhinoceros) and x is heavier than y].

  $(\forall x)(\forall y)[(Dx \,\&\, Oy) \supset Hxy] \,\&\, \sim (\exists x)(\exists y)[(Dx \,\&\, Ry) \,\&\, Hxy]$

- Anything that is heavier than every gorilla is a rhinoceros.

  Each w is such that [if each y is such that (if y is a gorilla then w is heavier than y) then w is a rhinoceros].

  $(\forall w)[(\forall y)(Gy \supset Hwy) \supset Rw]$

Notice that the scope of '$(\forall w)$' is the entire sentence, while the scope of '$(\forall y)$' is just '$(\forall y)(Gy \supset Hwy)$'. Compare this sentence of *PL* to

$$(\forall w)(\forall y)[Gy \supset (Hwy \supset Rw)],$$

which says that each pair of members of the UD is such that if one is a gorilla then if the other is heavier than that gorilla then it is a rhinoceros. In better English, this comes to 'Anything that is heavier than *any* gorilla [even one gorilla] is a rhinoceros'.

We can also use our symbolization key to construct English readings of the following sentences of *PL*:

- $(\forall y)[Jy \supset (\forall x)(Tx \supset \sim Lyx)]$

  Each y is such that if [y is a jaguar then each x is such that (if x is a tiger then it is not the case that y likes x)].

Since we are talking about each y and each x, this comes to

> Each jaguar and each tiger are such that it is not the case that the jaguar likes the tiger,

or more idiomatically:

> The jaguars do not like the tigers.

Note that it would be a mistake to read the sentence of *PL* we are currently considering as 'All the jaguars don't like all the tigers', for this English sentence is ambiguous. It can be taken to mean that it is not the case that all the jaguars like all the tigers, which is consistent with some of the jaguars liking some or all of the tigers. Our next example is

- $(\exists w)[Gw \ \&\ (\forall x)(Bx \supset Lwx)] \ \&\ \sim (\forall z)[Gz \supset (\forall x)(Bx \supset Lzx)]$

The left conjunct can be read as

> There is a w such that [w is a gorilla and each x is such that (if x is a grizzly bear then w likes x)]

which comes to 'There is a gorilla that likes every grizzly bear'. The right conjunct can be read as

> It is not the case that each z is such that [if z is a gorilla then each x is such that (if x is a grizzly bear then z likes x)]

which comes to 'Not every gorilla likes every grizzly bear'. An appropriate reading of the entire conjunction is

Some, but not all, of the gorillas like all the grizzly bears.

While *PL* can be used to symbolize claims about almost anything, including people, animals, all living things, countries, numbers, and whatever else our ontology (those things we take to exist) includes, the positive integers (the whole numbers 1, 2, 3, . . .) constitute an especially interesting UD for at least two reasons, and we will frequently use them as our UD in examples and exercises in the rest of this chapter as well as throughout Chapter 8. First, once one is familiar with the basic nature of the positive integers, symbolizing claims about them becomes fairly straightforward. Many claims about the positive integers and the relations among them are clear and unambiguous. This is often not true of sentences about other kinds of things. Second, if there is an interpretation of a set of sentences of *PL* on which all the members of the set are true then there is such an interpretation that uses the positive integers as the universe of discourse. This will be of considerable importance when we are working with the semantics of *PL*, as we will see in Chapter 8.

We will next symbolize a number of sentences about the positive integers. Readers may find it useful to consult Appendix 1, which details some simple facts about the positive integers, before proceeding. We will use the following symbolization key:

UD:	The set of positive integers
Lxy:	x is less than y
Ox:	x is odd
Ex:	x is even
Exy:	x times y is even
Oxy:	x times y is odd
Px:	x is a prime number
Sxy:	x is the successor of y (x = y + 1)
a:	2

- There is a smallest positive integer.

Symbolizing this sentence is fairly straightforward. All we need say is that there is a positive integer such that no positive integer is smaller than it. And this is what

$$(\exists y) \sim (\exists x)Lxy$$

says.

- There is no largest positive integer.

We do not have a predicate for 'x is larger than y' in our symbolization key, and we do not need one to symbolize this sentence. What would a largest positive integer be? It would be an integer such that there is no positive integer it is less than. And we do have a predicate for 'x is less than y': 'Lxy'. So the following says that there is a positive integer y such that there is no positive integer x such that y is less than x. That is, it says there is a largest positive integer.

$$(\exists y) \sim (\exists x)Lyx$$

So the negation of this sentence

$$\sim (\exists y) \sim (\exists x)Lxy$$

symbolizes 'There is no largest positive integer'.

- An odd positive integer times an odd positive integer is odd.

We can paraphrase and symbolize this sentence as follows:

> <u>Each x and each y are such that</u> [<u>if</u> (x is odd <u>and</u> y is odd) <u>then</u> x times y is odd]

$$(\forall x)(\forall y)[(Ox \,\&\, Oy) \supset Oxy]$$

- There is a pair of primes such that one member of the pair is the successor of the other member of the pair.

Our paraphrase and symbolization are

> <u>There is an x</u> and <u>there is a y such that</u> [(x is prime <u>and</u> y is prime) <u>and</u> y is the successor of x]

$$(\exists x)(\exists y)[(Px \,\&\, Py) \,\&\, Syx]$$

(This claim is true; 2 and 3 are both primes and 3 is the successor of 2.)

- An even positive integer times an even positive integer is an even positive integer.

Our paraphrase and symbolization are

> <u>Each x and each y are such that</u> [(<u>if</u> x is even <u>and</u> y is even) <u>then</u> x times y is even]

$$(\forall x)(\forall y)[(Ex \,\&\, Ey) \supset Exy]$$

- An even positive integer times an odd positive integer is an even positive integer.

Our symbolization of this sentence is like that of the preceding one, substituting 'Oy' for 'Ey', thus specifying that the second member of the pair is an odd, not an even, positive integer:

$$(\forall x)(\forall y)[(Ex \;\&\; Oy) \supset Exy]$$

Our next example combines the claims of the preceding two examples:

- An even positive integer times an even or an odd positive integer is even.

Our symbolization is

$$(\forall x)(\forall y)[(Ex \;\&\; (Ey \vee Oy)) \supset Exy]$$

Our final three sentences concern prime numbers. The first is

- 2 is prime and 2 has a prime successor.

$$Pa \;\&\; (\exists y)(Sya \;\&\; Py)$$

A literal reading of this sentence is '2 is prime and there is a successor of 2 and it is prime'.

Our second sentence about primes is

- 2 is prime and no prime number is less than 2.

Our symbolization is straightforward:

$$Pa \;\&\; {\sim}\, (\exists x)(Px \;\&\; Lxa)$$

Our last sentence concerning primes is

- 2 is an even prime and every prime greater than 2 is odd.

We can symbolize this sentence as a conjunction:

$$(Pa \;\&\; Ea) \;\&\; (\forall y)[(Py \;\&\; Lay) \supset Oy]$$

Before concluding this section, we note a limited parallel between *PL* and Aristotelian logic. Aristotelian logic recognizes four kinds of quantity claims, traditionally termed '**A-**', '**E-**', '**I-**', and '**O-sentences**':[6]

A-sentences	All **A**s are **B**s.
E-sentences	No **A**s are **B**s.

[6]The use of 'A', 'E', 'I', and 'O' to designate kinds of sentences apparently dates to the Middle Ages. A- and I-sentences are thought of as affirmations and match the first two vowels in the Latin verb 'affirmo' (which means 'I affirm') while E- and O-sentences are thought of as denials and match the first two vowels in the Latin verb 'nego' (which means 'I deny'). See Francis Garden, *Outline of Logic: For the Use of Teachers and Students*, 2nd ed. (Oxford and London: Rivingtons, 1871, p. 65).

I-sentences	Some **A**s are **B**s.
O-sentences	Some **A**s are not **B**s.

Here '**A**' and '**B**' are metavariables ranging over general terms, that is, terms such as 'people', 'horses', 'orators', 'fish', 'voters', and 'Athenians'. Here are examples of each kind of sentence:

A-sentence	All horses are mammals.
E-sentence	No horses are mammals.
I-sentence	Some horses are mammals.
O-sentence	Some horses are not mammals.

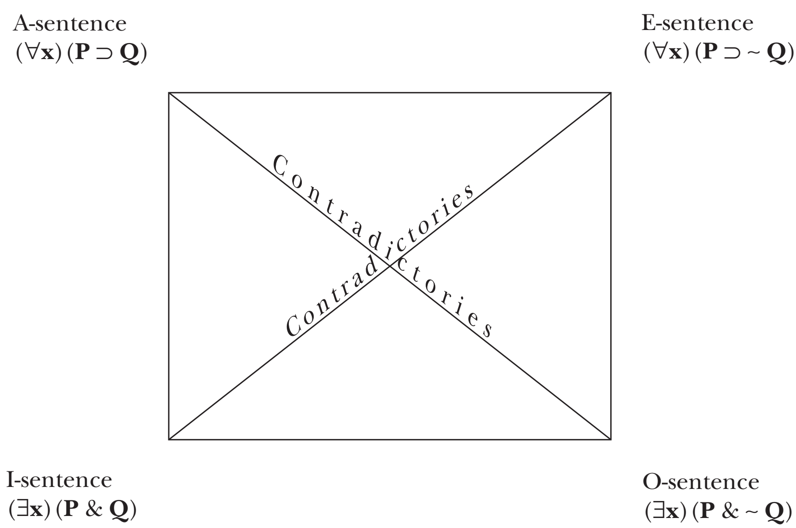
PL contains analogues to each of these kinds of sentences. Where **x** is a variable of *PL* and **P** and **Q** are open sentences of *PL*, each of which contains at least one occurrence of **x** and no **x**-quantifier, the *PL* analogues are

A-sentence	$(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$
E-sentence	$(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$
I-sentence	$(\exists \mathbf{x})(\mathbf{P} \,\&\, \mathbf{Q})$
O-sentence	$(\exists \mathbf{x})(\mathbf{P} \,\&\, \sim \mathbf{Q})$

