HOMEWORK #0

Exercise 1 (Atlas on a sphere)**.** Recall that on $S^n = \{(x_0, \ldots, x_n) \mid \sum_i x_i^2 = 1\}$ one can define the differentiable structure given by the atlas $\mathcal{A} = \{(U^+, \mathbf{x}_+), (U^-, \mathbf{x}_-)\}$ where $U^\pm = S^n \setminus (\pm 1, 0, \ldots, 0)$, and

$$\mathbf{x}_\pm(x_0, \ldots, x_n) = \left(\frac{x_1}{1 \mp x_0}, \ldots \frac{x_n}{1 \mp x_0} \right)$$

Prove that $\mathcal{A}$ is indeed an atlas. Is it a maximal atlas?

Exercise 2 (Product manifold)**.** Let M, M' be manifolds of dimension n, n' respectively, with charts $\{(U_\alpha, x_\alpha)\}_\alpha$ and $\{(U'_a, x'_a)\}_a$ respectively. Show that the product $M \times M'$ is a manifold of dimension $n + n'$, with charts $\{(U_\alpha \times U'_a, x_\alpha \times x'_a)\}_{(\alpha, a)}$.

Exercise 3 (Covering space)**.** Let M be a smooth manifold of dimension n, and $\hat{M}$ a topological space with $\pi : \hat{M} \to M$ a covering map. Show that:

(1) $\hat{M}$ can be given the structure of a smooth manifold of dimension n, such that π is a smooth map.
(2) If M is orientable, so is $\hat{M}$.

Exercise 4 (Quotient)**.** Let $G \times M \to M$ be a properly discontinuous action of a group G on a smooth manifold M. Prove that:

(1) The quotient M/G, with its quotient topology, can be given a differentiable structure such that the projection $\pi : M \to M/G$ is smooth.
(2) M/G is orientable if and only if there exists an orientation of M which is preserved by the elements of G.

Exercise 5 (Diffeomorphic differentiable structures)**.** Consider the following differentiable structures on the real line $\mathbb{R}$: $(\mathbb{R}, x_1)$, where $x_1 : \mathbb{R} \to \mathbb{R}$ is $x_1(t) = t$; and $(\mathbb{R}, x_2)$ where $x_2 : \mathbb{R} \to \mathbb{R}$ is $x_2(t) = t^3$. Show that

(1) The charts x_1 and x_2 are not compatible, so the maximal structures determined by them are distinct.
(2) The map $\phi : (\mathbb{R}, x_2) \to (\mathbb{R}, x_1)$ given by $\phi(t) = t^3$ is a diffeomorphism. That is, even though the differentiable structures $(\mathbb{R}, x_1)$ and $(\mathbb{R}, x_2)$ are distinct, they induce diffeomorphic differentiable manifolds.