

SAT Math: Tips! Tricks! Traps! Techniques!

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This guide will seem like a really random list of tricks, techniques, and facts, and, indeed, it is not much more than that. The purpose of this guide is to augment the reader's mathematical background and is predicated on an average understanding of math that every high school upper-classperson should have in order to be successful on these standardized exams. This is not meant to be a one-stop-shop for all tricks, and is not meant as a substitute for a solid basis in high school math. Some sections are incomplete at the time of this printing, but I am filling them in, ongoing. Please check back.

0 Remarks on test-taking techniques

0.1 Skip questions you have no idea about, for now

Skipping questions you have no idea how to start is a pretty good strategy. It ensures that you will get to solve and grid the questions you *do* know how to do and get credit for them. Something that happens often on these exams is that working a slightly different question later on in the test will remind us of an approach that we can use in a question that we skipped. Finally, when all is said and done, we can guess on such questions, and, remember, probability of getting a guess correct goes up, the more choices you can eliminate, so eliminate before guessing! Yes, eliminating even *one* choice helps!

0.2 Read the choices

After reading the question, it's a good idea to scan the choices right away. Sometimes, you can eliminate some or all of the wrong answers and proceed to gridding the correct one quickly. Reading the choices also helps set our expectations for the answer that we should get, and, often, we can tell partway through calculations that we are not headed to a correct answer, *or!* that we misread something. Finally, if you read the question and have no idea what to do, reading the choices sometimes gives us a hint about how we might proceed.

0.3 Skip over very wordy-looking questions

Cannot stress this one enough. If a math problem looks like a reading passage, skip it on your first pass through the section. The idea is to maximize points by picking off the more straightforward-looking algebra problems and knock them out. You can also scan visually and look for sparse ink - those are the questions to do first.

0.4 Watch for tricky wording; vigilance¹

We are so used to seeing that we should solve for x or one value or one variable. The SAT employs a classic move for which, even if you solve for x correctly, you get the question wrong. That is, they ask for something that requires that you *know* x , like, What is x^2 ? or What is half of x ? or What would the sum of the remaining two angles be in that triangle? Etc. Solve for x , yes. But making sure you know what to *do* with x - *by looking at what's being asked for* - is key. For sure, the value of x will be one of the choices, just to catch you! Don't fall for this trickery! Be vigilant!

¹Free vocab practice: vigilance: noun. state or quality of being vigilant; watchfulness

0.5 Saved seconds accumulate!

I make mention of many things in this handout that save a few seconds on calculations. Remember to keep the long game and the complete time management game in mind. If you save 5 seconds 5 times, you have 25 seconds. If you save 5 seconds 12 times, you have a full minute. Some of these techniques shave off more than 5 seconds. Don't underestimate how valuable saving time is!

0.6 Calculator use: Be. Judicious.² Yes, even on the Calculator section.

0.6.1 Trust yourself on basic arithmetic

At this point in your math career, please take a second to breathe and verify that you know the results of simple arithmetic operations: $2 + 3 = 5$, $4 \cdot 25 = 100$, half of 1000 is 500, etc. Trust yourself. Furthermore, **calculator usage slows you down**. Try to knock out the simple operations in any problem and then use the calculator for only the most difficult operations for mental math. This is particularly important on the free-response questions, when, say, you are making a decimal representation of your answer, as in, you got $\frac{\sqrt{2}}{2}$ as your answer, but you wish to grid 0.7071.

0.6.2 False sense of security & source of errors!

Being able to use a calculator often tempts test-takers to feel overly secure, when, in reality, calculators sometimes make errors happen more often. Typing mistakes, parentheses mistakes, errors in settings, etc. Combine this with (0.6.1), above. Use the calculator only as the last step or two of the problem.

0.7 Replicating test conditions during practice

I cannot stress this one enough. Not just doing timed tests, but *doing timed tests on paper, with practice gridding answers*. Books are fine, too. I'd say that practice tests on the Web, etc. are fine when doing drills that refresh/test various skills, but nothing is like paper and pencil, managing the booklet and answer sheet and calculator, managing time, etc. Many of the things I discuss in this document are geared toward time management (0.5). When practicing via the Web, take a moment to think of the time you spend:

- Clicking the mouse on the correct choice and clicking Next.
- Scrolling up and down to get the accompanying diagram in focus when the diagram and choices cannot fit on the same page nor in the same window.
- Paging back and forth between questions and web pages. As of this writing, I am not sure if one can even skip around in a particular section to employ section 0.3's advice.
- Writing down problems on a piece of paper so you can do algebra.
- Copying a complex geometry question onto paper so you can begin solving. Verifying that you drew the replica correctly.
- This even extends to the reading and grammar sections! *How does one underline phrases and annotate a reading passage on a web page????* (Right.)

Practice gridding answers on paper is important. Practice gridding and practice managing skipped questions are skills intimately tied to success on these sorts of tests. The answer sheet scanner is absolutely unforgiving in this sense. Consider: If you answered all questions correctly, but gridded incorrectly by failing to leave a space for even *one* skipped question, you have basically *randomized* your answers; only probability can help you. You don't want to leave all of this hard work to chance on test day, do you?

²Free vocab practice: judicious: adjective. 1. using or showing judgment as to action or practical expediency; discreet, prudent, or politic. 2. having, exercising, or characterized by good or discriminating judgment; wise, sensible, or well-advised.

0.8 If a technique here is new to you, practice it

This is advice for college and for life. Mastering any new technique takes practice. If you feel consonance or *Aha!* with any of the techniques mentioned in this guide, but have never seen or used them, go back to some problems you've been given on practice tests and try them out. Try them over and over. Real practice for high scores is on the order of 25 to 50 problems using a particular technique. There are many practice books out there, practice web sites, etc., that offer practice problems. For example, if you feel that the calculator section of this study guide (8) points out some new and interesting techniques, try to use those techniques on this week's homework for your math class. This practice is simple enough, and I promise eventual proficiency with this approach. Try it!

1 The nature of numbers and numerical operations

I would say that part of this test is about test takers' comfort level with numbers, using numbers, and noticing how they interact.

1.1 Combinations of odd/even integers when doing binary operations

Even if you do not memorize these rules, you can always cook up some quick examples in the margin of your exam booklet as you work. For some, this exam is maybe the last situation in which knowing this stuff is relevant.

1.1.1 Addition and subtraction

It's worth it to remind ourselves of the results here.

$$\text{odd} \pm \text{odd} = \text{even}$$

$$\text{even} \pm \text{even} = \text{even}$$

$$\text{odd} \pm \text{even} = \text{odd}$$

1.1.2 Multiplication

For these:

$$\text{odd} \times \text{odd} = \text{odd}$$

$$\text{even} \times \text{even} = \text{even}$$

$$\text{odd} \times \text{even} = \text{even}$$

1.2 Percentages and how percentages stack up

A percentage is nothing more than a fraction - *any* fraction - expressed with a denominator of 100. Calculating percentages always feels like a difficult topic. Markups, markdowns, %increase, %decrease - these all have calculation shortcuts.

