HOMEWORK #2 - DUE FEB 9, AT NOON

Exercise 1 (Tensor fields)**.** Prove the following:

(1) Let $F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Omega^1(M) \to C^\infty(M)$ be a $C^\infty(M)$-multilinear map. Given $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$, prove that the value of $f = F(X, Y, \omega)$ at p only depends on $X_p, Y_p \in T_pM$ and $\Omega_p \in T^*M$.

(2) Prove that the map

$$\Gamma(T^{2,1}M) \to \{F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Omega^1(M) \to C^\infty(M) \quad C^\infty(M)\text{- multilinear}\}$$

sending $\phi \in \Gamma(T^{2,1}M)$ to $F_\phi(X, Y, \omega)(p) := \phi_p(X_p, Y_p, \omega_p)$ is a $C^\infty(M)$-module isomorphism.

For the next exercise, recall that a *Lie group* is a smooth manifold G, together with a neutral element e, a smooth (multiplication) map $m : G \times G \to G$ (denote $m(g, h) =: gh$) and a smooth (inverse) map $\iota : G \to G$ (denote $\iota(g) =: g^{-1}$), which satisfy the axions of groups. In other words, Lie groups are both manifolds and groups at the same time, and the two structures are compatible. For any $g \in G$, denote $L_g : G \to G$ the map $L_g(h) = gh$ (*left multiplication*) and $R_g : G \to G$ the map $R_g(h) = hg$ (*right multiplication*). Denote $\mathfrak{g} := T_eG$.

Exercise 2 (Lie groups)**.** Prove the following:

(1) Letting $G \times \mathfrak{g} \overset{\pi_1}{\to} G$ denote the trivial vector bundle over G, with fiber $\mathfrak{g}$, prove that the map $\psi : TG \to G \times \mathfrak{g}$ sending $(g, v) \in T_gG$ to $(g, dL_{g^{-1}})$ is a diffeomorphism, such that $\pi_1 \circ \psi = \pi$ and such that the restriction to each fiber is a linear isomorphism (TG and $G \times \mathfrak{g}$ are *isomorphic smooth vector bundles*).

(2) A vector field $X \in \mathcal{X}(G)$ is called *left* invariant if $L_g(X_h) = X_{gh}$ for every $g, h \in G$. Prove that the set of left invariant vector fields is a vector space, isomorphic to $\mathfrak{g}$.

(3) Prove that the Lie bracket of two left invariant vector fields is again left invariant. Use this to prove that $\mathfrak{g}$ can be given the structure of a finite-dimensional Lie algebra.

(4) Prove that every inner product $\langle , \rangle_e : \mathfrak{g} \times \mathfrak{g}$ can be extended to a Riemannian metric $\langle , \rangle$ on G that is *left-invariant*, that is, such that every left multiplication $L_g : G \to G$ is an isometry.

Exercise 3 (Real projective space)**.** Let $a : S^n \to S^n$ denote the antipodal map, sending p to $-p$.

(1) Prove that $\{id_{S^n}, a\}$ is a group of diffeomorphisms of S^n, isomorphic to $\mathbb{Z}_2$, and that the action of $\mathbb{Z}_2$ on S^n induced by a is properly discontinuous. Therefore, the quotient $S^n/\mathbb{Z}_2$ is a manifold, called *real projective space* and denoted $\mathbb{RP}^n$.

(2) Prove that a is an isometry of S^n with its round metric g_0. Use this fact to produce a metric g on $\mathbb{RP}^n$ such that the quotient $\pi : S^n \to \mathbb{RP}^n$ satisfies $g_0 = \pi^*g$.