HOMEWORK #1

Exercise 1 (Orientability of tangent bundle)**.** Prove that for every manifold M, the tangent bundle TM is always orientable, independently of the orientability of M.

[**Hint:** use the charts shown in class]

Solution 1. Let $\{(U_\alpha, \mathbf{x}_\alpha)\}$ be any atlas of M, and let $\hat{\mathcal{A}}$ be the atlas of TM given by $\{(V_\alpha := \pi^{-1}(U_\alpha), \hat{\mathbf{x}}_\alpha := (\mathbf{x}_\alpha \times id) \circ \phi_\alpha)\}$, where $\phi_\alpha : V_\alpha \to U_\alpha \times \mathbb{R}^n$ are the trivializing maps. We claim that this is an oriented atlas. Define $\hat{\Phi} := \hat{\mathbf{x}}_\beta \circ \hat{\mathbf{x}}_\alpha^{-1}$. Recall from class that, given a vector $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $\hat{\Phi}(u, v) = (\Phi(u), A(u) \cdot v)$, where $\Phi = \mathbf{x}_\beta \circ \mathbf{x}_\alpha^{-1}$, and $A_{ij}(u) = \frac{\partial \Phi^i}{\partial x^j}|_u$.

Let $(u^1, \dots u^n, v^1, \dots v^n)$ be the coordinates around (u, v) in $\mathbb{R}^n \times \mathbb{R}^n$. Given $i, j = 1, \dots n$, one has:

- $\frac{\partial \hat{\Phi}^i}{\partial u^j}\Big|_{(u,v)} = \frac{\partial \Phi^i}{\partial x^j}\Big|_u = A_{ij}(u)$.
- $\frac{\partial \hat{\Phi}^i}{\partial v^j}\Big|_{(u,v)} = 0$.
- $\frac{\partial \hat{\Phi}^{n+i}}{\partial v^j}\Big|_{(u,v)} = \frac{\partial}{\partial v^j}\Big|_{(u,v)}\big(A(u) \cdot v\big) = A_{ij}(u)$.

Therefore, The Jacobian matrix of $\hat{\Phi}$ has the block form:

$$\begin{pmatrix} A & * \\ 0 & A \end{pmatrix}$$

and its determinant equals $\det(A)^2 > 0$.

Exercise 2 (Orientability and sections)**.** Given a manifold M of dimension n, prove that it admits an orientable atlas if and only if $\Lambda^n TM$ admits a nowhere-zero global section.

[**Hints:** for the 'if' part, recall that for any vector space V of dimension n, $\dim \Lambda^n V = 1$ and given bases $\{v_1, \dots v_n\}$, $\{w_1, \dots, w_n\}$ for V with $v_i = \sum_j a_{ij} w_j$, then $v_1 \wedge \dots \wedge v_n = \det(a_{ij}) w_1 \wedge \dots \wedge w_n$. For the 'only if' part, recall that any open cover $\{U_i\}_i$ of a manifold M admits a *partition of unity*, that is, a collection of functions $\phi_i : M \to \mathbb{R}$ with $Supp(\phi_i) \subset U_i$, and $\sum_i \phi_i = 1$.]

Solution 2. Suppose first that M is orientable, and let $\mathcal{A} = \{(U_\alpha, \mathbf{x}_\alpha)\}_\alpha$ denote an oriented atlas. On each U_α take the nowhere-zero section $\eta_\alpha = \frac{\partial}{\partial x_\alpha^1} \wedge \dots \wedge \frac{\partial}{\partial x_\alpha^n}$. Given any two charts $(U_\alpha, \mathbf{x}_\alpha)$, $(U_\beta, \mathbf{x}_\beta)$ in $\mathcal{A}$, on $U_{\alpha\beta}$ we have $\frac{\partial}{\partial x_\alpha^i} = \sum_j \frac{\partial x_\beta^j}{\partial x_\alpha^i} \frac{\partial}{\partial x_\beta^j}$ and thus