- %increase and markups: Multiply the starting value by $1 + \frac{\text{pct}}{100}$. Example: Markup of 13% on \$10 is $10 \cdot 1.13 = \$11.30$.
- %decrease and markdowns: After taking off a percentage, *part of the price remains*, so think about *the amount that is left after a discount is applied*. A discount of 30% means that $100 - 30 = 70\%$ of the price still remains. This is the price you pay. Example: 30% markdown on \$10 is $10 \cdot 0.7 = \$7$.
- Percentage increases and decreases stack up over time as in this example: If we start with a price of \$13, increase it by 11%, then 3 months later, decrease it by 25%, increase it again by 1%, then decrease by 66%, the calculation is: $0.34 \cdot 1.01 \cdot 0.75 \cdot 1.11 \cdot \$13 = \$3.72$

- **Super important thing to notice!** Since applying percentages is done by multiplication, the percentages can be applied *in any order!* Totally counter-intuitive, but this is true. Take the example from the last bullet and just rearrange the multiplications. Multiplication is commutative, which means it can be done in any order.

For example: Which raise is more desirable, a 2% salary increase after a salary increase of 10%, or a salary increase of 10% after a 2% salary increase? Most would reason that they want the larger percentage increase first, in order to maximize salary first, before applying the next increase. Let's have a look:

First way: 2% increase after 10% salary increase:

$$\text{new salary} = 1.02 \cdot (1.10 \cdot s) = 1.122 \cdot s$$

Second way: 10% increase after 2% salary increase:

$$\text{new salary} = 1.10 \cdot (1.02 \cdot s) = 1.122 \cdot s$$

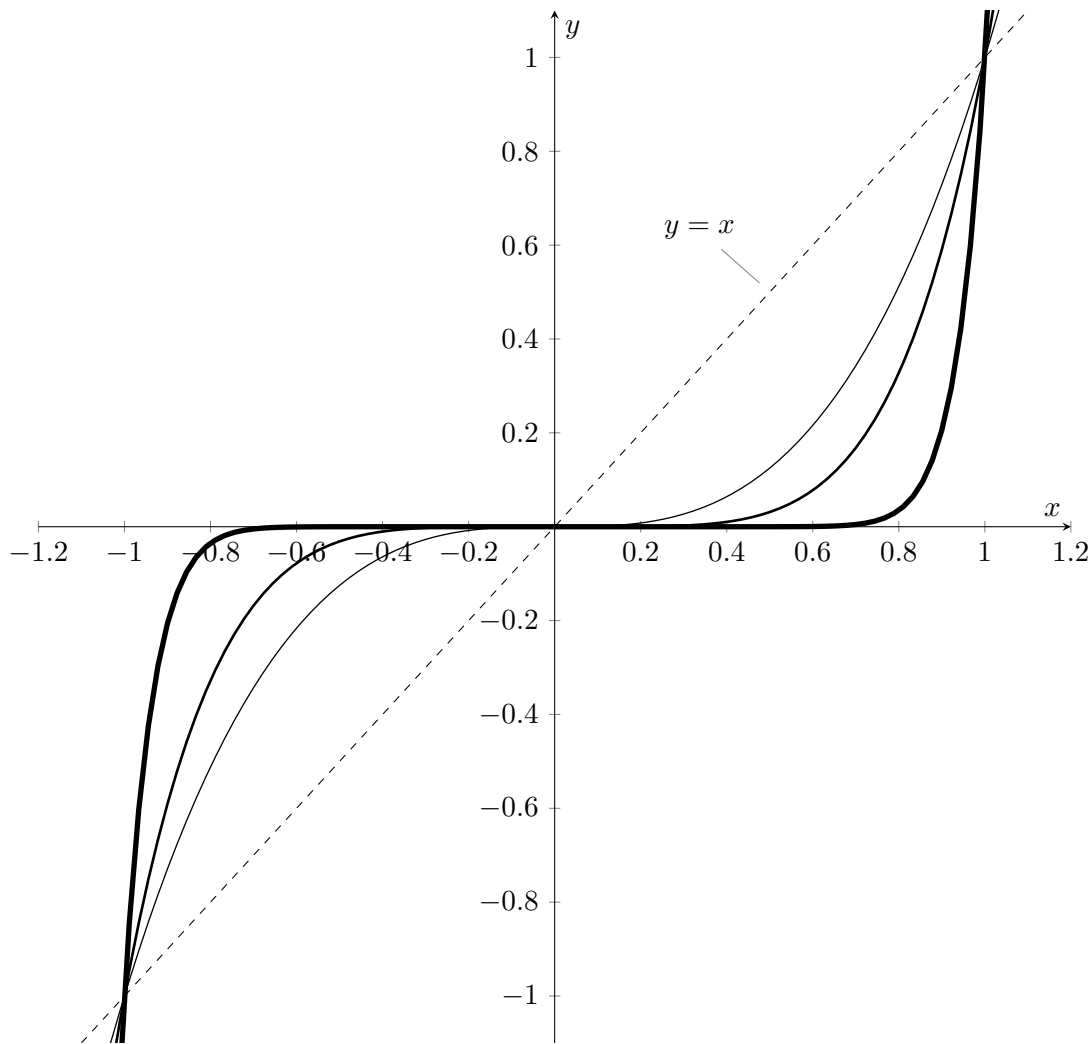
Exactly the same, because multiplication can be done in any order.

1.3 Exponents, roots, and $-1 < x < 1$

We probably are pretty used to thinking of exponentiation as an operation that produces bigger numbers than any input value of x . Examples: $2^2 = 4, 10^3 = 1000$. Even in the vernacular, we say that things, "increase exponentially," to indicate really rapid growth, all the time. But for fractional values of x , i.e., values of x that are between 0 and 1, this is not true. In fact, just a few calculations can convince us that raising these values of x to positive exponents make them smaller:

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad \frac{1}{4} < \frac{1}{2} \quad \text{and} \quad \left(\frac{2}{5}\right)^4 = \frac{16}{625}, \quad \frac{16}{625} < \frac{2}{5} \quad \text{and} \quad \left(\frac{2}{3}\right)^3 = \frac{8}{27}, \quad \frac{8}{27} < \frac{2}{3}$$

It is worth having a look at a graph of this phenomenon. Here, the lines get more bold with increase in an odd power of x .