We can use these templates to provide symbolizations of the above four claims about horses:

A-sentence	$(\forall x)(Hx \supset Mx)$
E-sentence	$(\forall x)(Hx \supset \sim Mx)$
I-sentence	$(\exists x)(Hx \,\&\, Mx)$
O-sentence	$(\exists x)(Hx \,\&\, \sim Mx)$

We are here taking our UD to be the set of living things, and using 'Hx' to symbolize 'x is a horse' and 'Mx' to symbolize 'x is a mammal'. The relations among these kinds of claims are often presented through a **square of opposition:**



Aristotle held that there are several interesting relationships among the four types of sentences displayed at the corners of the square of opposition. In *PL* the interesting relations are those between sentence types at opposite ends of the diagonal lines. These constitute contradictory sentence pairs. That is, an A-sentence is equivalent to the *negation* of the corresponding O-sentence and vice versa. And an E-sentence is equivalent to the *negation* of the corresponding I-sentence, and vice versa, yielding the following pairs of equivalent sentence forms:

$$(\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q}) \quad \text{and} \quad \sim (\exists \mathbf{x})(\mathbf{P} \;\&\; \sim \mathbf{Q})$$
$$(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q}) \quad \text{and} \quad \sim (\exists \mathbf{x})(\mathbf{P} \;\&\; \mathbf{Q})$$
$$(\exists \mathbf{x})(\mathbf{P} \;\&\; \mathbf{Q}) \quad \text{and} \quad \sim (\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$$
$$(\exists \mathbf{x})(\mathbf{P} \;\&\; \sim \mathbf{Q}) \quad \text{and} \quad \sim (\forall \mathbf{x})(\mathbf{P} \supset \mathbf{Q})$$

Knowing the foregoing equivalences can be helpful in symbolizing English sentences in *PL*, for these equivalences provide alternative patterns for symbolizing sentences that can be symbolized as A-, E-, I-, or O-sentences.

7.3E EXERCISES

1. Symbolize the following sentences in *PL*, without using quantifiers, using the following symbolization key:

UD: The set {Bob, Carol, David, Emily}
Gy: y will graduate
Jy: y will get a job
Ay: y will join the Army
Ly: y will become a longshoreman
Mxy: x will make more money than y
b: Bob
c: Carol
d: David
e: Emily

a. Bob and Carol will graduate and so will either David or Emily.
*b. If David doesn't graduate he will join the Army and if Emily doesn't graduate she will become a longshoreman.
c. If David joins the Army and Emily becomes a longshoreman, she will make more money than he will.
*d. All of those who graduate will get jobs.
e. If David will graduate they will all graduate.
*f. If at least one of them graduates they will all graduate.

2. Symbolize the following sentences in *PL*.

UD: The set of positive integers
Ex: x is even
Ox: x is odd

Lxy: x is less than y
Px: x is prime
a: 1
b: 2
c: 4
d: 100

a. Some positive integers are odd and some are even.
*b. Some positive integers are prime but not all positive integers are prime.
c. No positive integer is less than 1.
*d. No positive integer is less than itself.
e. 2 is less than 4 and 4 is less than some positive integer.
*f. Not every positive integer is less than 100.
g. Not all positive integers are prime and not all positive integers are even.
*h. Not all positive integers are primes and not all positive integers are non-primes.
i. All positive integers are even if and only if all positive integers are not odd.
*j. 1 is not prime and no positive integer is less than 1.
k. There is a positive integer that is less than 100.

3. Symbolize the following sentences in *PL*, using quantifiers wherever appropriate, using the following symbolization key:

UD: The set of seniors at Dartmouth College
Gy: y will graduate
Jy: y will get a job
Ay: y will join the Army
Ly: y will become a longshoreman
Mxy: x will make more money than y
b: Bob
c: Carol
d: David
e: Emily

a. All of those who graduate will get jobs.
*b. If David will graduate, all seniors will graduate.
c. If at least one senior graduates, they will all graduate.
*d. Everyone who doesn't graduate will join the Army.
e. If anyone joins the Army both Carol and David will.
*f. Everyone will graduate or no one will graduate.
g. Each senior will either graduate or not graduate.
*h. If anyone who graduates becomes a longshoreman Emily will become a longshoreman.
i. Everyone who becomes a longshoreman will make more money than will everyone who does not.
*j. Each senior will join the Army if and only if he or she does not graduate.

4. Using the following symbolization key, symbolize the following sentences in *PL*. (Note: Not all of these sentences are true.)

UD: The set of positive integers
Px: x is a prime

Ox:	x is odd
Ex:	x is even
Lxy:	x is less than y
Txy:	x times y is prime
Dxy:	x is evenly divisible by y (x is divisible by y without remainder)
a:	2

a. There is a positive integer that is less than all primes.
*b. A positive integer is even if and only if it is evenly divisible by 2.
c. A prime times a prime is not prime.
*d. A prime times an even positive integer is not prime.
e. A prime times any positive integer greater than 1 is not prime.
*f. If a pair of positive integers is such that the first is evenly divisible by the second, then either both integers are even or both are odd.
g. If a pair of positive integers is such that the first is evenly divisible by the second and the second is greater than 1, then either both integers are even or both are odd.
*h. For each prime, there is a greater non-prime.

7.4 SYMBOLIZATION FINE-TUNED

In this section we discuss some missteps that need to be avoided in symbolizing English sentences in *PL*, and we symbolize some sentences that are more complex than the ones we have so far dealt with.

There are contexts in English and other natural languages in which singular terms cannot be interpreted as denoting or referring to anything, and there are contexts in which predicates cannot be interpreted as we have been interpreting them. These contexts arise because we can think, dream, speculate, hunt for, and believe in (and give names to) things that do not exist. Consider, for example, the following claims:

Ponce de Leon is hunting for the Fountain of Youth.
Max is looking for trolls.

Ponce de Leon was a Spanish explorer of the fifteenth century who allegedly spent a lot of time looking for the Fountain of Youth. But of course there is no such thing. The nonexistence of such a fountain does not keep people from looking for it, though of course that nonexistence does prevent anyone from finding it. So too, although Norse mythology contains numerous descriptions of trolls there are no trolls. Nonetheless, it may well be true that our benighted friend Max is out looking for trolls.

Because there is no Fountain of Youth we cannot symbolize the sentence concerning Ponce de Leon as

Hpf

where ‘Hxy’ symbolizes ‘x is hunting for y’, ‘p’ designates Ponce de Leon, and ‘f’ designates the Fountain of Youth, because there is no such thing for ‘f’ to designate. Nor can we symbolize ‘Max is looking for trolls’ as

(∃y)(Ty & Lmy)

where ‘Tx’ symbolizes ‘x is a troll’, ‘Lxy’ symbolizes ‘x is looking for y’, and ‘m’ designates Max, because the English sentence ‘Max is looking for trolls’ does not entail ‘There are trolls’. For these reasons, we should instead symbolize the claim about Ponce de Leon as an atomic sentence of *PL* such as

Fp

where ‘Fx’ symbolizes ‘x is looking for the Fountain of Youth’ and ‘p’ designates Ponce de Leon. And we should symbolize our sentence about Max as an atomic sentence such as

Tm

where ‘m’ designates ‘Max’ and ‘Tx’ symbolizes ‘x is looking for trolls’. In the first case we have embedded the non-referring expression ‘the Fountain of Youth’ in a predicate, thus keeping it out of referential position. In the second case we have embedded ‘trolls’ in a larger predicate to avoid the problematic existential quantification.

A related problem arises when someone is looking for or seeking an object of a kind of which there are instances, but no particular instance is being looked for. Suppose that an orangutan—Sally, to be specific—has gone missing from the Saint Louis Zoo and the zookeeper, Mike by name, is in pursuit of her. In this situation the zookeeper is looking for a particular orangutan. Finding another orangutan might be a surprise, and perhaps even a pleasant surprise (for the zoo is short on orangutans), but this will not bring Mike’s search to an end. He is after Sally, not just any orangutan. In this situation, we can symbolize ‘Mike is looking for an orangutan missing from the Saint Louis Zoo’ as

(∃z)[(Oz & Mz) & Lmz],

where ‘Oz’ symbolizes ‘z is an orangutan’, ‘Mz’ symbolizes ‘z is missing from the Saint Louis Zoo’, ‘Lwz’ symbolizes ‘w is looking for z’, and ‘m’ designates Mike. Similarly, if all of the zoo’s orangutans have gone missing and Mike is in pursuit, we can accurately say that Mike is looking for all of the missing orangutans and symbolize this claim as

(∀y)[(Oy & My) ⊃ Lmy]

In the envisioned situation this sentence of *PL* accurately says ‘<u>Each y is such that</u> <u>if</u> y is an orangutan <u>and</u> y is missing from the Saint Louis Zoo <u>then</u> Mike is looking for y'. But the situation is quite different if Mike has been sent to Indonesia to acquire an orangutan for the zoo. In this context

Mike is looking for an orangutan
cannot be symbolized as
$(\exists x)(Ox \ \& \ Lmx)$

because it is not true that there is a particular orangutan that Mike is looking for. Nor is

$(\forall x)(Ox \supset Lmx)$

an appropriate characterization of Mike's activity, for this sentence says that he is looking for *all* orangutans, and he is not. Mike is neither looking for one particular orangutan nor looking for all orangutans. He does want to acquire an orangutan, but any orangutan will suffice. So we should symbolize 'Mike is looking for an orangutan' as an atomic sentence of *PL,* say, 'Lm, where 'Lx' symbolizes 'x is looking for an orangutan' and 'm' again designates Mike.

The general point is that we can look for, believe in, and dream about things that do not exist and we can look for, speculate about, and hope to find a certain sort of thing without there being a particular thing that we are looking for, speculating about, or hoping to find. We must symbolize sentences concerning these activities as we have just done, by embedding the problematic language in predicates that specify the relevant activity (thinking about, searching for, hoping to find, and so on).

We turn now to a more general discussion of how to decide what predicates it is appropriate to use in symbolizing sentences in *PL*. Usually this is a straightforward matter. But consider sentences such as the following:

There are rabid bats in the attic.

