#5: IMPORTANT SECTIONS

Vector fields

Prop. There is an isomorph of $C^\infty(M)$-modules

$\mathcal{X}(M) \to Der(C^\infty(M))$: $\{D: C^\infty(M) \to C^\infty(M)$ linear maps w/ $D(fg) = f D(g) + g \cdot D(f)$

"Proof": $X \longmapsto D_X(f)(p) := X_p(f)$

Check: 1. $D_X(f) \in C^\infty(M)$

2. $D_X \in Der(C^\infty(M))$

3. The map is an isomorphism.

$[D_X f =: X(f)]$

Def (Lie bracket): $X, Y \in \mathcal{X}(M)$. Then the map $f \mapsto XY(f) - YX(f)$ is a derivation.

$\downarrow$

$\exists$ v. field $[X,Y] \in \mathcal{X}(M)$ st $[X,Y](f) = XY(f) - YX(f)$ called (Lie) bracket.

Prop: Lie bracket satisfies:

1. bilinearity: $[aX + bX', Y] = a[X,Y] + b[X', Y]$
2. $[X,Y] = -[Y,X]$
3. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

$(\mathcal{X}(M), [\cdot,\cdot])$ is a LIE ALGEBRA.

Flow of vector fields

Prop: $X \in \mathfrak{X}(M)$, $p \in M$. Then $\exists!$ curve $\gamma : [0, t_{MAX}) \to M$, $t_{MAX} \leq \infty$

s.t.

$$(*) \begin{cases} \gamma'(t) = X_{\gamma(t)} \text{ in } T_{\gamma(t)}M & \forall t \in [0, t_{MAX}) \\ \gamma(0) = p \end{cases}$$

"Proof" Chart ~~φ~~ $(U, \underline{x})$ around p, $\underline{x}(p) = 0$

denote $\underline{x} \circ \gamma(t) = (x^1(t) \dots x^n(t)) \to \gamma'(t) = \sum x^i(t)' \cdot \frac{\partial}{\partial x_i^i}\Big|_{\gamma(t)}$

$X = \sum a_i \frac{\partial}{\partial x_i}$ $\qquad a_i : U \to \mathbb{R}$ fns.

Then $(*)$ ~~implies that x^i~~ becomes a system of ODE's

$$\begin{cases} x^i(t)' = a_i(\gamma(t)) \\ x^i(0) = 0 \end{cases}$$

ODE fact: solutions exist and are unique $\blacksquare$

Def: γ satisfying $(*)$ is the integral curve of X at p. $(\gamma_{X,p})$

Prop: $X \in \mathfrak{X}(M)$ and $p \in M$. Then $\exists$ U nbd of p, and interval $(-\delta, \delta)$, with a map.

$$\varphi^X : U \times (-\delta, \delta) \longrightarrow M$$
$$(p, t) \longmapsto \varphi_t(p) := \gamma_{X,p}(t)$$

$\forall t \in (-\delta, \delta)$,

$\varphi_t^X : U \to M$ is a diffeo onto the image.

φ^X flow of X.

Tensor fields

Prop: There is an ~~1-1 corres~~ isomorphism of $C^\infty(M)$-modules

$$\Gamma(T^{k,l}M) \longrightarrow \Big\{F : \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k \text{ times}} \times \underbrace{\Omega^1(M)\times\cdots\Omega^1(M)}_{l\text{-times}} \longrightarrow C^\infty(M) \quad C^\infty(M)\text{-multilinear}\Big\}$$

"Proof": $\phi \in \Gamma(T^{k,l}M)$

$$\mapsto F_\phi(X_1 \dots X_k, \omega^1, \dots \omega^l)(p) := \underset{\substack{\in \\ T_p^{k,l}M}}{\phi_p}\big(X_1(p) \dots X_k(p), \omega^1(p) \dots \omega^l(p)\big)$$

Check: $F_\phi^{(\dots)} \in C^\infty(M)$, F_ϕ multilinear, the map is an isomorphism.

Prop: $\exists$ isomorphism of $C^\infty(M)$-modules

$$\Gamma(\mathrm{Sym}^2 T^*M) \longrightarrow \{g : \mathfrak{X}(M)\times\mathfrak{X}(M) \to C^\infty(M) \quad C^\infty(M)\text{-bilinear, } \text{~~bil~~ symmetric fns}\}$$

[RIEMANNIAN GEOMETRY]

Def: M mfd. A RIEMANNIAN METRIC is a section $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ s.t. $\forall p \in M$ ~~one of~~, $g_p : T_pM \times T_pM \to \mathbb{R}$ is positive definite. $(g_p(x,x) > 0 \ \forall x \neq 0)$

Equivalently, $g : \mathfrak{X}(M)\times\mathfrak{X}(M) \to C^\infty(M)$ is s.t. $\forall X \in \mathfrak{X}(M)$, $g(X,X)(p) \geq 0$ whenever $X_p \neq 0$.

Notation: $g(X,Y) = \langle X, Y\rangle$, $g_p(X,Y) : \langle X, Y\rangle_p$

Given $(U, \underline{x})$ chart, $g = g_{ij}\, dx^i dx^j$

$G = (g_{ij})$ is a symmetric, positive definite matrix, with entries in $C^\infty(U)$.

<u>Def</u> ~~(M,g)~~ Riem. mfd is a pair (M,g) of a mfd M, with a Riem metric g.

<u>Prop:</u> $\varphi : M \longrightarrow N$ diffeo, g Riem metric on N.
Define $\varphi^* g \in \Gamma(\mathrm{Sym}^2 M)$ by
$\underset{\substack{\cap \\ M}}{\varphi^* g_p}(\underset{T_pM}{\underset{\cap\ \cap}{x, y}}) := g_{\varphi(p)}(\varphi_* x, \varphi_* y)$. Then $\varphi^* g$ is a metric on M.

<u>Def:</u> $\varphi : (M, \overset{\langle , \rangle_M}{g_M}) \to (N, \overset{\langle , \rangle_N}{g_N})$ is called ISOMETRY if

$g_M = \varphi^* g_N$

$\Updownarrow$

$\forall\, x, y \in T_pM, \quad \langle \varphi_* x, \varphi_* y \rangle_N = \langle x, y \rangle_M$

EXAMPLES:
1. $(\mathbb{R}^n, g = \sum dx^{i^2})$ $\quad x^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the i-th projection.
2. (M, g) Riemannian mfd, $\imath : N \longrightarrow M$ immersion (map s.t. $d_p \imath : T_pN \longrightarrow T_pM$ is injective). Then $\exists$ metric (N, g_N) $\quad g_{N\,p}(x, y) = g_{\imath(p)}(\imath_* x, \imath_* y)$

Ex: $\imath : S^n \hookrightarrow \mathbb{R}^{n+1}$ inclusion, is an immersion
$\to \sum dx^{i^2}$ induces a metric on S^n (round metric)