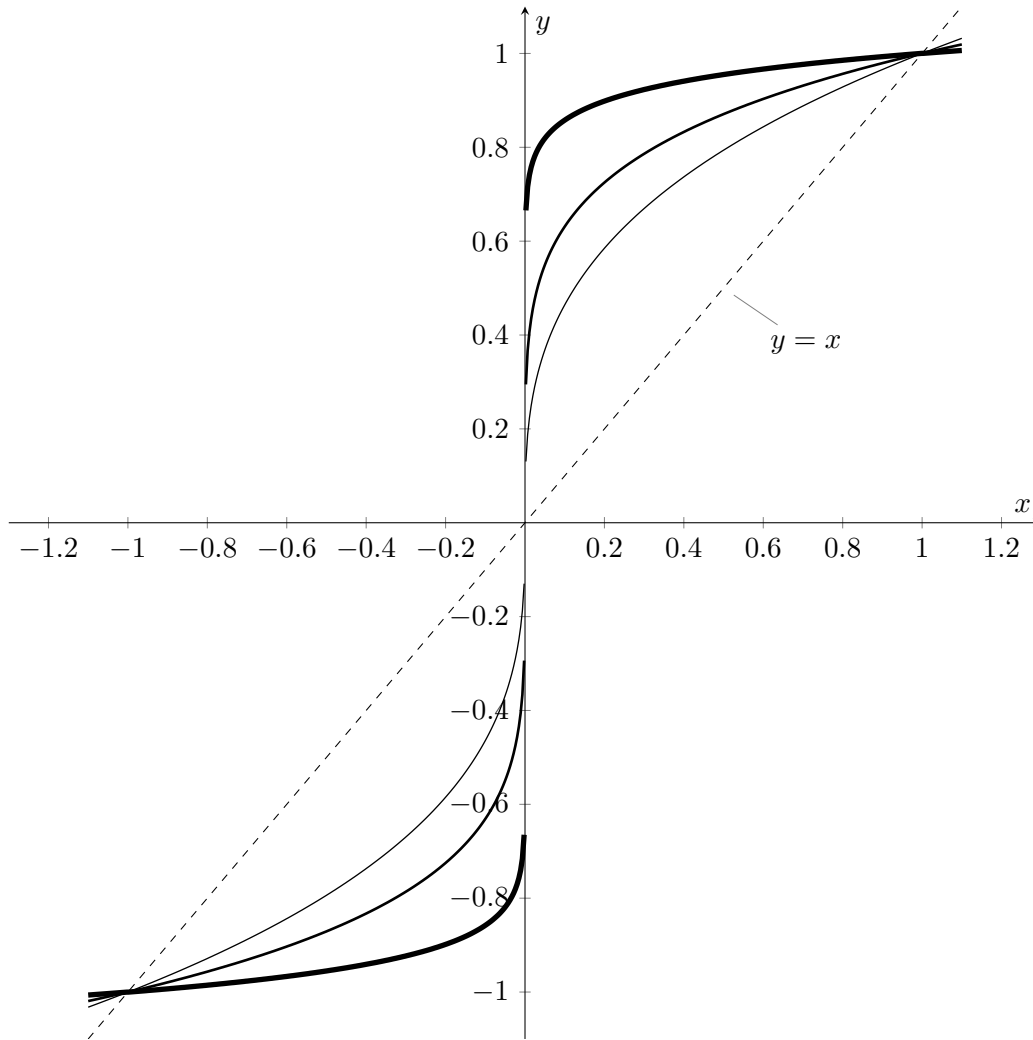


That the curves are *below* the line $y = x$ on the interval $(0, 1)$ indicates that the values of the powers of these x -values are less than the the x -values themselves. Also notice how, on the interval $(-1, 0)$, the powers of those numbers *are* bigger than the x -values. The situation changes at $(-1, -1)$ and $(1, 1)$, however, indicated at the rightmost and leftmost parts of the graph. On the interval $(1, \infty)$, the powers of the numbers again are greater than the the x -values themselves, and on $(-\infty, -1)$, they are smaller than the the x -values themselves.

Similarly and inversely, we are quite used to smaller results when we take roots of numbers. Examples: $\sqrt{25} = \pm 5$, $\sqrt[3]{-27} = -3$. And, similarly, when we take roots of fractions between 0 and 1, the values of the roots are bigger than the input values. Examples:

$$\sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10} = 0.3162\dots, \quad 0.3162 > \frac{1}{10} \quad \text{and} \quad \sqrt[3]{\frac{1}{4}} = \frac{(\sqrt[3]{4})^2}{4} = 0.62996\dots, \quad 0.62996 > \frac{1}{4}$$

Let's have a look at the graphs of roots, where here, the lines get more bold with increasing roots of x .



That the curves are *above* the line $y = x$ on the interval $(0,1)$ indicates that the values of the roots of these x -values are greater than the the x -values themselves. On the interval $(-1,0)$, the values are smaller than the the x -values themselves. The situation changes at $(-1,-1)$ and $(1,1)$, however, indicated at the rightmost and leftmost parts of the graph.

1.4 Complex/imaginary numbers and the trick

1.4.1 The basics of complex numbers and i

The very basis of complex arithmetic, you may remember, is the definition:

$$i = \sqrt{-1}$$

A complex number is any number that has both a real part and an imaginary part. Any complex number, denoted by z , is defined as:

$$z = a \pm bi,$$

Where a and b are real-number constants.

1.4.2 The trick for high powers of i

I feel like this one isn't covered quite enough in coursework to make it stick. There is a trick for evaluating what seems to be impossible exponents involving i . Let's have a look at some values involving i :

$$\begin{array}{cccc}
 i = \sqrt{-1} & i^2 = (\sqrt{-1})^2 = -1 & i^3 = i^2i = -i & i^4 = i^2i^2 = 1 \\
 i^5 = i^4i = i & i^6 = i^4i^2 = -1 & i^7 = i^4i^3 = -i & i^8 = i^4i^4 = 1 \\
 i^9 = i^8i = i & i^{10} = i^8i^2 = -1 & i^{11} = i^8i^3 = -i & i^{12} = i^8i^4 = 1 \\
 \dots & \dots & \dots & \dots
 \end{array}$$

The pattern we see emerging here is that there are only 4 possible answers for powers of i : i , -1 , $-i$, and 1 . The trick for higher powers of i takes advantage of this pattern. Specifically, all the powers of i for powers greater than 4, can be thought of as some power of i^4 multiplied by some other power of i . Since $i^4 = 1$, that only leaves i , i^2 , i^3 , and i^4 over and over. The trick here is, simply: take the power of i you're given and divide it by 4. Whatever the *remainder* is, that is the power of i that it reduces to. The other thing that makes this work is that $i^4 = 1$, which is the same as $i^0 = 1$. So, for i^{3459} , the remainder when dividing by 4 is 3, so $i^{3459} = i^3 = -i$. $i^{37} = i^1 = i$. $i^{101} = i^1 = i$.

1.4.3 Realizing the denominator: difference of squares idea makes it work!

We all remember that it's convention to not have roots in the denominators of fractions. When it comes to complex arithmetic (arithmetic involving numbers in the form $a \pm bi$), there is also a convention of complex numbers not being in the denominator of fractions. The method we use, called, *realizing the denominator*, is similar to rationalizing the denominator, the method we use with roots in denominators. The technique is to multiply the fraction, top and bottom, by the complex conjugate of the denominator. The complex conjugate of any complex number $a \pm bi$ is the same number with the sign in between changed to its opposite. Examples: $-7 + 8i$'s conjugate is $-7 - 8i$. $1 - \sqrt{2}i$'s conjugate is $1 + \sqrt{2}i$.

Multiplying any complex number $a + bi$ by its conjugate $a - bi$ looks like:

$$(a + bi)(a - bi)$$

If you check out (2.3.2) in this handout, you'll see that this is pretty-much the same thing. Carrying out the multiplication:

$$\begin{aligned}
 (a + bi)(a - bi) &= a^2 - b^2i^2 && \text{(by difference of squares pattern)} \\
 &= a^2 - b^2(-1) \\
 &= a^2 - (-b^2) && \text{(Recall: } i^2 = -1) \\
 &= a^2 + b^2
 \end{aligned}$$

Now, when we need to realize a denominator, we can proceed a little more quickly:

$$\begin{aligned}
 \frac{1}{-7 + 3i} &= \frac{1}{-7 + 3i} \cdot \frac{-7 - 3i}{-7 - 3i} \\
 &= \frac{-7 - 3i}{(-7)^2 + 3^2} && \text{(by difference of squares pattern in the denominator)} \\
 &= \frac{-7 - 3i}{49 + 9} \\
 &= \frac{-7 - 3i}{58} \\
 &= -\frac{7}{58} - \frac{3}{58}i && \text{(splitting up the fraction into } a \pm bi \text{ pattern)}
 \end{aligned}$$

And, yes, the end result is quite a bit different than what we started with!