In symbolizing sentences such as this, where an adjective modifies a noun, we must decide how many predicates we should use. Should we use a single predicate, 'x is a rabid bat in the attic', two predicates, 'x is a rabid bat' and 'x is in the attic', or three, 'x is a bat, 'x is rabid', and 'x is in the attic'? Using just one predicate will yield '$(\exists x)Ix$' where 'Ix' symbolizes 'x is a rabid bat in the attic'. If we use two predicates our symbolization might be

$(\exists x)(Bx \ \& \ Ax)$,

where 'Bx' symbolizes 'x is a rabid bat' and 'Ax' symbolizes 'x is in the attic'. An appropriate symbolization using three predicates is

$(\exists x)[(Rx \ \& \ Bx) \ \& \ Ax]$,

here using 'Rx' to symbolize 'x is rabid', 'Bx' to symbolize 'x is a bat', and 'Ax' to symbolize 'x is in the attic'.

When the sentence 'There are rabid bats in the attic' is taken in isolation, or as part of a set of symbolization exercises, all three ways of symbolizing the sentence are correct symbolizations of 'There are rabid bats in the attic'. But there are contexts in which one symbolization is clearly preferable to the others. Consider this simple and clearly valid argument:

Rabid animals are dangerous.
There are rabid bats in the attic.

There are dangerous animals in the attic.

The symbolization key

UD:	The set of all animals
Rx:	x is rabid
Dx:	x is dangerous
Bx:	x is a rabid bat
Ax:	x is in the attic

yields an argument that is not valid in *PL*:

$(\forall x)(Rx \supset Dx)$
$(\exists x)(Bx \;\&\; Ax)$

$(\exists x)(Dx \;\&\; Ax)$

This argument is invalid because its component sentences do not reveal the connection between there being rabid bats in the attic and there being rabid animals in the attic. But if we use 'Bx' to symbolize 'x is a bat', rather than 'x is a rabid bat' the resulting symbolization of our argument *is* valid in *PL*:

$(\forall x)(Rx \supset Dx)$
$(\exists x)[(Rx \;\&\; Bx) \;\&\; Ax]$

$(\exists x)(Dx \;\&\; Ax)$

The lesson to be learned here is that when we are symbolizing a number of sentences and are interested in the relations among them it is advisable to select predicates that will capture as many of the connections among the English sentences as possible. But we must be careful. It is not always correct to extract two separate predicates when an adjective modifies a noun. Consider:

Sue is a ninety-eight pound gymnast.
Ed is an attractive candidate.
Stan is a meticulous accountant.

We can extract two predicates from the first of these sentences, parsing it as 'Sue is a gymnast and Sue weighs ninety-eight pounds', and symbolize it as 'Gs & Ns', using 's' to designate Sue, 'Gx' to symbolize 'x is a gymnast', and 'Nx' to symbolize 'x weighs ninety-eight pounds'. But we cannot similarly parse 'Ed is an attractive candidate', at least not in every context. Suppose Ed is running for state office and that he is fiscally conservative and a wounded war veteran. These traits may well make him an attractive candidate for state office, but they don't have anything to do with his being attractive in the sense of being a handsome man. In this case, we must treat 'is an attractive candidate' as *one* predicate. Similarly, it is probably unwise to parse 'Stan is a meticulous accountant' as 'Stan is meticulous and Stan is an accountant', for although Stan is a meticulous accountant, he may be anything but meticulous in the rest of his life.

We now turn our attention to finer issues concerning quantifiers. The syntax of *PL* requires that each variable occurring in a sentence of *PL* be bound, that is, fall within the scope of a matching quantifier. So '$(\forall x)(Fx \supset Gy)$' is a formula but not a sentence of *PL*, because 'y' is free in '$(\forall x)(Fx \supset Gy)$'. We have also seen that quantifiers can have overlapping scope. For example, we can transform '$(\forall x)(Fx \supset Gy)$' into a sentence by adding a universal y-quantifier. The three sentences we can obtain in this way are

$$(\forall x)(Fx \supset (\forall y)Gy)$$
$$(\forall x)(\forall y)(Fx \supset Gy)$$
$$(\forall y)(\forall x)(Fx \supset Gy)$$

The question now arises: are these three sentences equivalent? In subsequent chapters we will present techniques for answering this question but at present our concern is with symbolizing sentences in *PL*, and in symbolizing sentences in *PL* we need to understand the effects of placing quantifiers in different positions.

In fact, the three sentences of *PL* are equivalent. The second and third sentences are equivalent because whenever a sentence begins with multiple universal quantifiers or with multiple existential quantifiers and the rest of the sentence is in the scope of all of these quantifiers, the order in which the quantifiers appear does not matter. That is, changing the order does not change what the sentence says. So

$$(\exists x)(\exists y)(\exists z)Fxyz$$
$$(\exists y)(\exists x)(\exists z)Fxyz$$
$$(\exists z)(\exists y)(\exists x)Fxyz$$

are also equivalent sentences of *PL*, as are the results of placing the three existential quantifiers in any order. And as all of '$Fy \supset Gx$' falls within the scope of both quantifiers in

$$(\exists x)(\exists y)(Fy \supset Gx)$$

reversing the order of the quantifiers produces an equivalent sentence:

$$(\exists y)(\exists x)(Fy \supset Gx)$$

But our rule about reversing the order of initial quantifiers does not apply to '$(\exists x)[(\exists y)Fy \supset Gx]$', as this sentence does not begin with two existential quantifiers both having scope over the rest of the sentence. The scope of '$(\exists y)$' in '$(\exists x)[(\exists y)Fy \supset Gx]$' is just '$(\exists y)Fy$'.

When a sentence contains consecutive quantifiers of different types, existential and universal, such as '$(\forall x)(\exists y)$', we cannot in general change the order of those quantifiers. '$(\forall x)(\exists y)Lxy$' and '$(\exists y)(\forall x)Lxy$' *are not* equivalent sentences. Suppose we are using the set of positive integers as the UD and using 'Lxy' to symbolize 'x is less than y'. Then the first sentence says that every positive integer is less than some positive integer, while the second sentence says that there is a specific positive integer such that every positive integer is less than it.

Quantifiers can often be moved without producing nonequivalent sentences. They can, of course, only be moved if the result is *not* a formula containing a free variable. Consider the following pairs of sentences:

$Fa \;\&\; (\exists x)Gx$	$(\exists x)(Fa \;\&\; Gx)$
$Fa \;\&\; (\forall x)Gx$	$(\forall x)(Fa \;\&\; Gx)$
$Fa \vee (\exists x)Gx$	$(\exists x)(Fa \vee Gx)$
$Fa \vee (\forall x)Gx$	$(\forall x)(Fa \vee Gx)$
$Fa \supset (\exists x)Gx$	$(\exists x)(Fa \supset Gx)$
$Fa \supset (\forall x)Gx$	$(\forall x)(Fa \supset Gx)$

Careful reflection should convince the reader that all of these are pairs of equivalent sentences.

But there are two cases in which changing the scope of a quantifier requires changing the quantifier: in these cases if we broaden the scope of an existential quantifier we must replace it with a universal quantifier, and if we broaden the scope of a universal quantifier we must replace it with an existential quantifier. Here is an example of the first case:

$(\exists x)Gx \supset Fa$	$(\forall x)(Gx \supset Fa)$

These sentences are equivalent and it is fairly easy to see why they are. We discussed such a case when we symbolized 'If anyone respects Rita, Henry does'. We saw that this sentence can be correctly symbolized either as '$(\exists x)Rxr \supset Rhr$' or as '$(\forall x)(Rxr \supset Rhr)$'. Both will be true if either the UD does not contain anyone who respects Rita, or it contains at least one person who respects Rita and Henry respects Rita. So we can add '$(\exists x)Gx \supset Fa$' and '$(\forall x)(Gx \supset Fa)$' to our list of pairs of equivalent sentences.

The second case in which extending the scope of a quantifier requires changing the quantifier is illustrated by the following pair of sentences:

$(\forall x)Gx \supset Fa$ $\qquad\qquad$ $(\exists x)(Gx \supset Fa)$

It turns out, perhaps surprisingly, that these sentences are equivalent. It should be apparent that the first of these sentences is equivalent to '$\sim (\forall x)Gx \vee Fa$' on truth-functional grounds. Because '$\sim (\forall x)Gx$' is equivalent to '$(\exists x) \sim Gx$', '$\sim (\forall x)Gx \vee Fa$' is equivalent to '$(\exists x) \sim Gx \vee Fa$'. And since we can extend the scope of an existential quantifier over a wedge (providing the result *is* a sentence of *PL*), this sentence is equivalent to '$(\exists x)(\sim Gx \vee Fa)$', which, again on truth-functional grounds, is equivalent to '$(\exists x)(Gx \supset Fa)$'. So '$(\forall x)Gx \supset Fa$' and '$(\exists x)(Gx \supset Fa)$' are equivalent sentences.

The following table displays equivalent sentence forms. Here **P** is a formula containing at least one free occurrence of **x** and **Q** is a sentence of *PL* in which **x** *does not* occur.

$(\exists x)\mathbf{P} \supset \mathbf{Q}$	$(\forall x)(\mathbf{P} \supset \mathbf{Q})$
$(\forall x)\mathbf{P} \supset \mathbf{Q}$	$(\exists x)(\mathbf{P} \supset \mathbf{Q})$
$\mathbf{Q} \supset (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \supset \mathbf{P})$
$\mathbf{Q} \supset (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \supset \mathbf{P})$
$(\exists x)\mathbf{P} \vee \mathbf{Q}$	$(\exists x)(\mathbf{P} \vee \mathbf{Q})$
$(\forall x)\mathbf{P} \vee \mathbf{Q}$	$(\forall x)(\mathbf{P} \vee \mathbf{Q})$
$\mathbf{Q} \vee (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \vee \mathbf{P})$
$\mathbf{Q} \vee (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \vee \mathbf{P})$
$(\exists x)\mathbf{P} \ \& \ \mathbf{Q}$	$(\exists x)(\mathbf{P} \ \& \ \mathbf{Q})$
$(\forall x)\mathbf{P} \ \& \ \mathbf{Q}$	$(\forall x)(\mathbf{P} \ \& \ \mathbf{Q})$
$\mathbf{Q} \ \& \ (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \ \& \ \mathbf{P})$
$\mathbf{Q} \ \& \ (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \ \& \ \mathbf{P})$

Conspicuously absent from this table are sentence forms containing the triple bar. It turns out that in general, a sentence of the form $(\forall \mathbf{x})\mathbf{Px} \equiv \mathbf{Q}$ is equivalent neither to the corresponding sentence of the form $(\forall \mathbf{x})(\mathbf{Px} \equiv \mathbf{Q})$ nor to the corresponding sentence of the form $(\exists \mathbf{x})(\mathbf{Px} \equiv \mathbf{Q})$. Hence, the scope of a quantifier that includes only one side of a material biconditional *cannot* in general be broadened to have scope over the entire biconditional without creating a nonequivalent sentence.