$$\eta_\alpha(p) = \det \left(\frac{\partial x_\beta^j}{\partial x_\alpha^i}\Big|_{\mathbf{x}_\alpha(p)} \right) \eta_\beta(p) =: J_{\alpha\beta}(\mathbf{x}_\alpha(p)) \cdot \eta_\beta(p) \qquad \forall p \in U_{\alpha\beta}$$

where $J_{\alpha\beta}(\mathbf{x}_\alpha(\cdot)) > 0$ because $\mathcal{A}$ is oriented. Take now a partition of unity $\{\phi_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$, and let η be the global section of $\Lambda^n TM$ given by $\eta(p) = \sum_\alpha \phi_\alpha(p)\eta_\alpha(p)$. We claim that this section is nowhere-zero: in fact, for any $p \in M$, we can write $\eta(p) = \sum_{\alpha \in A} \phi_\alpha(p)\eta_\alpha(p)$ where A is the set of all α such that $p \in U_\alpha$. Fix one such $\alpha_0 \in A$, and for any $\alpha \in A$, $\eta_\alpha = J_{\alpha\alpha_0}\eta_{\alpha_0}$. Therefore:

$$\eta = \sum_{\alpha \in A} \phi_\alpha(p)\eta_\alpha(p) = \left(\sum_{\alpha \in A} \phi_\alpha(p) J_{\alpha\alpha_0}(p) \right) \eta_{\alpha_0}(p) \neq 0$$

because $\sum_{\alpha \in A} \phi_\alpha(p) J_{\alpha\alpha_0}(p) > 0$ and $\eta_{\alpha_0}(p) \neq 0$.

Suppose now that there exists a nowhere-zero section η of $\Lambda^n TM$. For any chart $(U_\alpha, \mathbf{x}_\alpha)$, the local section η_α defined above is nowhere zero in U_α, and therefore the function $f_\alpha : U_\alpha \to \mathbb{R}$ such that $\eta_\alpha = f_\alpha \eta$ is also nowhere zero. In particular, it is either everywhere positive or everywhere negative. Take now the atlas $\mathcal{A}$ consisting of those charts $(U_\alpha, \mathbf{x}_\alpha)$ such that $f_\alpha > 0$. It is clear that $\mathcal{A}$ is an atlas so let us prove that $\mathcal{A}$ is oriented. Given two overlapping charts $(U_\alpha, \mathbf{x}_\alpha)$, $(U_\beta, \mathbf{x}_\beta)$, on $U_{\alpha\beta}$ one has $\eta_\alpha = (f_\alpha/f_\beta)\eta_\beta$ with $f_\alpha/f_\beta > 0$ on $U_{\alpha\beta}$. However, from the computations before one has $\eta_\alpha = J_{\alpha\beta}\eta_\beta$, so $J_{\alpha\beta} = f_\alpha/f_\beta > 0$, and this proves that $\mathcal{A}$ is oriented.

Exercise 3. Let A be a section of $T^{1,1}M$, let $(U, \mathbf{x})$ be a chart. In this coordinate system, let

$$A = \sum_{i,a} A^i_a \ dx^a \otimes \frac{\partial}{\partial x^j}.$$

Prove that A is smooth in U if and only if the functions A^i_a are smooth for all $i, a + 1, \ldots, n$.

Solution 3. Recall that A is a map $M \to T^{1,1}M$.

Let $(U, \mathbf{x})$ be a chart in M, let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^2}$ be the local trivialization of $\pi^{-1}(U)$ and $\hat{x} = (\mathbf{x} \times id) \circ \phi$ the chart on $\pi^{-1}(U)$. By definition, A is smooth in U if and only if $\hat{\mathbf{x}} \circ A \circ \mathbf{x}^{-1} = (\mathbf{x} \times id) \circ (\phi \circ A) \circ \mathbf{x}^{-1}$ is smooth. Now, $\phi \circ A : U \to U \times \mathbb{R}^{n^2}$ is given by:

$$\phi \circ A(p) = \phi \left(\sum_{i,j} A^i_j(p) \ d_p x^i \otimes \frac{\partial}{\partial x^j}\Big|_p \right) = \left(p, \left(A^i_j(p) \right)_{i,j} \right)$$