2 Important algebra: forms, rules, techniques

2.1 Splitting apart fractions

I mention this only because a common thing that happens is that, in calculations, we get an answer that looks nothing like the choices. A lot of times, the choices contain the answer we got but in a different form. Fractions can be split apart like this:

$$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}$$

Please note that this is *not* allowable:

$$\frac{a}{b \pm c} \neq \frac{a}{b} \pm \frac{a}{c}$$

In words: we can split apart a fraction across a + or - sign in the numerator only, and must keep the denominator intact for each term. More generally:

$$\frac{a \pm b}{f(x)} = \frac{a}{f(x)} \pm \frac{b}{f(x)}$$

where $f(x)$ is some *expression* involving x . That is, $f(x)$ must be kept intact in the denominator of each fraction.

2.2 Cancelling in fractions

Following on from (2.1), above, a comment on factoring, reducing, cancelling in fractions. If we have a sum in the numerator, we must cancel *each* of the terms in the numerator with the denominator. This is not allowed:

$$\frac{5x + 4}{8} \neq \frac{5x + 1}{2}$$

via reducing/cancelling the 4 and the 8. If we think of it as first splitting the fraction and then reducing each of the two fractions in the next step, we get the correct answer:

$$\frac{5x + 4}{8} = \frac{5x}{8} + \frac{4}{8} \quad \text{(split fraction properly)}$$

$$= \frac{5x}{8} + \frac{1}{2} \quad \text{(can only reduce the 2nd fraction)}$$

$$= \frac{5}{8}x + \frac{1}{2} \quad \text{(rearrange, optional)}$$

Here is an example where both terms can be cancelled/reduced, and two methods for doing so.

Method 1:

$$\frac{16x - 4}{8} = \frac{16x}{8} - \frac{4}{8} \quad \text{(properly splitting up fraction)}$$

$$= 2x - \frac{1}{2} \quad \text{(reducing each term)}$$

Method 2:

$$\begin{aligned}\frac{16x - 4}{8} &= \frac{4(4x - 1)}{8} && \text{(factor the numerator)} \\ &= \frac{4x - 1}{2} && \text{(reduce the 4 \& 8)} \\ &= \frac{4x}{2} - \frac{1}{2} && \text{(split up fraction)} \\ &= 2x - \frac{1}{2} && \text{(reduce)}\end{aligned}$$

2.3 Quadratic equations

2.3.1 Forms of quadratic equations

- Standard form: $ax^2 + bx + c = 0$
- Factored form: $k(x - a)(x - b) = 0$, where the roots are $x = a$ and $x = b$, and k is a constant
- Vertex form: $y - k = 4p(x - h)^2$, where the vertex is (h, k)

2.3.2 The difference of squares pattern for quadratics

This will save some time in factoring and multiplying binomials, if you can spot it:

$$(x - a)(x + a) = x^2 - a^2$$

We are really used to seeing things like $x^2 - 16 = (x + 4)(x - 4)$ and $x^2 - 100 = (x + 10)(x - 10)$, but where this *really* helps save time is with something like this:

$$\begin{aligned}x^2 - 2 &= (x + \sqrt{2})(x - \sqrt{2}) \\ \text{or this: } 16x^2 - 4 &= (4x + 2)(4x - 2) \\ \text{or, most generally: } ax^2 - b &= (\sqrt{a} \cdot x + \sqrt{b})(\sqrt{a} \cdot x - \sqrt{b})\end{aligned}$$

2.3.3 The Quadratic Formula

This is one of the formulas you *must* remember for the test. Of all the formulas that are included, this is *not* one of them. Given $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a , b , and c are the coefficients in the standard form equation.

2.3.4 The nature of the roots of quadratics using the quadratic formula

Recall that we use the **discriminant** of the quadratic formula to determine the nature of the roots of a quadratic. The discriminant is the $b^2 - 4ac$ term. Three cases:

- $b^2 - 4ac > 0$ (positive): Two real roots.
- $b^2 - 4ac = 0$: One real (repeated) root.
- $b^2 - 4ac < 0$ (negative): Two imaginary roots, because we would be taking the square root of a negative number.

2.3.5 Complex/imaginary roots

- Imaginary roots travel in conjugate pairs. **Always.** If a problem tells you that some $x = a + bi$ is a root, then you know that $x = a - bi$ is also a root. The converse is also true.
- One simple source of imaginary roots is from a quadratic that is the *sum* of squares: $ax^2 + b^2 = 0$. This produces roots of the form $x = \sqrt{a} \pm \sqrt{b} \cdot i$. Contrast with results for difference of squares (2.3.2, this guide). This fact can help save some time on some questions.

2.4 "find x in terms of y and n "

All this means is to get x alone on one side of an equation or inequality, and y , n , and any number of variables and constants on the other. That is all. This is really a test of how well you can manipulate numbers and symbols while adhering to the rules of calculation. Nothing more.

2.5 Needing to solve for (what *appears* to be) more than one variable

You may have learned that it's not possible to solve for more than one variable when you do not have enough expressions. This is still true. Things get confusing when there are lots of things to do *in symbols*, and that's understandable. My best advice here is: **push through it**. You will see that, as you manage and combine equations, things will drop out as you proceed. They will likely will become clearer and more manageable, too.

2.6 Directly proportional and inversely proportional

Writing these expressions correctly is key!

- y is directly proportional to x : $y = k \cdot x$, where k is a constant.
- y is inversely proportional to x : $x \cdot y = k$, where k is a constant.

Sometimes, we get a problem that is worded like, *Distance of a moving object varies directly with time and inversely with another force...* The equation might look like $df = kt$, and we can simplify to $d = \frac{kt}{f}$. In steps, say:

$$d = k_1 t, \quad \text{then}$$
$$df = k_2, \quad \text{which yields } d = \frac{k_2}{f}.$$

We can combine these to give: $d = k_1 t \cdot \frac{k_2}{f}$

Rearrange: $d = k_1 k_2 \frac{t}{f}$

Then, we can say: $d = \frac{Kt}{f}$, if we combine the constants.

We see that the resulting formula has d varying directly with t in the Kt part of the expression, and inversely with f in the $\frac{K}{f}$ part.

2.7 Function notation, weird symbols

$f(x)$ and $f(a, b)$ is notation we are used to seeing. Phrasing like, *if we define [some weird symbol] as ...* this is a strong hint that this is a function notation problem.

Suppose $a !!! b = ab + 2b$. What is $-6 !!! 7$?

This is nothing more than:

Suppose $f(a, b) = ab + 2b$. What is $f(-6, 7)$?

3 Basic statistics & data reasoning: mean, median, mode, central tendency

Definitions:

- Mean: This is the average. Sum of all the data points divided by the number of data points.
- Median: This is the value of the middle data point in a list of data points. The data points must be in numerical order before you do this. For an odd number of data points, the median is the middle point. For an even number of numbers, the median is the average of the middle two numbers.
- Mode: This is the value that shows up the most in a data set.
- Mean, median, and mode are all called *measures of central tendency*, that is, they are used to describe what the "typical" value in a set of data might look like. They usually all vary from each other and are qualified against each other to help us make sense of sets of data.