We now turn to more complex symbolizations. Recall the argument we considered at the beginning of this chapter:

> None of David's friends supports Republicans. Sarah supports Breitlow and Breitlow is a Republican. So Sarah is no friend of David's.

We saw that symbolizations of this argument in *SL* are not valid. We are now in a position to provide a symbolization in *PL* that is valid. We will use the following symbolization key:

UD:	The set of all people
Fxy:	x is a friend of y
Sxy:	x supports y
Rx:	x is a Republican
d:	David
b:	Breitlow
s:	Sarah

The second premise is readily symbolized as the conjunction 'Ssb & Rb'. The conclusion is also easy to symbolize since it simply amounts to the claim that Sarah is not a friend of David's: '~ Fsd'. The first premise, however, may pose difficulties. An appropriate paraphrase is

<u>It is not the case that</u> <u>there is an x such that</u> [x is a friend of David's <u>and</u> (<u>there is a y such that</u> y is a Republican <u>and</u> x supports y)].

The expressions '<u>there is an x</u>' and '<u>there is a y</u>' are standing proxy for existential quantifiers. The structure of our paraphrase indicates that our symbolization will be a negation containing two existential quantifiers and two occurrences of '&'. Our symbolization mirrors the syntax of our paraphrase:

$$\sim (\exists x)[Fxd \ \& \ (\exists y)(Ry \ \& \ Sxy)]$$

This is a somewhat complicated case of a negated I-sentence. Our English argument can thus be symbolized as the following argument of *PL*:

$$\begin{array}{l} \sim (\exists x)[Fxd \ \& \ (\exists y)(Ry \ \& \ Sxy)] \\ \underline{Ssb \ \& \ Rb \qquad\qquad\qquad\qquad} \\ \sim Fsd \end{array}$$

The techniques presented in subsequent chapters can be used to show that this is a valid argument of *PL*.

We know that the negation of an I-sentence is equivalent to the corresponding E-sentence. This suggests that there is an alternative but equally correct symbolization of the first premise of our argument that has the form $(\forall \mathbf{x})(\mathbf{P} \supset \sim \mathbf{Q})$, and there is. We can alternatively paraphrase the argument's first premise as

<u>Each x is such that</u> [<u>if</u> x is a friend of David's <u>then</u> <u>it is not the case that</u> <u>there is a y such that</u> (y is a Republican <u>and</u> x supports y)].

Our symbolization of this paraphrase is

$$(\forall x)[Fxd \supset \sim (\exists y)(Ry \ \& \ Sxy)]$$

Here is a somewhat more interesting, and convoluted, argument:

> Anyone who is proud of anyone is proud of Samantha. Rhoda isn't proud of anyone who's proud of him- or herself, but she is proud of everyone who has mastered calculus. Therefore if Art has mastered calculus, Samantha isn't proud of herself.

We will use the following symbolization key:

UD:	The set of students in Samantha's class
Pxy:	x is proud of y
Mx:	x has mastered calculus
a:	Art
r:	Rhoda
s:	Samantha

The first premise can be paraphrased as

> <u>Each x is such that</u> [<u>if</u> <u>there is a y such that</u> x is proud of y <u>then</u> x is proud of Samantha]

and can be symbolized as

$$(\forall x)[(\exists y)Pxy \supset Pxs]$$

The second premise of our argument is a conjunction. The first conjunct is

> Rhoda isn't proud of anyone who's proud of him- or herself.

Although the quantity expression 'anyone' does not occur at the beginning of this sentence, it is clear that the sentence is saying something about anyone who is proud of him- or herself. And 'anyone' in this sentence will go over to a universal quantifier in our symbolic sentence, for the sentence says something about *all* those individuals in Samantha's class who are proud of themselves. Our paraphrase is

> <u>Each x is such that</u> (<u>if</u> x is proud of x <u>then</u> <u>it is not the case that</u> Rhoda is proud of x).

The second conjunct of the second premise is

> she (Rhoda) is proud of everyone who has mastered calculus

This is a claim about everyone in Samantha's class who has mastered calculus. Our paraphrase is

<u>Each x is such that</u> (<u>if</u> x has mastered calculus <u>then</u> Rhoda is proud of x).

Our symbolization of the entire second premise is thus

$(\forall x)(Pxx \supset \sim Prx) \ \& \ (\forall x)(Mx \supset Prx)$

The conclusion, 'If Art has mastered calculus, Samantha isn't proud of herself' is a simple truth-functional claim and can be symbolized as

$Ma \supset \sim Pss$

Our complete argument of *PL* is therefore

$$\begin{array}{l}(\forall x)[(\exists y)Pxy \supset Pxs] \\ \underline{(\forall x)(Pxx \supset \sim Prx) \ \& \ (\forall x)(Mx \supset Prx)} \\ Ma \supset \sim Pss\end{array}$$

Techniques developed in subsequent chapters can be used to show that this is a valid argument of *PL*.

We will next paraphrase and symbolize a number of sentences about the positive integers. We will paraphrase each sentence before we symbolize it, and we will classify our paraphrases and symbolizations according to the Aristotelian classification system introduced at the end of the last section. But our classification is arbitrary in the sense that, as the square of opposition illustrates, an English sentence that can be symbolized as an A-sentence can also be symbolized as the negation of an O-sentence, one that can be symbolized as an E-sentence can also be symbolized as the negation of an I-sentence, and so on. Some readers will find identifying sentences in terms of the Aristotelian classification system useful; others will not. We will use the following symbolization key:

UD:	The set of positive integers
Px:	x is prime
Ex:	x is even
Ox:	x is odd
Gxy:	x is greater than y
Dxy:	x is evenly divisible by y
Sxyz:	x is the sum of y and z
Txyz:	x is the product of y and z
a:	1
b:	2

- Not every positive integer is prime.

  Negation of an A-sentence

It is not the case that each x is such that x is prime.

$\sim (\forall x)Px$

- Every prime greater than 2 is odd.
  A-sentence

  Each x is such that [(if x is prime and x is greater than 2) then x is odd].

  $(\forall x)[(Px \,\&\, Gxb) \supset Ox]$

- The sum of two primes each of which is greater than 2 is even.
  A-sentence

  Each x, each y, and each z are such that if [([(x is prime and y is prime) and (x is greater than 2 and y is greater than 2)] and z is the sum of x and y) then z is even].

  $(\forall x)(\forall y)(\forall z)[([(Px \,\&\, Py) \,\&\, (Gxb \,\&\, Gyb)] \,\&\, Szxy) \supset Ez]$

- The sum of 2 and a prime greater than 2 is odd.
  A-sentence

  Each x and each y are such that if ([(x is prime and x is greater than 2) and y is the sum of 2 and x] then y is odd).

  $(\forall x)(\forall y)([(Px \,\&\, Gxb) \,\&\, Sybx] \supset Oy)$

- No product of primes is a prime.
  Negation of an I-sentence

  It is not the case that there is an x and a y and a z such that [(x is prime and y is prime) and (z is the product of x and y and z is prime)].

  $\sim (\exists x)(\exists y)(\exists z)[(Px \,\&\, Py) \,\&\, (Tzxy \,\&\, Pz)]$

- No product of a prime and a non-prime greater than 1 is prime.
  Negation of an I-sentence

  It is not the case that there is an x and a y and a z such that [(x is prime and it is not the case that y is prime) and [(y is greater than 1 and (z is the product of x and y and z is prime)]].

  $\sim (\exists x)(\exists y)(\exists z)[(Px \,\&\, \sim Py) \,\&\, [Gya \,\&\, (Tzxy \,\&\, Pz)]]$

- No prime greater than 2 is evenly divisible by 2.
  Negation of an I-sentence

It is not the case that there is a y such that [(y is prime and y is greater than 2) and y is evenly divisible by 2].

$\sim (\exists y)[(Py \ \& \ Gyb) \ \& \ Dyb]$

- No prime greater than 2 is evenly divisible by an even number.
  E-sentence

  Each w is such that [if (w is prime and w is greater than 2) then it is not the case that there is a z such that (z is even and w is evenly divisible by z)].

  $(\forall w)[(Pw \ \& \ Gwb) \supset \sim (\exists z)(Ez \ \& \ Dwz)]$.

- There are pairs of primes whose sum is prime.
  I-sentence

  There is an x and a y and a z such that [(x is prime and y is prime) and (z is the sum of x and y and z is prime)].

  $(\exists x)(\exists y)(\exists z)[(Px \ \& \ Py) \ \& \ (Szxy \ \& \ Pz)]$

The last group of sentences we symbolize are more complex than those we have so far dealt with, and the last of these is very complex. We will use the symbolization key:

UD:	The set of all books and all people
Uxy:	x understands y
Lxy:	x likes y
Axy:	x admires y
Rxy:	x reads y
Lx:	x is a logician
Px:	x is a person
p:	*Principia Mathematica*
a:	*Alice in Wonderland*
g:	*Green Eggs and Ham*

We note that the universe of discourse does not consist exclusively of people. This means that when we want to say something about people we will have to use the predicate ‘Px’ to distinguish them from other members of the UD.