Therefore, $\hat{\mathbf{x}} \circ A \circ \mathbf{x}^{-1} = (\mathbf{x} \times id) \circ (\phi \circ A) \circ \mathbf{x}^{-1}$ is given by

$$\hat{\mathbf{x}} \circ A \circ \mathbf{x}^{-1}(u) = (u, (A^i_j \circ \mathbf{x}^{-1}(u))_{i,j}) \qquad \forall u \in \mathbf{x}(U) \subset \mathbb{R}^n.$$

The map above is smooth if and only if all $A^i_j \circ \mathbf{x}^{-1}$ are smooth, which IS the definition of $A^i_j : U \to \mathbb{R}$ being smooth.

Exercise 4 (Change of coordinates for tensor bundles)**.** Let $\{x^i\}$ and $\{y^a\}$ be local coordinates defined on $U \subset M$, and $A \in T^{2,1}(M)$. Suppose in the two coordinate systems, a (k, ℓ)-tensor $A \in \Gamma(T^{k,\ell}M)$ is given by

$$(A_x)^k_{ij} \frac{\partial}{\partial x^k} \otimes dx^i \otimes dx^j, \qquad (A_y)^c_{ab} \frac{\partial}{\partial y^c} \otimes dy^a \otimes dy^b$$

Show that

$$(A_y)^c_{ab} = \frac{\partial y^c}{\partial x^k} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} (A_x)^k_{ij}.$$

where we are using Einstein summation convention.

Solution 4. Clearly $(A_y)^c_{ab} = A\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}, dy^c\right)$ and $(A_x)^k_{ij} = A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, dx^k\right)$. Thus we can write, always using the summation convention,

$$\begin{aligned}(A_y)^c_{ab} &= A\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}, dy^c\right)\\ &= A\left(\frac{\partial x^i}{\partial y^a}\frac{\partial}{\partial x^i}, \frac{\partial x^j}{\partial y^b}\frac{\partial}{\partial x^j}, \frac{\partial y^c}{\partial x^z} dx^k\right)\\ &= \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\frac{\partial y^c}{\partial x^k} A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, dx^z\right)\\ &= \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}\frac{\partial y^c}{\partial x^k}(A_x)^k_{ij},\end{aligned}$$

Exercise 5 (Extending vectors to sections)**.** Let $\pi : E \to M$ be a smooth vector bundle, and let $x \in E_p(= \pi^{-1}(p))$ for some $p \in E$. Prove that there exists a global section $\sigma \in \Gamma(E)$ such that $\sigma(p) = x$.

Solution 5. Let $U \subset M$ be a trivializing neighbourhood of p, with trivializing map $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Let $\phi(x) = (p, v)$. Let $U' \subset U$ be an open set containing p with $\overline{U'} \subset U$, and take a function $f \in C^\infty(M)$ such that $f \equiv 0$ outside of U', and $f \equiv 1$ in a neighbourhood of p, and define the function $\sigma' : U \times \mathbb{R}^k$ by $\sigma'(q) = (q, f(q)v)$. Finally, define the global section $\sigma : M \to E$ by:

$$\sigma(q) = \begin{cases} \phi^{-1}(\sigma'(q)) & \text{if } q \in U \\ 0_q & \text{otherwise.} \end{cases}$$

In other words, σ is the zero section outside of U, and $\phi^{-1} \circ \sigma'$ in U. Notice however that, since $\phi^{-1} \circ \sigma'$ is already the zero section near the boundary of U, the two pieces meet smoothly and give rise to a smooth global section. Moreover, $\sigma(p) = \phi^{-1}\sigma'(p) = \phi^{-1}(p, v) = x$.

Important! The fundamental property used here, is the existence of bump functions, which holds in the case of smooth function. In fact, if instead we were dealing with more rigid functions, such as analytic or holomorphic functions, then in these setups there are bundles with NO global sections.