Other things to remember:

- In a **symmetrical distribution**, the mean, median, and mode are all in the same place. In a **skewed distribution**, skewness is described as *left-skewed* and *right-skewed*, and is *where the tail is*, relative to the hump (*not* where the hump itself is!).
- In a left-skewed distribution, the mean, median, and mode appear left-to-right, in alphabetical order. The mean is located in the tail, the median is located between the mean and the hump, and the hump is where we find the mode. In symbols: $mean < median < mode$ for a left-skewed distribution.
- In a right-skewed distribution, the mean, median, and mode appear left-to-right, in reverse alphabetical order. The mode is located at the hump, the mean is located in the tail, and the median is located between the mode and mean. In symbols: $mode < median < mean$ in a right-skewed distribution.
- The median is what we call *resistant* to outliers and change. That is, the median remains the middle value or average of the middle two values in an ordered list of data. The more numbers in a list of data, the more representative of the data the median is. When adding new values to the data, the median only ever moves one value to the left or right for each value we add to the set.
- The mean (average) is what we call *non-resistant* to change or outliers. That is, unlike the median, it is possible to drastically change the mean of the data in a set by adding one value that is very much bigger than or very much less than where most of the data is clustered.

4 Counting problems

Recall that there are a few ways to count things. Permutations and Combinations are the two major ways, and questions on the test are usually no more advanced than that. This relates somewhat to probability, in that we are usually calculating some permutation as the total number of ways a particular outcome can happen.

4.1 Permutations

Permutations are groupings where order of selection matters. Typical signals for calculations using permutations are things like assigning ranks within a company, finishing order in a race, etc. Note that **P**ermutation and **P**lace both begin with **P**. If you think about people receiving awards for running a race, we award 1st **P**lace, 2nd **P**lace, 3rd **P**lace, etc. If you think order doesn't matter, ask the 1st Place person to switch with the 3rd Place person. Think about the reactions you would expect.

4.2 Calculating permutations

4.2.1 P-notation

This shortcut is hardly ever taught, but it's faster and more intuitive:

$${}_n P_r = \text{first } r \text{ factors of } n!$$

Examples:

$$\begin{aligned} {}_{10} P_2 &= 10 \cdot 9 && = 90 \\ {}_{10} P_7 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 && = 604800 \\ {}_{100} P_5 &= 100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 && = 9034502400 \end{aligned}$$

4.3 Calculating permutations "positionally"

"Position" also begins with **P**. There are certain kinds of problems where it's not really worth, in a time-pressed sense, doing things with P-notation. This works with ordered things like license plate numbers, social security numbers, and other things. It works for many things on these exams. Some examples:

Example: How many license plates can be made with 3 letters followed by 3 digits?

Answer: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^3 = 17576 \cdot 1000 = 17,576,000$

26 letters in the alphabet, 10 digits [0...9]. Can use any of them in their positions.

Example: How many license plates can be made with 2 letters followed by an A, two digits, and ending in 3?

Answer: $26 \cdot 26 \cdot 1 \cdot 10 \cdot 10 \cdot 1 = 67,600$

Same choices, but some positions (the A and the 3) only have one choice.

Example: How many phone numbers are possible in any one area code?

Answer: Thinking about this one is most helpful, perhaps, if we think about phone numbers that are usually NOT allowed. Ones beginning with 911, for instance. Phone numbers beginning with 0, too. We have to calculate the total number of possible phone numbers, then subtract out the ones not allowed.

4.4 Combinations

Combinations are groupings where order of selection doesn't matter. Typical signals for calculations using combinations are things like coming up with teams, delegations, words, handshakes, diagonals of polygons, etc., where all positions in the selection order are equal. Contrast this with section (4.1)'s definition of Permutations. In addition, Combinations are useful when there is a chance that we might count something more than once in our multiplication. For example, I shook your hand and you shook mine. Sounds like two handshakes, but it's actually only *one* that we *counted* twice. Note that **C**ombination and **C**hoose both begin with **C**.

4.5 Calculating combinations

4.5.1 C-notation

The real magic of (4.2.1)'s shortcut happens here, in my opinion. The shortcut is:

$${}_n C_r = \frac{{}_n P_r}{r!} = \frac{\text{first } r \text{ factors of } n!}{r!}$$

Using the results from the examples in (4.2.1), we have: Examples:

$$\begin{aligned} {}_{10}C_2 &= \frac{10 \cdot 9}{2!} &= \frac{90}{2} &= 45 \\ {}_{10}C_7 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{7!} &= \frac{604800}{5040} &= 120 \\ {}_{100}C_5 &= \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5!} &= \frac{9034502400}{120} &= 75287520 \end{aligned}$$

5 Probability problems

As far as this exam is concerned, probability is the likelihood of an outcome or set of outcomes compared to the total number of outcomes, and can be thought of as:

$$\text{Probability of event X} = P(X) = \frac{\text{number of ways event X can occur}}{\text{total ways all events can occur}}$$

5.1 AND means multiply

Example: Given a normal, fair 6-sided die, what is the probability of getting a 1 and then a 6 on two consecutive rolls?

Answer: $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

5.2 OR means add

Example: Given a normal, fair 6-sided die, what is the probability of getting a 1 or a 6 on two consecutive rolls?

Answer: $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

5.3 AND with OR means some combo of multiplying and adding

Example: Given a normal, fair 6-sided die, what is the probability of getting a 1 and then a 6 on two consecutive rolls, or a 2 and a 5 on two consecutive rolls?

Answer: $(\frac{1}{6} \cdot \frac{1}{6}) + (\frac{1}{6} \cdot \frac{1}{6}) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$

5.4 Ratios and odds

Odds are slightly different than probability calculations, and the exam writers are banking on test-takers not knowing the difference. While probability can always be thought of in the way that (5) discusses, odds are actually comparisons of possibilities against other possibilities, instead of each against the whole. Examples are wins:losses, red:blue, males:females, etc. These are a little different than ratios and certainly different than probabilities.

Example: An urn contains 3 blue marbles and 3 red marbles. Here are the things we can say/calculate about this urn and these marbles:

- Total number of blue marbles: 3
- Total number of red marbles: 2
- Total number of marbles: 5
- Ratio of blue marbles to red marbles (blue:red) = 3 : 2

- Ratio of red marbles to blue (red:blue) = 2 : 3
- $P(\text{pulling a blue marble on one draw}) = \frac{3}{5}$
- $P(\text{pulling a red marble on one draw}) = \frac{2}{5}$
- Percentage of blue marbles in the urn = $\frac{3}{5} = 0.6 = 60\%$
- Percentage of red marbles in the urn = $\frac{2}{5} = 0.4 = 40\%$
- Odds of picking a blue marble: 3 : 2 = 1.5 in favor of. Note that this cannot be a probability because it's greater than 1.
- Odds of picking a red marble 2 : 3 = 0.66 in favor of. Note that this is not the same as the probability, above.
- In comparing the odds in the previous two bullets, it stands to reason that the odds are higher (more in favor of) picking a blue marble than a red one if all 5 marbles are in the urn.

Example: A room has a male:female ratio of 23:7. What percentage of the group is female?