Our first example can be symbolized as an I-sentence; our second example can be symbolized as the conjunction of two I-sentences:

- Someone understands *Principia Mathematica* and *Alice in Wonderland.*

  There is an x such that [x is a person and (x understands *Principia Mathematica* and x understands *Alice in Wonderland*)].

  $(\exists x)[Px \ \& \ (Uxp \ \& \ Uxa)]$

- Someone understands *Principia Mathematica* and someone understands *Alice in Wonderland.*

  <u>There is an x such that</u> (x is a person <u>and</u> x understands *Principia Mathematica*) <u>and</u> <u>there is an x such that</u> (x is a person <u>and</u> x understands *Alice in Wonderland*).

  $(\exists x)(Px \,\&\, Uxp) \,\&\, (\exists x)(Px \,\&\, Uxa)$

The difference between the two sentences we have just paraphrased and symbolized is that the first says that there is some one person who understands both the books in question. The second sentence says only that there is someone who understands *Principia Mathematica* and that there is someone who understands *Alice in Wonderland.* It does not say whether these are one and the same person.

The third example can be symbolized as an A-sentence, and the fourth as the negation of an I-sentence:

- Everyone who reads *Green Eggs and Ham* both understands it and likes it.

  <u>Each y is such that</u> [<u>if</u> (y is a person <u>and</u> y reads *Green Eggs and Ham*) <u>then</u> (y is understands *Green Eggs and Ham* <u>and</u> y likes *Green Eggs and Ham*)].

  $(\forall y)[(Py \,\&\, Ryg) \supset (Uyg \,\&\, Lyg)]$

- No one who reads *Principia Mathematica* either understands it or likes it.

  <u>It is not the case that</u> <u>there is a w such that</u> [(w is a person <u>and</u> w reads *Principia Mathematica*) & (w understands *Principia Mathematica* <u>or</u> w likes *Principia Mathematica*)].

  $\sim (\exists w)[(Pw \,\&\, Rwp) \,\&\, (Uwp \vee Lwp)]$

Our fifth example can be symbolized as an E-sentence and our sixth example as an A-sentence:

- Anyone who reads *Green Eggs and Ham* and likes it doesn't understand anyone who reads it and doesn't like it.

  <u>Each z is such that</u> <u>if</u> ([z is a person <u>and</u> (z reads *Green Eggs and Ham* <u>and</u> z likes *Green Eggs and Ham*)] <u>then</u> <u>it is not the case that</u> <u>there is a w such that</u> ([(w is a person <u>and</u> w reads *Green Eggs and Ham*) <u>and</u> <u>it is not the case that</u> w likes *Green Eggs and Ham*] <u>and</u> z understands w)]).

  $(\forall z)([Pz \,\&\, (Rzg \,\&\, Lzg)] \supset \sim (\exists w)([(Pw \,\&\, Rwg) \,\&\, \sim Lwg] \,\&\, Uzw))$

- Anyone who understands both *Principia Mathematica* and *Alice in Wonderland* is admired by every logician.

  Each x is such that if ([x is a person and (x understands *Principia Mathematica* and x understands *Alice in Wonderland*) then each y is such that (if y is a logician then y admires x))

  $(\forall x)([Px \& (Uxp \& Uxa)] \supset (\forall y)(Ly \supset Ayx))$

Our seventh example can be paraphrased and symbolized as the negation of an I-sentence:

- No one who is not a logician understands either *Principia Mathematica* or *Alice in Wonderland.*

  It is not the case that there is an x such that [x is a person and it is not the case that x is a logician and (x understands *Principia Mathematica* or x understands *Alice in Wonderland*)].

  $\sim (\exists x)[(Px \& \sim Lx) \& (Uxp \vee Uxa)]$

We can symbolize our eighth example as a conjunction of an O-sentence and the negation of an O-sentence:

- There are logicians who understand but do not like *Principia Mathematica* but there are no logicians who understand but do not like *Alice in Wonderland.*

  There is a z such that [(z is a logician and z understands *Principia MathematicaI)* and it is not the case that z likes *Principia Mathematica*] and it is not the case that there is a y such that [(y is a logician and y understands *Alice in Wonderland*) and it is not the case that y likes *Alice in Wonderland*]

  $(\exists z)[Lz \& (Uzp \& \sim Lza)] \& \sim (\exists y)[Ly \& (Uya \& \sim Lya)]$

We can symbolize our last example as a conjunction whose left conjunct is a negation of an A-sentence and whose right conjunct is an A-sentence.

- Not everyone admires those who understand *Principia Mathematica,* but those who do also admire those who understand *Alice in Wonderland.*

  It is not the case that each w is such that [if w is a person then each z is such that (if (z is a person and z understands *Principia Mathematica*) then w admires z)] and each x is such that [if [x is a person and each y is such that [if (y is a person and y understands *Principia Mathematica*) then x admires y]] then each z is such that [if (z is a person and z understands *Alice in Wonderland*) then x admires z]].

$\sim (\forall w)[Pw \supset (\forall z)((Pz \ \& \ Uzp) \supset Awz)] \ \& \ (\forall x)[[Px \ \& \ (\forall y)[(Py \ \& \ Uyp) \supset Axy]] \supset (\forall z)[(Pz \ \& \ Uza) \supset Pxz]]$

7.5 THE LANGUAGE *PLE* (PREDICATE LOGIC EXTENDED)

The language *PLE* is an expansion of *PL* and as such includes all the vocabulary of *PL*. In addition, *PLE* includes a two-place predicate that is defined as the identity predicate, and functors (used to express functions).

Our standard reading of 'some' is 'at least one'. Some may object that this is not an accurate reading, that 'some' sometimes means something like 'at least two'. It is alleged, for example, that to say

There are still some apples in the basket

when there is only one apple in the basket is at best misleading and at worst false. In any event we clearly do want a means of symbolizing such claims as

There are at least two apples in the basket.

We can do this by interpreting one of the two-place predicates of *PL* as expressing the identity relation. For example, we could interpret 'Ixy' as 'x is identical with y'. Given the symbolization key

UD:	The set of items in a basket of fruit
Nxy:	x is in y
Ixy:	x is identical with y
Ax:	x is an apple
b:	the basket

both

$(\exists x)(Ax \ \& \ Nxb)$

and

$(\exists x)[(Ax \ \& \ Nxb) \ \& \ (\exists y)(Ay \ \& \ Nyb)]$

say 'There is at least one apple in the basket'. The latter merely says it twice. But

$(\exists x)(\exists y)([(Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)] \ \& \sim Ixy)$

does say ‘There are at least two apples in the basket’. This sentence of *PL* can be paraphrased as ‘There is an x and there is a y such that ([(x is an apple and y is an apple) and (x is in the basket and y is in the basket)] and it is not the case that x is identical to y)’. This last clause is not redundant because using different variables does not commit us to there being more than one thing of the specified sort.

THE IDENTITY PREDICATE

An alternative to interpreting one of the two-place predicates of *PL* as expressing identity is to introduce a special two-place predicate and specify that it *always* be interpreted as expressing the identity relation. This is the course we shall follow. In adding this predicate to *PL*, we generate a new language, *PLE*. As an extension of *PL*, it includes all the vocabulary of *PL* and an additional two-place predicate. *PLE* also includes, as we detail later in this section, functors (used to express functions). The formulas and sentences of *PL* are also formulas and sentences of *PLE*.

The new two-place predicate that is distinctive of *PLE* is the **identity predicate,**

$=''$

When using this predicate we shall, as we have been doing with other predicates, omit the two primes as the number of individual terms used (two) will show that this is a two-place predicate. This predicate is always interpreted as the identity predicate. For example, ‘= ab’ says that a is identical to b. However, it is customary to write, informally, ‘a = b’, rather than ‘= ab’—that is to place one individual term before the predicate and one after it—and we shall follow this custom.

So, instead of ‘= ab’, ‘ = xy’, and ‘ = aa’, we write ‘a = b’, ‘x = y’, and ‘a = a’. And in place of, for example, ‘~ = ab’, we write ‘~ a = b’. Since the interpretation of ‘=’ is fixed, we never have to include an interpretation of this predicate in a symbolization key.

We can now symbolize ‘There are at least two apples in the basket’ in *PLE*, using the preceding symbolization key (but dispensing with the now superfluous ‘Ixy’), as

$(\exists x)(\exists y)([(Ax \;\&\; Ay) \;\&\; (Nxb \;\&\; Nyb)] \;\&\; \sim x = y)$

In *PLE* we can also say that there are just so many apples in the basket and no more—for example, that there is exactly one apple in the basket. An appropriate paraphrase is

> There is a y such that [(y is an apple and y is in the basket) and each thing z is such that [(if z is an apple and z is in the basket) then z is identical to y]].

A full symbolization is

$(\exists y)[(Ay \;\&\; Nyb) \;\&\; (\forall z)[(Az \;\&\; Nzb) \supset z = y]]$

What we are saying is that there is at least one apple in the basket and that anything that is an apple and is in the basket is *that very apple*.

Consider next

Henry hasn't read *Alice in Wonderland* but everyone else in the class has.

If we limit our universe of discourse to the students in the class in question, let 'h' designate Henry, and interpret 'Ax' as 'x has read *Alice in Wonderland*', we can symbolize this claim as

$$\sim Ah \ \& \ (\forall y)[\sim y = h \supset Ay]$$

And, using 'b' to designate Bob, we can symbolize 'Only Henry and Bob have not read *Alice in Wonderland*', as

$$\sim (Ah \vee Ab) \ \& \ (\forall x)[\sim (x = h \vee x = b) \supset Ax]$$

This says that neither Henry nor Bob has read *Alice in Wonderland* and that everyone else—that is, each person in the class who is neither identical to Henry nor identical to Bob—has read it.

We can also use the identity predicate to symbolize the following sentences of *PLE*:

1. There are apples and pears in the basket.
2. The only pear in the basket is rotten.
3. There are at least two apples in the basket.
4. There are two (and only two) apples in the basket.
5. There are no more than two pears in the basket.
6. There are at least three apples in the basket.

