#7 - CONNECTIONS ON VECTOR BDLES. (DC, Ch. 2)

Last time:

$\Omega \subseteq \mathbb{R}^n$, $\underline{x}: \Omega \to D$ chart, $\omega = f\, dx^1 \wedge \dots \wedge dx^n$

$\Rightarrow \int_\Omega \omega := \int_D f\, dx^1 \dots dx^n$ well defined.

M mfd, $\mathcal{A} = \{(U_\alpha, \underline{x}_\alpha)\}$ atlas, $\omega \in \Omega^n(M)$. Take partition of unity $\{\varphi_\alpha\}$, write $\omega|_{\underline{x}_\alpha} = f(x^1 \dots x^n)\, dx^1 \wedge \dots \wedge dx^n$ and define

- $$\int_{U_\alpha} \varphi_\alpha \omega := \int_{\underline{x}_\alpha(U_\alpha)} \varphi_\alpha \cdot f \cdot dx^1 \dots dx^n$$

[on $U_{\alpha\beta} \neq 0$, $\int_{U_{\alpha\beta}} \omega$ does not depend on $\underline{x}_\alpha$ or $\underline{x}_\beta$]

- $\int_M \omega := \sum_\alpha \int_{U_\alpha} \varphi_\alpha \omega$ [does not depend on φ_α, $\underline{x}_\alpha$, U_α]

If (M,g) Riem mfd $\Rightarrow$ can integrate functions $\int_M f := \int_M f \cdot d\mathrm{vol}_g$.

$E \to M$ vector bdle.

<u>Def</u>: connection on E is a map
$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$
$$(X, \sigma) \longmapsto \nabla_X \sigma$$
st.

1) $f, g \in C^\infty(M)$, $\nabla_{(fX + gY)} \sigma = f \cdot \nabla_X \sigma + g \nabla_Y \sigma$
2) $a, b \in \mathbb{R}$, $\nabla_X (a\sigma + b\tau) = a \nabla_X \sigma + b \nabla_X \tau$
3) $\nabla_X f\sigma = X(f)\sigma + f \cdot \nabla_X \sigma$.

$\nabla : X \in \mathfrak{X}(M) \longrightarrow \nabla_X \in Der(\Gamma(E))$ ~~map of~~ morphism of $C^\infty(M)$-modules.

Take (U, x) chart, with trivializ. $\varphi : \pi^{-1}U \to U \times \mathbb{R}^m$, and take $\sigma_i \in \Gamma(E)$ $\quad \sigma_i(p) = \varphi^{-1}(p, e_i)$ $\quad$ ($\sigma_1 \dots \sigma_m$ frame of sections of E)

~~given $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$~~, $\nabla_{\frac{\partial}{\partial x^i}} \sigma_j = \underbrace{\Gamma^k_{ij}}_{\text{Christoffel symbols of } \nabla} \sigma_k$

Given
$X = x^i \frac{\partial}{\partial x^i}$, $\quad \sigma = y^j \sigma_j$ $\qquad x^i, y^j : U \to \mathbb{R}$, Then

$$\nabla_X \sigma = \nabla_X y^j \sigma_j = X(y^j)\sigma_j + y^j \nabla_X \sigma_j =$$

$$= X(y^j)\sigma_j + y^j x^i \nabla_{\frac{\partial}{\partial x^i}} \sigma_j = X(y^j)\sigma_j + x^i y^j \Gamma^k_{ij} \sigma_k$$

$$= \left(X(y^k) + x^i y^j \Gamma^k_{ij} \right) \sigma_k$$

$\downarrow$
We get

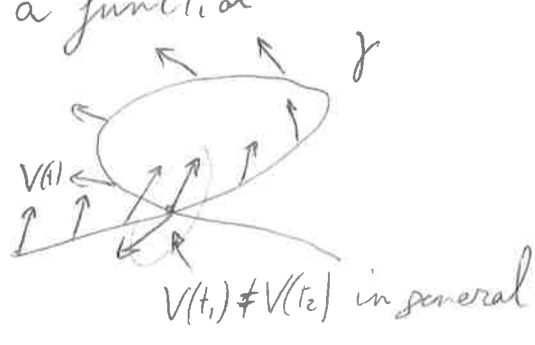
<u>Prop</u>: 1. $\nabla_X \sigma(p)$ depends on X_p, and on the first derivative of σ in the direction on X_p.

2. ∇ is determined, in U, by the Christoffel symbols.

$\downarrow$

Let $\gamma : [0,1] \longrightarrow M$ smooth curve, $E \to M$ bdle.

<u>Def</u> A ~~vector field~~ section of E along γ is a function
$V : t \longmapsto V(t) \in E_{\gamma(t)}$



~~$\Gamma(E_\gamma)$~~ $\quad \Gamma(\gamma^* E) = \{ V \text{ sect of } E \text{ along } \gamma \}$

Def: A covariant derivative is a ~~map~~ linear map

$$\frac{D}{dt} : \Gamma(\gamma^*E) \longrightarrow \Gamma(\gamma^*E) \quad \text{st. } \frac{D}{dt}\left(f(t) \cdot V(t)\right) = f'(t)\, V(t) + f(t) \frac{D}{dt} V(t)$$

PROP: a connection ~~on~~ ∇ on E induces, for every smooth curve $\gamma : [0,1] \to M$, a unique covariant derivative $\frac{D}{dt}$ st. for $V(t) = \sigma|_{\gamma(t)}$, $\frac{D}{dt} V(t) = \nabla_{\gamma'(t)} \sigma$ (does not depend on the specific σ)

PROP. $E \xrightarrow{\pi} M$ v. bdle, $\gamma : [0,1] \to M$ curve, $\frac{D}{dt}$ cov. der. of γ^*E

$\Rightarrow \forall\, v_0 \in E_{\gamma(0)}\ \exists\, V(t) \in \Gamma(\gamma^*E)$ st

$$\circledast \begin{cases} \frac{D}{dt} V = 0 \quad \forall t \\ V(0) = v_0 \end{cases}$$

V called parallel along γ

Pf: Assume $Im(\gamma) \subseteq U$ $\quad (U, x)$ chart. Fix a frame of sections of E, $\sigma_1, \dots, \sigma_m$

$$\gamma'(t) = \sum a^i(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}, \quad V_j(t) = \sigma_j|_{\gamma(t)}, \quad V(t) = y^j(t)\, V_j(t)$$

$$\Rightarrow \frac{D}{dt} V(t) = (y^j)'(t)\, V_j(t) + \frac{D}{dt} V_j(t) = (y^j)'(t)\, V_j(t) + y^j \nabla_{\gamma'(t)} V_j =$$

$$= (y^j)'(t)\, V_j(t) + y^j(t) a^i(t)\, \Gamma_{ij}^k(\gamma(t)) V_k(t)$$

$$= \left((y^k)'(t) + y^j(t) a^i(t)\, \Gamma_{ij}^k(\gamma(t)) \right) V_k(t)$$

write $v_0 = v_0^k V_k(0)$. Then V satisfying $\circledast$ equals the system of linear 1st order ODE's:

$$\begin{cases} (y^k)' = -\, y^j a^i\, \Gamma_{ij}^k \circ \gamma \\ y^k(0) = v_0^k \end{cases}$$

$\exists!$ solution.

Def: parallel transport: $\gamma : [a,b] \to M$, def $P_\gamma^{a,b} : E_{\gamma(a)} \longrightarrow E_{\gamma(b)}$

$P_\gamma^{a,b}(v) = V(b)$, where $V \in \Gamma(\gamma^*E)$ is the unique section, st $\frac{D}{dt} V = 0$, $V(a) = v$.