Answer:

Don't be tempted to make a fraction out of the given information, that is, not $\frac{23}{7}$, nor $\frac{7}{23}$. Neither are correct, as the 7 and 23 describe the sizes of the subgroups relative to each other, not to the whole. This is a key distinction.

total number of people in the room = males + females = $23 + 7 = 30$ (needed for the denominator)

females make up 7 out of 30 people, = $\frac{7}{30} = 0.233 = 23.3\%$

5.5 Physical probability

[this section will be completed soon]

6 Systems of equations tricks

These always take some extra time, so we need to be aware of signals to shortcuts quickly. As you practice, take note of how a brief, "working smart" step before you start calculating can make all of the difference sometimes!

6.1 One equation is a multiple of the other, all terms

This case is super easy. Look for it first. Infinite number of solutions. Done.

6.2 Both equations can be graphed really easily

The solution exists where both graphs intersect. As long as you feel like you will be able to see the point of intersection easily, graphing is always a viable choice. The difficulty, however, is that not every equation is already in $y = mx + b$ form, which is the simplest form to graph quickly.

6.3 One variable is sort of by itself (has a coefficient of ± 1)

This is a great candidate for substitution. If you see equations that look like:

$$\begin{array}{ll} x + 7y = -8 & \text{solve for } x! \\ -9x - y = 37 & \text{solve for } y! \\ y = -\frac{2}{3}x - 1 & \\ x = \frac{y}{2} + 11 & \\ x = 40 & \\ y = -2 & \end{array}$$

solve for one variable and use substitution.

6.4 At least one variable in each can be related by a constant

This is a great candidate for elimination. This is the classic case, but it also takes the longest time to do, usually.

7 Important geometry

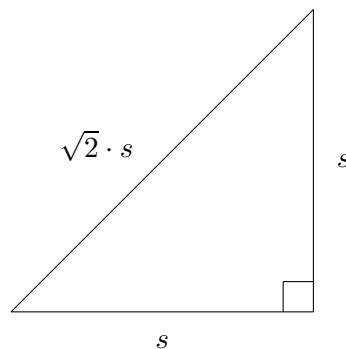
7.1 Common Pythagorean triples

This can save time. There are 2 sets of Pythagorean triples that show up on the SAT exam over and over: (3, 4, 5) and (5, 12, 13). The key to using these effectively is to remember that we can scale them up and down by multiplying or dividing all the lengths by the same number. This is a basic triangle similarity rule. So, (5, 12, 13), (10, 24, 26), and $(5\sqrt{2}, 12\sqrt{2}, 13\sqrt{2})$ are all Pythagorean triples, as are $(6, 8, 10)$ and $(1, \frac{4}{3}, \frac{5}{3})$. Just remember that the first two numbers are the lengths of the legs and the last number is the length of the hypotenuse. Position in the triple is important! For example, the Pythagorean triple (5, 13, x) has a *hypotenuse* of length $x = \sqrt{194}$, since *the two legs* are 5 and 13.

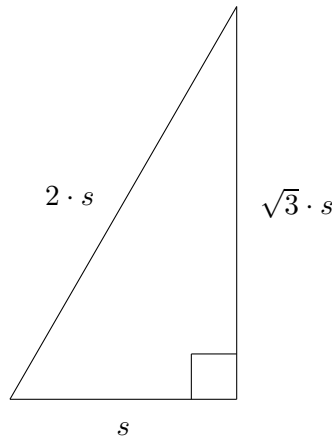
7.2 Special right triangles

These are given on the exam, but it is worth it to review them.

- 45° - 45° - 90° triangles. Remember that if the legs are length 1, then the hypotenuse has length $\sqrt{2}$. Side lengths can be scaled up and down by multiplying and dividing. So, more generally, in a 45° - 45° - 90° triangle, if the legs each have length s , then the hypotenuse has length $\sqrt{2} \cdot s$. It is also worth remembering that all isosceles right triangles have this relationship between their legs and hypotenuse.



- 30° - 60° - 90° triangles. The shortest leg is (always!) opposite the 30° angle. If the shortest leg is 1, then the length of the other leg is $\sqrt{3}$, and the length of the hypotenuse is 2. Again, side lengths can be scaled up and down by multiplying and dividing. So, more generally, in a 30° - 60° - 90° triangle, if the shortest leg has length s , then the length of the other leg is $\sqrt{3} \cdot s$, and the length of the hypotenuse is $2s$.



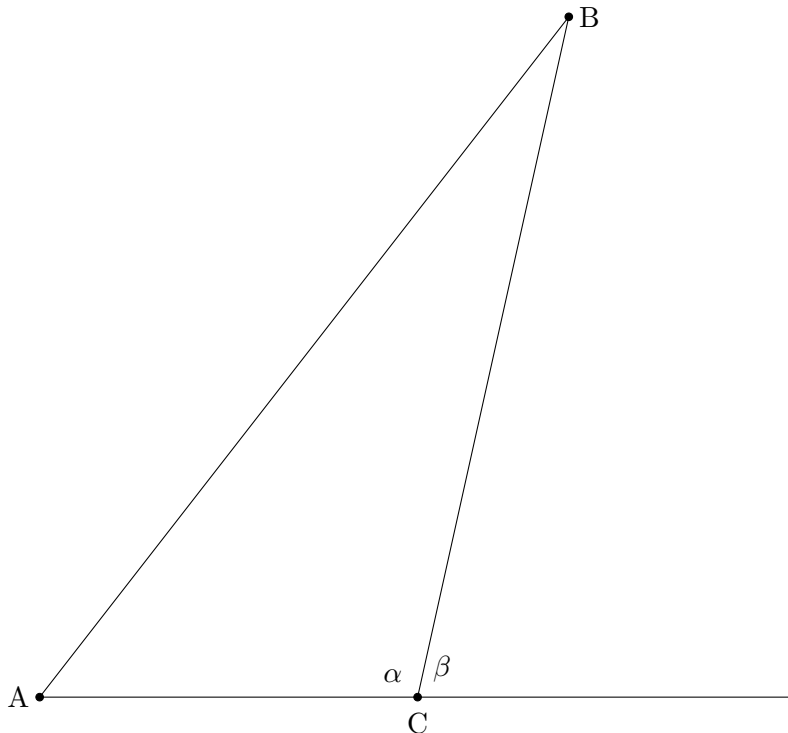
7.3 Finding right triangles in geometric figures, 2D & 3D

This is classic SAT territory. Some frequent problems involve:

- A cube inscribed in a sphere. The diameter of the sphere is the same as the diagonal of the inscribed cube. The sphere touches the cube at each of its 8 vertices.
- A sphere inscribed in a cube. The diameter of the sphere is always parallel to any edge of the cube. The sphere touches the cube at the centers of each of the 6 faces.
- Pyramids with rectangular bases. A right triangle is always formed by the height (leg), *half* the diagonal of the base (leg), and slant height (hypotenuse) of the pyramid.