UD:	The set of items in a fruit bowl
Ax:	x is an apple
Nxy:	x is in y
Px:	x is a pear
Rx:	x is rotten
b:	the basket

We can recast sentence 1 as 'There is at least one apple and at least one pear in the basket', and symbolize it without using the identity predicate:

$$(\exists x)(\exists y)[(Ax \ \& \ Py) \ \& \ (Nxb \ \& \ Nyb)]$$

However, if we take sentence 1 to assert that there are at least two apples and at least two pears in the basket, we do need the identity predicate:

$$(\exists x)(\exists y)[((Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)) \ \& \sim x = y] \ \&$$
$$(\exists x)(\exists y)[((Px \ \& \ Py) \ \& \ (Nxb \ \& \ Nyb)) \ \& \sim x = y]$$

Sentence 2 says that there is one and only one pear in the basket and that that one pear is rotten:

$$(\exists x)[((Px \ \& \ Nxb) \ \& \ Rx) \ \& \ (\forall y)[(Py \ \& \ Nyb) \supset y = x]]$$

Sentence 3 says only that there are *at least* two apples in the basket, not that there are *exactly* two. Hence

$$(\exists x)(\exists y)[((Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)) \ \& \ {\sim}\, x = y]$$

To symbolize sentence 4 we start with the symbolization for sentence 3 and add a clause saying there are no additional apples in the basket:

$$(\exists x)(\exists y)([((Ax \ \& \ Ay) \ \& \ (Nxb \ \& \ Nyb)) \ \& \ {\sim}\, x = y] \ \& \\ (\forall z)[(Az \ \& \ Nzb) \supset (z = x \vee z = y)])$$

The added clause says, in effect, 'and anything that is an apple and is in the basket is either x or y'. Sentence 5 does not say that there are two pears in the basket; rather, it says that there are *at most* two pears in the basket. We can express this in *PLE* by saying that of any pears, x, y, and z that are in the basket these are really at most two; that is, either x is identical to y, or x is identical to z, or y is identical to z. In other words

$$(\forall x)(\forall y)(\forall z)[([(Px \ \& \ Py) \ \& \ Pz] \ \& \ [(Nxb \ \& \ Nyb) \ \& \ Nzb]) \supset \\ ((x = y \vee x = z) \vee y = z)]$$

Finally sentence 6 can be symbolized by building on the symbolization for sentence 3:

$$(\exists x)(\exists y)(\exists z)(([(Ax \ \& \ Ay) \ \& \ Az] \ \& \ [(Nxb \ \& \ Nyb) \ \& \ Nzb]) \ \& \\ [({\sim}\, x = y \ \& \ {\sim}\, y = z) \ \& \ {\sim}\, x = z)]$$

We now return to our discussion of positive integers. This time we will use this symbolization key for the sentences that follow.

UD:	The set of positive integers
Bxyz:	x is between y and z
Lxy:	x is larger than y
Sxy:	x is a successor of y
Ex:	x is even
Px:	x is prime
a:	1
b:	2
c:	10
d:	14

1. There is no largest positive integer.
2. There is a unique smallest positive integer.
3. 2 is the only even prime.
4. Every positive integer has exactly one successor.
5. 2 is the only prime whose successor is prime.

As we saw in our earlier discussion, we can symbolize sentence 1 without using the identity predicate, for to say that there is no largest positive integer it suffices to say that for every integer there is a larger integer (no matter what integer one might pick, there is an integer larger than it):

$$(\forall x)(\exists y)Lyx$$

It is also tempting to symbolize sentence 2 without using the identity predicate, for to say that there is a smallest positive integer seems to be to say that there is an integer that is not larger than any integer:

$$(\exists x) \sim (\exists y)Lxy$$

But while the foregoing does say that there is a smallest positive integer, it does not say that there is a unique such integer. So a better symbolization is

$$(\exists x)(\forall y)(\sim y = x \supset Lyx)$$

This sentence of *PL* says that there is an integer such that every integer not identical to it is larger than it. This does imply uniqueness.

Sentence 3, '2 is the only even prime', says that 2 is prime and is even and that all other primes are not even:

> 2 is prime <u>and</u> 2 is even, <u>and</u> <u>each</u> z is such that <u>if</u> z is prime <u>and</u> z is <u>not</u> identical with 2 <u>then</u> z is <u>not</u> even.

In *PLE*

$$(Pb \;\&\; Eb) \;\&\; (\forall z)[(Pz \;\&\; \sim z = b) \supset \sim Ez]$$

This is equivalent to

$$(Pb \;\&\; Eb) \;\&\; (\forall z)[(Pz \;\&\; Ez) \supset z = b]$$

Notice that we could equally well have paraphrased and symbolized sentence 3 as

> 2 is prime <u>and</u> 2 is even, <u>and</u> <u>it is not the case that</u> <u>there is</u> a z such that z is prime <u>and</u> z is even, <u>and</u> z is not identical with 2

and symbolized this claim as

$$(Pb \ \& \ Eb) \ \& \ {\sim} \ (\exists z)[(Pz \ \& \ Ez) \ \& \ {\sim} \ z = b]$$

Notice, too, that all three symbolic versions of sentence 3 are truth-functional compounds, not quantified sentences.

Sentence 4, ‘Every positive integer has exactly one successor’, can be symbolized as

$$(\forall x)(\exists y)[Syx \ \& \ (\forall z)(Szx \supset z = y)]$$

This says that each positive integer x has a successor y and that any integer that is a successor of x is identical to y—that is, that each positive integer has exactly one successor.

Sentence 5, ‘2 is the only prime whose (only) successor is prime’, can be paraphrased as a conjunction:

> (2 is prime <u>and</u> <u>there is an x such that</u> [(x is the successor of 2 <u>and</u> <u>each y is such that</u> (<u>if</u> y is the successor of 2 <u>then</u> y = x)) <u>and</u> x is prime]) <u>and</u> <u>each x and each y</u> are such that [(<u>if</u> x is the successor of y <u>and</u> (y is prime <u>and</u> <u>it is not the case that</u> y = b)) <u>then it is not the case that</u> x is prime]

The first conjunct can be symbolized as

$$Pb \ \& \ (\exists x)[(Sxb \ \& \ (\forall y)(Syb \supset y = x)) \ \& \ Px]$$

The second conjunct can be symbolized as

$$(\forall x)(\forall y)[(Sxy \ \& \ (Py \ \& \ {\sim} \ y = b)) \supset {\sim} \ Px]$$

Putting these together we obtain

$$(Pb \ \& \ (\exists x)[(Sxb \ \& \ (\forall y)(Syb \supset y = x)) \ \& \ Px]) \ \& \ (\forall x)(\forall y)$$
$$[(Sxy \ \& \ (Py \ \& \ {\sim} \ y = b)) \supset {\sim} \ Px]$$

DEFINITE DESCRIPTIONS

In Section 7.1 we discussed three kinds of singular terms of English: proper names, pronouns, and **definite descriptions.** We subsequently noted that individual constants of *PL* can be used analogously to singular terms of English that do refer. But following this practice means that the internal structure of definite descriptions is not represented in *PL*. Consider, by way of illustration, this argument:

> The Roman general who defeated Pompey invaded both Gaul and Germany. Therefore Pompey was defeated by someone who invaded both Gaul and Germany.

This is fairly obviously a valid argument. But its symbolization in *PL* is not valid:

UD:	The set of persons and countries
Ixy:	x invaded y
Dxy:	x defeated y
r:	The Roman general who defeated Pompey
p:	Pompey
g:	Gaul
e:	Germany

Treating 'The Roman general who defeated Pompey' as an unanalyzable unit, to be symbolized by 'r,' and paraphrasing the conclusion as <u>'There is an x such that</u> [x defeated Pompey <u>and</u> (x invaded Gaul <u>and</u> x invaded Germany)]' yields the following symbolization:

$$\frac{\text{Irg \& Ire}}{(\exists x)[\text{Dxp \& (Ixg \& Ixe)}]}$$

The techniques we develop for testing arguments of *PL* will show that this argument of *PL* is invalid. This should not be surprising, for the premise tells us only that the thing designated by 'r' invaded both Gaul and Germany; it does not tell us that that thing is a thing that defeated Pompey, as the conclusion claims.

By using the identity predicate we can capture the structure of definite descriptions within *PLE*. Suppose we paraphrase the first premise of the preceding argument as

> <u>There is an x such that</u> [[(x is a Roman general <u>and</u> x defeated Pompey) <u>and each y is such that</u> [<u>if</u> (y is a Roman general <u>and</u> y defeated Pompey) <u>then</u> y = x]] <u>and</u> (x invaded Gaul <u>and</u> x invaded Germany)].

Definite descriptions are, after all, descriptions that purport to specify conditions that are satisfied by exactly one thing. Using our current symbolization key, plus 'Rx' for 'x is a Roman general', we can symbolize the first premise as

$$(\exists x)[[(Rx \,\&\, Dxp) \,\&\, (\forall y)[(Ry \,\&\, Dyp) \supset y = x]] \,\&\, (Ixg \,\&\, Ixe)]$$

We shall later show that in *PLE* the conclusion '$(\exists x)[Dxp \,\&\, (Ixg \,\&\, Ixe)]$' does follow from this premise.

By transforming definite descriptions into unique existence claims, that is, claims that there is exactly one object of such-and-such a sort, we gain the further benefit of being able to symbolize English language definite descriptions that may, in fact, not designate anything. For example, taking the UD to be persons and using 'Dxy' for 'x is a daughter of y', 'Bx' for 'x is a biochemist', and 'j' to designate John, we might symbolize 'John's only daughter is a biochemist' as

$$(\exists x)[(Dxj \,\&\, (\forall y)(Dyj \supset y = x)) \,\&\, Bx]$$

If it turns out that John has no, or more than one, daughter, or that his only daughter is not a biochemist, the above sentence of *PLE* will be false, not meaningless or truth-valueless. This is an acceptable result.

