HOMEWORK #1 - DUE FEB 2, AT NOON

Exercise 1 (Orientability of tangent bundle)**.** Prove that for every manifold M, the tangent bundle TM is always orientable, independently of the orientability of M.

[**Hint:** use the charts shown in class]

Exercise 2 (Orientability and sections)**.** Given a manifold M of dimension n, prove that it admits an orientable atlas if and only if $\Lambda^n TM$ admits a nowhere-zero global section.

[**Hints:** for the 'if' part, use the fact that for any vector space V of dimension n, $\dim \Lambda^n V = 1$ and given bases $\{v_1, \ldots v_n\}$, $\{w_1, \ldots, w_n\}$ for V with $v_i = \sum_j a_{ij} w_j$, then $v_1 \wedge \ldots \wedge v_n = \det(a_{ij}) w_1 \wedge \ldots \wedge w_n$. For the 'only if' part, recall that any open cover $\{U_i\}_i$ of a manifold M admits a *partition of unity*, that is, a collection of functions $\phi_i : M \to \mathbb{R}$ with $Supp(\phi_i) \subset U_i$, and $\sum_i \phi_i = 1$.]

Exercise 3. Let A be a section of $T^{1,1}M$, let $(U, \mathbf{x})$ be a chart. In this coordinate system, let

$$A = \sum_{i,a} A^i_a \; dx^a \otimes \frac{\partial}{\partial x^j}.$$

Prove that A is smooth in U if and only if the functions A^i_a are smooth for all $i, a + 1, \ldots, n$.

Exercise 4 (Change of coordinates for tensor bundles)**.** Let $\{x^i\}$ and $\{y^a\}$ be local coordinates defined on $U \subset M$. Suppose in the two coordinate systems, a $(2,1)$-tensor $A \in \Gamma(T^{2,1}M)$ is given by

$$(A_x)^k_{ij} \frac{\partial}{\partial x^k} \otimes dx^i \otimes dx^j, \qquad (A_y)^c_{ab} \frac{\partial}{\partial y^c} \otimes dy^a \otimes dy^b$$

Show that

$$(A_y)^c_{ab} = \frac{\partial y^c}{\partial x^k} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} (A_x)^k_{ij}.$$

where we are using Einstein summation convention.

Exercise 5 (Extending vectors to sections)**.** Let $\pi : E \to M$ be a smooth vector bundle, and let $x \in E_p (= \pi^{-1}(p))$ for some $p \in E$. Prove that there exists a global section $\sigma \in \Gamma(E)$ such that $\sigma(p) = x$.