7.4 External angles of triangles

This is a *BIG* time-saver! Consider:



We know that $m\angle A + m\angle B + m\angle\alpha = 180^\circ$. Since $\angle\alpha$ and $\angle\beta$ are also supplementary, $m\angle\alpha + m\angle\beta = 180^\circ$. Note that $180^\circ - m\angle\alpha = m\angle\beta$, but inside the triangle, $180^\circ - m\angle\alpha = m\angle A + m\angle B$! So, by transitive property of equality, we can say $m\angle\beta = m\angle A + m\angle B$ directly, as a result of the "overlap" of the sum of the angles in a triangle and supplementary angles; the overlap is at $\angle\alpha$. You can go *straight* to this result. Saves a few seconds of calculations.

7.5 Useful properties of n-gons

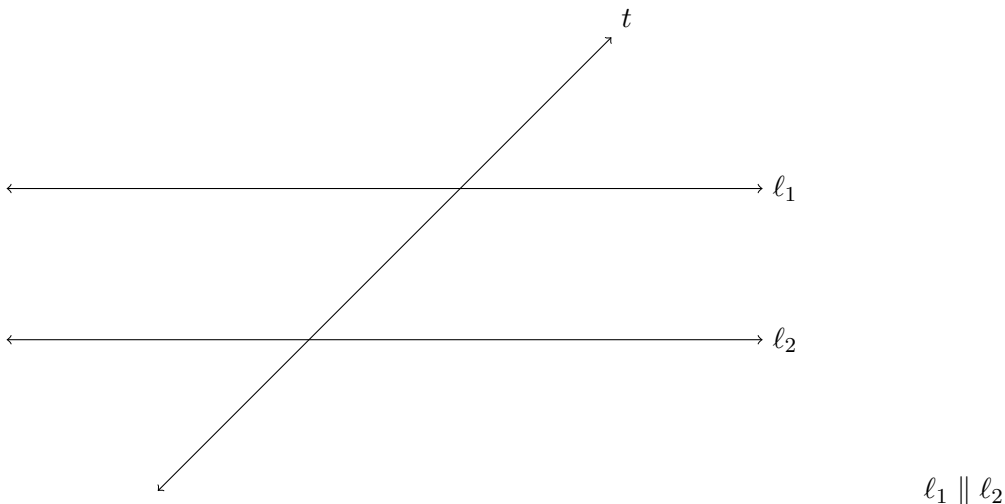
- An n-gon is simply a convex polygon with n sides and n interior angles. A *regular* n-gon is special case of n-gon, where all n sides are congruent and n angles are congruent.
- The sum of all the interior angles of *any* n-gon is $180 \cdot (n - 2)$.
The easiest way to remember this is perhaps through a triangle. We know that all angles in a triangle add up to 180° , and $n = 3$ in this formula.
- The sum of all exterior angles of an n-gon always adds up to 360° , so each exterior angle of a *regular* n-gon is $\frac{360}{n}$.

7.6 Doing "shape math"

Any problem involving differences of areas and volumes, inner/outer rings in circles, differences of areas in geometric figures, scaling factors, etc. are all examples of shape math. A great way to reposition these problems involves coming up with expressions or numbers for each of the pieces in a problem. For example, area of outer circle minus area of inner circle for concentric circles, yields area of a torus/ring/donut.

7.7 Non-obvious parallel lines cut by transversals

Remember: parallelogram, square, rhombus, trapezoid, $AB \parallel CD$, $\ell \parallel m$, etc. all mean that there are parallel lines somewhere. Draw a diagram if the problem is just stated in words. Once you have a diagram (one may be given), extend the lines in order to reveal non-obvious relationships between lines and angles. Something this simple can change things dramatically! This diagram and its relationships should be familiar to you. It's worth reviewing.



7.8 Circles

7.8.1 Degrees and radians

Sometimes, we need to convert between degrees and radians. The most straightforward formula for this is:

$$\frac{\pi}{180} = \frac{r}{d}$$

where d is degrees and r is radians. You're given one and you just have to solve for the other. In addition, if your calculator has a degrees mode and a radians mode, to make sure you are in the proper mode when doing angle and trigonometric calculations. Other calculations will not be affected.

7.8.2 Quickly going from radians to degrees

This works for radians only. Since π is 180° , we can treat π as a variable and just substitute 180° for it. This saves a few seconds on calculations.

Example: What is $\frac{7\pi}{8}$ in degrees? $d = \frac{7 \cdot 180}{8} = 157.5^\circ$

Example: What is $\frac{11\pi}{16}$ in degrees? $d = \frac{11 \cdot 180}{16} = 123.75^\circ$

7.8.3 Area of a sector

This can be thought of as largely a shape math problem where we need to find a proportion where the angle subtended by the sector (the angle, from the center, with two radii as sides) to 360° versus the total area. A sector is basically a fraction of the total area ($A = \pi r^2$).

$$\frac{m\angle_{sector}}{360^\circ} = \frac{A_{sector}}{\pi r^2}$$

This is one calculation where radians make things easier. The area of a sector where the central angle is θ radians is

$$A_{sector} = \theta \cdot r$$

7.8.4 Angles in a circle

[this section will be completed soon]

8 Useful calculator moves

8.1 Math / Frac (TI-84 family)

[this section will be completed soon]

8.2 Finding remainders when dividing integers - any calculator

This is a really simple technique, as finding the remainder of a division is often useful. Calculators, however, do not excel at this. The algorithm is:

1. Divide the dividend by the divisor as normal.
2. Take the integer part of the result (to the left of the decimal point) and multiply it by the divisor.
3. Subtract the result from the original number.
4. The result is the remainder.

Note that the remainder of any division of integers has a range (allowable values) of 0 to the divisor-1. If you get a number outside of that range, you did something wrong. This algorithm is extensible to non-integer values, however, I just recommend scaling the problem to divide only integers. For example, if you need to divide 1234.56 by 78.9, just multiply both numbers by 100 and divide the integers. Same exact result; the remainder from dividing the integers just needs to be divided by the multiplication factor.

Example:

What is the quotient and remainder of 1209 divided by 7?

Calculator says that 1209 divided by 7 is 172.7142857. This means that the whole number part of the quotient is 172. $172 \cdot 7 = 1204$. $1209 - 1204 = 5$, so the remainder is 5.

Answer is: 172r5.

Check: $172 \cdot 7 + 5 = 1209$.

8.3 Finding remainders when dividing integers - TI-84 family

To see just how much time this saves, read (8.2), above. The algorithm is simply this:

1. Divide as normal.
2. Take the answer and do Math (button) | Math (submenu) | ► Frac (or hit 1 key). Should result in a fraction of the form $\frac{n}{d}$.
3. Take that answer and do Math (button) | Frac (submenu) | ► n/d ◀► Un/d (or hit 4 key)
4. The result is now in mixed number form: an integer and a fraction. The integer is the whole number part of the division. The numerator of the fraction is the remainder of the division.

ONE HUGE CAVEAT REGARDING THIS ALGORITHM! The TI-84 works in *reduced fractions*. That is, every answer involving a fraction is already in its lowest terms. So, if you look at your answer and the denominator of the fraction is different, remember to *unreduce* the fraction, i.e., multiply top and bottom by whatever factor makes the bottom equal to your original divisor. Then the top will be the proper remainder.

Example:

What is the quotient and remainder of 249 divided by 6?

Calculator says that 249 divided by 6 is 41.5. If we convert this to mixed number form using the algorithm above, the calculator tells us it's $41\frac{1}{2}$, but 2 is not our original divisor. We must convert $\frac{1}{2}$ to sixths. $\frac{1}{2} = \frac{3}{6}$, so our remainder is 3.