PROPERTIES OF RELATIONS

Identity is a relation with three rather special properties. First, identity is a **transitive** relation. That is, if an object x is identical with an object y, and y is identical with an object z, then x is identical with z. The following sentence of *PLE* says, in effect, that identity is transitive:

$$(\forall x)(\forall y)(\forall z)[(x = y \& y = z) \supset x = z]$$

Many relations other than identity are also transitive relations. The predicates

x is larger than y
x is taller than y
x is an ancestor of y
x is heavier than y
x occurs before y

all express transitive relations. But, 'x is a friend of y' does not represent a transitive relation. That is, 'Any friend of a friend of mine is a friend of mine' is a substantive claim, and one that is generally false. Where x, y, and z are all variables of *PL* or *PLE* and A is a two-place predicate of *PL* or *PLE*, the following says that A expresses a transitive relation:

$$(\forall x)(\forall y)(\forall z)[(Axy \& Ayz) \supset Axz]$$

Identity is also a **symmetric** relation; that is, if an object x is identical with an object y, then y is identical with x. The following says that A is a symmetric relation:

$$(\forall x)(\forall y)(Axy \supset Ayx)$$

The following predicates also express symmetric relations:

x is a sibling of y
x is a classmate of y
x is a relative of y
x has the same father as does y

Note that neither 'x is a sister of y' nor 'x loves y' expresses a symmetric relation. Jane Fonda is a sister of Peter Fonda, but Peter Fonda is not a sister of

Jane Fonda. And, alas, it may be that Manfred loves Hildegard even though Hildegard does not love Manfred.

A relation is **reflexive** if and only if each object stands in that relation to itself. In *PL* and *PLE* the following says that A expresses a reflexive relation:

$(\forall x)Axx$

Identity is a reflexive relation. In an unrestricted UD it is rather hard to find other reflexive relations. For example, a little thought should show that none of the following expresses a reflexive relation in an unrestricted universe of discourse:

x is the same age as y
x is the same height as y
x is in the same place as y

Since the number 48 is not of any age, it is not the same age as itself nor the same height as itself. Numbers have neither age nor height, though inscriptions of numerals usually have both. So, too, neither the number 93 nor the set of human beings is in any place. Numbers and sets do not have spatial positions; hence neither is in the same place as itself. However, the relations just discussed are reflexive relations in suitably restricted universes of discourse. For example, if the universe of discourse consists exclusively of people, then

x is the same age as y

expresses a reflexive relation (it is also transitive and symmetric). Every person is the same age as him- or herself. In this restricted universe 'x is the same height as y' and 'x is in the same place as y' also represent reflexive relations. Each person is the same height as him- or herself and is in the same place as him- or herself. And, if the universe of discourse is restricted to the positive integers, then

x is evenly divisible by y

expresses a reflexive relation, for every positive integer is evenly divisible by itself. This relation is not symmetric (not every positive integer evenly divides all the positive integers it is evenly divisible by). However, 'x is evenly divisible by y' does express a transitive relation.

FUNCTIONS

A **function** is an operation that takes one or more element of a set as arguments and returns a single value. *Addition, subtraction, multiplication, square,* and *successor* are all common functions of arithmetic. Each returns, for each number or pair of numbers, a single value. *Addition* takes a pair of numbers as arguments and returns their sum; *multiplication* takes a pair of numbers and returns the product of those numbers; *subtraction* returns, for each pair of numbers, the first number minus the second. The *square* function returns, for each number, the result of multiplying that number by itself; the *successor* function returns, for any positive integer **n**, the integer **n** + 1.

Not all functions are arithmetic functions. We have already encountered truth-functions—functions that map values from the set consisting of the truth-values (the set {**T**, **F**}) to truth-values. *Negation* is a function of one argument that returns **F** when given **T** as an argument and returns **T** when given **F** as an argument. *Conjunction, disjunction,* the *material conditional,* and the *material biconditional* are all functions that take two arguments (two truth-values) and return a single truth-value. Characteristic truth-tables display the value of each of these functions for each pair of truth-values.

Functions are also found outside of formal logic and mathematics. Consider a set of monogamously married individuals.[7] Here *spouse* is a function that takes a single member of the set as an argument and returns that person's spouse as its value. For the set of all twins, the function *twin* returns, for each member of the set, that member's twin. In *PLE* we shall use lowercase italicized Roman letters *a*–*z*, with or without a positive-integer subscript, followed by one or more prime marks to symbolize functions. We call these symbols **functors.** Where **n** is the number of prime marks after the functor, the function assigned to the functor takes **n** arguments. For example, in talking about the set of positive integers, we might assign the successor function to the functor *f*.[8] We specify this assignment in a symbolization key much the way we have been assigning interpretations to predicates. The following symbolization key assigns the successor function to f':

UD:	The set of positive integers
f'(x):	the successor of x
Ex:	x is even
Ox:	x is odd
a:	2
b:	3

The variable x in parentheses indicates that we are assigning to f' a function that takes a single argument. The expression to the right of the colon assigns the successor function to f'. Given the above symbolization key,

Ob

says 3 is odd. The sentence

Of'(a)

[7]The example is from Geoffrey Hunter, *Metalogic: An Introduction to the Metatheory of Standard First Order Logic*, Paperback ed. (Berkeley: University of California Press, 1996).

[8]It is customary to use, where only a few functors are needed, the letters 'f'', 'g', 'h', . . . We will follow this custom.

says the successor of 2, which is 3, is odd. Both claims are, of course, true. And

$$f'(a) = b$$

says the successor of 2 is 3, which it is. Similarly,

$$(\exists x)Of'(x) \ \& \ (\exists x)Ef'(x)$$

says there is a positive integer whose successor is odd and there is a positive integer whose successor is even. We can also use the symbolization key to symbolize 'The successor of an even number is odd'. A first step is the quasi-English

$(\forall x)(Ex \supset$ the successor of x is odd)

The successor of x is $f'(x)$, so the full symbolization is

$$(\forall x)(Ex \supset Of'(x))$$

We can add the following to our symbolization key

$h''(x,y)$: the sum of x and y

and symbolize 'The sum of an even number and an odd number is odd' as

$$(\forall x)(\forall y)[(Ex \ \& \ Oy) \supset Oh''(x,y)]$$

Since the number of distinct individual terms occurring within the parentheses after a functor indicates how many arguments the function assigned to that functor takes, we can informally omit the primes that officially follow functors, just as we do for predicate letters. Hereafter we will do so.

Returning to our example of the set of twins, we can use the following symbolization key

UD:	The set of all twins
$f(x)$:	the twin of x
c:	Cathy
h:	Henry
j:	Jose
s:	Simone

to symbolize

Simone is Henry's twin

as

$$s = f(h)$$

and

Jose is Cathy's twin

as

$$j = f(c)$$

Using 'Bx' for 'x is bald', we can symbolize 'A twin is bald if and only if her or his twin is bald' as

$$(\forall x)\,[Bx \equiv Bf(x)]$$

and 'Some bald twins have twins that are not bald' as

$$(\exists x)\,[Bx \;\&\; \sim Bf(x)]$$

The symbolization

$$(\forall x)\,(\forall y)\,[(\exists z)\,(z = f(x) \;\&\; z = f(y)) \supset x = y]$$

says, in quasi-English, 'Any members of the UD x and y who are such that if there is a z who is both a twin of x and a twin of y then x and y are identical', or 'No one is a twin of two different twins'.

We require that the functions we symbolize with functors have the following characteristics:

1. An **n**-place function must yield one and only one value for each **n**-tuple of arguments.[9]
2. The value of a function for an **n**-tuple of members of a UD must be a member of that UD.

If the UD is the set of integers, the square root operation does not meet condition 1 because it can yield more than one value for its arguments (there are two square roots of 4, 2 and −2.). (It also fails to meet condition 2 because not all square roots of integers are integers.) If the UD is the set of positive integers, the subtraction function does not meet condition 2, because when y is greater than x, x minus y yields a value that is not a positive integer (3 minus 9 is −6, and −6 is not a positive integer). Subtraction *does* meet condition 2 when the UD is the set of all integers—positive, zero, and negative. If the UD is the set of positive integers, division also fails to meet condition 2 (3 divided by 9 yields 1/3, which is not a positive integer). Division *does* meet condition 2 when the UD is the set of positive rational numbers (positive integers plus numbers expressible as the ratio between positive integers). Finally division does not meet condition 1 when the UD is the set of *all* integers because it is undefined when the divisor is zero.

[9] An **n**-tuple is an ordered set containing **n** members.

As we have just seen, functors can be used to generate a new kind of individual term (in addition to the individual constants and variables of *PL*). We call these new terms **complex terms.** Complex terms are of the form

$$\boldsymbol{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n)$$

where $\boldsymbol{f}$ is an **n**-place functor and $\mathbf{t}_1$, $\mathbf{t}_2$, . . . $\mathbf{t}_n$ are individual terms. Further examples of complex terms include

$f(a,b)$
$h(a,b,c)$
$g(a)$
$f(b,b)$
$f(x,y)$
$f(a,y)$
$f(y,a)$
$g(x)$
$f(g(a),b)$
$f(a,g(x))$

Complex terms are complex in that they are always formed from a functor and at least one individual term. Some complex terms contain variables, and some do not. We call individual terms that do not contain variables **closed terms,** and those that do **open terms.** This makes *both* individual constants and complex terms that contain no variables closed terms. Complex terms that do contain at least one variable, as well as variables themselves, are open terms. Individual terms that are not complex terms (the individual constants and individual variables) are **simple individual terms.** In the above list, the first four complex terms are closed, the next four open, the ninth closed, and the last open. Note the last two examples. In each, one of the individual terms from which the example is built is itself a complex term. This is wholly in order, as complex terms are individual terms and can occur anywhere a constant can occur. The kinds of individual terms included in *PLE* are summarized in the following table:

INDIVIDUAL TERMS OF *PLE*

	Open	*Closed*
Simple	Individual variables	Individual constants
Complex	Individual term formed from a functor and *at least one* individual variable—for example, $f(x)$, $f(a,x)$, $g(f(a),y)$, $g(h(x,y),a)$	Individual term formed from a functor and containing *no* individual variable—for example, $f(a)$, $g(a,b)$, $f(g(a,f(a,c)))$

All of the following are formulas of *PLE*:

Faf(x)

Ff(x)a

Ff(a)b

(∀x)Faf(x)

(∀x)(∃y)Fxf(y)

In each of these examples 'F' is a two-place predicate. The first and second are formulas of *PLE* but are not sentences (because the x in 'f(x)' is not bound). The third, fourth, and fifth examples are all both formulas and sentences of *PLE*. The third says that f(a) bears the relation F to b. The fourth says that each thing x in the UD is such that a bears the relation F to f(x), that is, to the value of the function f as applied to x. The fifth says that each thing x in the UD is such that there is a thing y such that x bears the relation F to f(y). Every example contains a complex individual term, and all but the third an open complex individual term.