Answer is: 41r3.

Check: $41 \cdot 6 + 3 = 249$.

Practice this a few times to get the hang of it! It really does save time.

9 Optional topics

9.1 Working with absolute value

Absolute value problems can be tricky. Equations are not so bad, but inequalities are where things get weird sometimes.

9.1.1 Absolute value equations

Absolute value equations are really two equations in one. We must keep in mind that there is an *algebraic* way to work out "oh, you just drop the sign." That is, we split up the absolute value equation into two equations.

$$|x| = a$$

means

$$x = a, \text{ for } x > 0 \text{ (positive } x) \qquad \text{and} \qquad -x = a, \text{ for } x < 0 \text{ (negative } x)$$

There are two answers: $x = a$, and $x = -a$. This is easy enough. Keep in mind that the absolute value bars act sort of like parentheses and we treat them that way when isolating a variable or expression that is between them. The idea is to get things into a $|f(x)| = a$ format and split that up, as above.

9.1.2 Absolute value inequalities

At some point before reading this guide, I am sure some readers have split the $|x| = a$ expression immediately into $x = a$ and $x = -a$ by negating the right-hand side of the equation. **This approach fails for inequalities**, which is why I say always negate the *argument* to the absolute value function, not the terms on the opposite side of the the equals/inequality sign.

9.2 The Rational Roots Theorem

The Rational Roots Theorem is a consequence of the Fundamental Theorem of Algebra, and is sometimes taught in Algebra II classes. It's most useful in helping to factor polynomials whose degree is greater than 2. It basically relates the constant term and the a terms of a polynomial written in standard form. You may recall that a is the coefficient of the highest-degree term of the polynomial. Some standard forms of polynomials:

$$\begin{aligned}ax^2 + bx + c &= 0 && \text{(quadratic, } c \text{ is constant)} \\ax^3 + bx^2 + cx + d &= 0 && \text{(cubic, } d \text{ is constant)} \\ax^4 + bx^3 + cx^2 + dx + f &= 0 && \text{(4th-degree, } f \text{ is constant)} \\&&& \text{etc.}\end{aligned}$$

The Rational Roots Theorem tells us that the possible values of x can be found by creating a fraction, where the numerator is a list of the integer factors of the constant term over a list of the integer factors of a . Constructing possible roots for the given polynomial is a matter of constructing a list of numbers that are combinations of one element from the list in the numerator divided by an element in the denominator. For example, if $a = 1$ and the constant term is 2, we construct a fraction:

$$x = \frac{\pm 1, \pm 2}{\pm 1}$$

And our possible list of roots would be:

$$x = \pm 1, \pm 2$$

Similarly, if $a = 12$ and the constant term is 49, the possible roots would be:

$$x = \frac{\pm 1, \pm 7, \pm 49}{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}$$

And our possible list of roots would be:

$$x = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}, \dots \quad (1)$$

and so on, with each of the top numbers over each of the bottom numbers. The main idea here is to first find the roots of a polynomial by substituting an element of possible x values into the expression and see if it produces zero (definition of root), then construct a binomial from that root. From there, we can do polynomial division or do synthetic division on our polynomial.

To construct binomials from (1), we simply say: $(x \pm 1)$, $(x \pm \frac{1}{2})$, $(x \pm \frac{1}{3})$, $(x \pm \frac{1}{4})$, $(x \pm \frac{1}{6})$, etc., and divide.

For the purposes of this exam, always start with the integer elements of something like (1).

9.3 The Polynomial Remainder Theorem

Following the information from section 9.2, above, we can comment on division of polynomials from the perspective of whether there will be a remainder when we divide. Remainders of zero when we divide indicate the same thing as division of two numbers does: the dividend can be evenly divided by the divisor.

The Polynomial Remainder Theorem states that for any polynomial $P(x)$, evaluating $P(r)$ produces the remainder when $P(x)$ is divided by the binomial $(x - r)$. Makes sense, since $P(r) = 0$ when r is a root of $P(x)$. This can make some factoring problems go a little quicker. Sometimes, the SAT directly asks what the remainder is when a polynomial is divided by some factor - this is the theorem to use.

9.4 The Binomial Theorem

This one comes up sporadically. It's usually positioned as something like, *What is the n^{th} term of (some binomial to a large power)?* The Binomial Theorem really generalizes what we know, say about an expression like $(x+1)^2 = x^2 + 2x + 1$. The name of the game here is to keep in mind the standard form of expressing a polynomial as discussed in (9.2), above.

9.4.1 Using Pascal's Triangle

This is probably the easiest way to handle these questions on this test. Pascal's triangle is how we find the values of the coefficients of each term of a binomial expansion. For simple binomial expansions as $(x + 1)^2$ above, all we do is go to the $n = 2$ row and read off the coefficients: 1, 2, 1. In standard form, the expansion is: $\underline{1}x^2 + \underline{2}x + \underline{1}(1^2)$.

$n = 0:$				1									
$n = 1:$			1		1								
$n = 2:$			1		2		1						
$n = 3:$			1		3		3		1				
$n = 4:$			1		4		6		4		1		
$n = 5:$			1		5		10		10		5		1

For more difficult expansions, it's a bit tricky, but not too bad. Let's have a look at $(3x - 7)^3$ for instance. First, we select the $n = 3$ row and find coefficients 1, 3, 3, 1. Standard form of a cubic is: $ax^3 + bx^2 + cx + d$, and Pascal's Triangle tells us (partially) how to calculate a , b , c , and d . What we need to do is use the four coefficients and the values of each part of the binomial - $3x$ and -7 - like this: $\underline{1}(3x)^3 + \underline{3}(3x)^2(-7) + \underline{3}(3x)(-7)^2 + \underline{1}(-7)^3$. Notice that the powers of $(3x)$ are descending, term-to-term, while the powers of (-7) are ascending. In standard form, $(3x - 7)^3$ is $27x^3 - 189x^2 + 441x - 343$

Pascal's Triangle is easy enough to cook up in the margin of the test booklet. When a question asks about the n^{th} term of a binomial expansion, this method is probably easiest and quickest.

10 Great places for reviewing concepts

I am not affiliated with any of these sites, but their reputations are great and their material is easy to access and simple to use.

- The College Board (<http://collegeboard.org>)
Authors of the SAT exam. They have free practice for all sections of the test.
- Khan Academy (<http://khanacademy.org>)
Partners with The College Board on free SAT practice. Can track your practice and progress here, too.
- Paul's Online Math Notes (<http://tutorial.math.lamar.edu/>)
Really really great and free.
- McGraw-Hill "Top 50 Skills for a Top Score: SAT Math"
(<https://www.amazon.com/gp/product/1259585670>)
- Desmos online graphing calculator (<http://desmos.com>)
Simply the best graphing tool available. Works right in your Web browser. Give it a try!

11 Reporting errors and giving feedback

I am so pleased that you have downloaded this study guide and have considered the techniques herein. I am really happy to help with a sometimes tricky exam that covers so many topics! To that end, I am the only writer and the only editor of these things, so if you find an error in the text or calculations, please email me and tell me about it! I am committed to prompt changes when something is inaccurate. I also really appreciate it when someone takes a moment to tell me how I'm doing with these sorts of things, so please do so, if you feel inclined.

My email address is: phil.petrocelli@gmail.com.

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