Consider this symbolization key:

UD:	The set of positive integers
Ox:	x is odd
Ex:	x is even
Px:	x is prime
Gxy:	x is greater than y
h(x,y):	the sum of x and y
f(x):	the successor of x
a:	1
b:	2

The sentence

(∀x)[Ex ⊃ Of(x)]

says, truly, that each positive integer is such that if it is even then its successor is odd. And

(∀x)[Ex ⊃ Ef(f(x))]

says, truly, that each positive integer is such that if it is even then the successor of its successor is also even. The sentence

(∀x)(∀y)[(Ex & Ey) ⊃ Eh(x,y)]

can be read in quasi-English as '<u>Each x and each y are such that</u> [<u>if</u> (x is even <u>and</u> y is even) <u>then</u> the sum of x and y is even]. This is, of course, true.

Here are further sentences of *PLE* that can be read in English using the above symbolization key. The sentence

$$(\forall x)(\forall y)[Gh(x,y)x \ \& \ Gh(x,y)y]$$

says that for any positive integers x and y the sum of x and y is greater than x, and the sum of x and y is greater than y. This is true. The sentence

$$(\exists x)Gxh(a,b)$$

says that there is a positive integer, x, that is greater than the sum of 1 and 2—that is, there is a positive integer that is greater than 3. This is also true. The sentence

$$(\forall x)(\forall y)[(Ex \ \& \ Oy) \supset Oh(x,y)]$$

says that, for any pair of positive integers x and y, if the first is even and the second is odd, then their sum is odd. This is true as well. Finally the sentence

$$(\forall x)(\forall y)[Ph(x,y) \supset \sim (Px \ \& \ Py)]$$

says that, for any pair of positive integers, if their sum is prime then it is not the case that they are both prime, or, in other words, that there are no prime numbers x and y such that their sum is also prime. This sentence is false; 2 and 3 are both prime, and so is their sum, 5.

THE SYNTAX OF PLE

In addition to the vocabulary of *PL*, the vocabulary of *PLE* also includes

> "=": The two-place identity predicate (fixed interpretation)
>
> *Functors of PLE:* Lowercase italicized Roman letters *a, b, c*, . . . , with or without a numeric subscript, followed by **n** primes.
>
> *Individual terms of PLE:*
>
> Individual constants are individual terms of *PLE*
>
> Individual variables are individual terms of *PLE*
>
> Expressions of the form $\boldsymbol{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n)$, where $\boldsymbol{f}$ is an **n**-place functor and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ are individual terms of *PLE*

We can classify the individual terms of *PLE* as follows:

> *Simple terms of PLE:* The individual constants and individual variables of *PLE*

Complex terms of PLE: Individual terms of the form $f(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$, where f is an $\mathbf{n}$-place functor

Closed individual term: An individual term in which no variable occurs

Open individual term: An individual term in which at least one variable occurs

Individual variables and functors that contain at least one individual variable are thus open terms. Individual constants and functors that contain no variables are thus closed terms.

In *PLE* a substitution instance is defined as follows:

Substitution instance of **P**: If **P** is a sentence of *PLE* of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$ and **t** is a closed individual term, then $\mathbf{Q}(\mathbf{t}/\mathbf{x})$ is a substitution instance of **P**. The individual term **t** is the *instantiating individual term.*

Note that every substitution instance of a sentence of *PL* is also a substitution instance of that same sentence in *PLE.*

7.5E EXERCISES

1. Symbolize the following sentences in *PLE* using the following symbolization key:

UD:	The set of all people
Sx:	x is a sailor
Lx:	x is lucky
Cx:	x is careless
Yx:	x dies young
Sxy:	x is a son of y
Dxy:	x is a daughter of y
Wx:	x is a Wilcox
d:	Daniel Wilcox
j:	Jacob Wilcox
r:	Rebecca Wilcox

a. Every Wilcox except Daniel is a sailor.
*b. Every Wilcox except Daniel is the offspring of a sailor.
c. Every Wilcox except Daniel is either a sailor or the offspring of sailor.
*d. Daniel is the only son of Jacob.
e. Daniel is the only child of Jacob.
*f. All the Wilcoxes except Daniel are sailors.
g. Rebecca's only son is Jacob's only son.
*h. Rebecca Wilcox has only one son who is a sailor.
i. Rebecca Wilcox has at least two daughters who are sailors.

*j. There are two and only two sailors in the Wilcox family.
k. Jacob Wilcox has one son and two daughters, and they are all sailors.

2. Give fluent English readings for the following sentences of *PLE* using the given symbolization key.

UD:	The set of positive integers
Lxy:	x is less than y
Gxy:	x is greater than y
Ex:	x is even
Ox:	x is odd
Px:	x is prime
f(x,y):	the product of x and y
t:	2
f:	5
n:	9

a. $(\forall x)(\exists y)Lxy$
*b. $(\exists x)(\forall y)(\sim x = y \supset Lxy)$
c. $(\exists x)(\forall y) \sim Lyx$
*d. $\sim (\exists x)(Ex \,\&\, Lxt)$
e. $(Pt \,\&\, Et) \,\&\, (\forall x)[(Px \,\&\, Ex) \supset x = t]$
*f. $\sim (\exists x)(\exists y)[(Px \,\&\, Py) \,\&\, Pf(x,y)]$
g. $(\forall y)(\forall z)[(Oy \,\&\, Oz) \supset Of(y,z)]$
*h. $(\forall y)(\forall z)[(Ey \,\&\, Ez) \supset Ef(z,y)]$
i. $(\forall y)(\forall z)[(Ey \vee Ez) \supset Ef(y,z)]$
*j. $(\forall x)[Ex \supset (\exists y)(Oy \,\&\, Gxy)] \,\&\, \sim (\forall x)[Ox \supset (\exists y)(Ey \,\&\, Gxy)]$
k. $(\exists x)[[Px \,\&\, (Gxf \,\&\, Lxn)] \,\&\, (\forall y)([Py \,\&\, (Gyf \,\&\, Lyn)] \supset y = x)]$

3. For a–p, decide whether the specified relation is reflexive, whether it is symmetric, and whether it is transitive (in suitably restricted universes of discourse). In each case give the sentences of *PL* that assert the appropriate properties of the relation in question. If the relation is reflexive, symmetric, or transitive only in a restricted universe of discourse, specify such a universe of discourse.

a. Nxy: x is a neighbor of y
*b. Mxy: x is married to y
c. Axy: x admires y
*d. Nxy: x is north of y
e. Rxy: x is a relative of y
*f. Sxy: x is the same size as y
g. Txy: x is at least as tall as y
*h. Cxy: x coauthors a book with y
i. Exy: x enrolls in the same course as y
*j. Fxy: x fights y
k. Wxy: x weighs the same as y
*l. Cxy: x contracts with y
m. Axy: x is an ancestor of y
*n. Cxy: x is a cousin of y
o. Lxy: x and y have the same taste in food
*p. Rxy: x respects y

4. Symbolize the following sentences in *PLE* using the given symbolization key.

UD:	The set of people in Doreen's hometown
Dxy:	x is a daughter of y
Sxy:	x is a son of y
Bxy:	x is a brother of y
Oxy:	x is older than y
Mxy:	x is married to y
Txy:	x is taller than y
Px:	x is a physician
Bx:	x is a baseball player
Mx:	x is a marine biologist
d:	Doreen
c:	Cory
j:	Jeremy
h:	Hal

a. Jeremy is Cory's son.
*b. Jeremy is Cory's only son.
c. Jeremy is Cory's oldest son.
*d. Doreen's only daughter is a physician.
e. Doreen's eldest daughter is a physician.
*f. Doreen is a physician and so is her eldest daughter.
g. Cory is Doreen's eldest daughter.
*h. Cory is married to Hal's only son.
i. Cory is married to Hal's tallest son.
*j. Doreen's eldest daughter is married to Hal's only son.
k. The only baseball player in town is the only marine biologist in town.
*l. The only baseball player in town is married to one of Jeremy's daughters.
m. Cory's husband is Jeremy's only brother.

5. Symbolize the following sentences in *PLE* using the given symbolization key.

UD:	The set of positive integers
Ox:	x is odd
Ex:	x is even
Px:	x is prime
a:	1
b:	2
$f(x)$:	the successor of x
$q(x)$:	x squared
$t(x,y)$:	the product of x and y
$s(x,y)$:	the sum of x and y

a. One is not the successor of any integer.
*b. One is not prime but its successor is.
c. There is a prime that is even.
*d. There is one and only one even prime.
e. Every integer has a successor.

*f. The square of a prime is not prime.
g. The successor of an odd integer is even.
*h. The successor of an even integer is odd.
i. If the product of a pair of positive integers is odd, then the product of the successors of those integers is even.
*j. If the product of a pair of positive integers is even, then one of those integers is even.
k. If the sum of a pair of positive integers is odd, then one member of the pair is odd and the other member is even.
*l. If the sum of a pair of positive integers is even, then either both members of the pair are even or both members are odd.
m. The product of a pair of prime integers is not prime.
*n. There are no primes such that their product is prime.
o. The square of an even number is even and the square of an odd number is odd.
*p. The successor of the square of an even number is odd.
q. The successor of the square of an odd number is even.
*r. 2 is the only even prime.
s. The sum of 2 and a prime other than 2 is odd.
*t. There is exactly one integer that is prime and is the successor of a prime.
u. There is a pair of primes such that their product is the successor of their sum.