HOMEWORK #2 - SOLUTIONS

Exercise 1 (Tensor fields)**.** Prove the following:

(1) Let $F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Omega^1(M) \to C^\infty(M)$ be a $C^\infty(M)$-multilinear map. Given $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$, prove that the value of $f = F(X, Y, \omega)$ at p only depends on $X_p, Y_p \in T_pM$ and $\Omega_p \in T^*M$.

(2) Prove that the map

$$\Gamma(T^{2,1}M) \to \{F : \mathcal{X}(M) \times \mathcal{X}(M) \times \Omega^1(M) \to C^\infty(M) \quad C^\infty(M)\text{- multilinear}\}$$

sending $\phi \in \Gamma(T^{2,1}M)$ to $F_\phi(X, Y, \omega)(p) := \phi_p(X_p, Y_p, \omega_p)$ is a $C^\infty(M)$-module isomorphism.

Solution 1. 1) Let $X', Y' \in \mathcal{X}(M)$, $\omega' \in \Omega_1(M)$ be such that $X_p = X'_p$, $Y_p = Y'_p$, $\omega_p = \omega'_p$. By multilinearity we have

$$\begin{aligned}
F(X, Y, \omega)(p) - F(X', Y', \omega')(p) =& \big(F(X, Y, \omega)(p) - F(X', Y, \omega)(p)\big) \\
&+ \big(F(X', Y, \omega)(p) - F(X', Y', \omega)(p)\big) \\
&+ \big(F(X', Y', \omega)(p) - F(X', Y', \omega')(p)\big) \\
=& F(X - X', Y, \omega)(p) + F(X', Y - Y', \omega)(p) \\
&+ F(X', Y', \omega - \omega')(p)
\end{aligned}$$

We want to prove that the difference is zero, and from the equality above it is enough to prove that $F(X, Y, \omega)(p) = 0$ whenever $X_p = 0$ or $Y_p = 0$ or $\omega_p = 0$.

Let's assume $X_p = 0$. Take $(U, \mathbf{x})$ a chart around p. Write $X = X^i \frac{\partial}{\partial x^i}$, so that $X^i_p = 0$ for all i, and let $\phi : M \to [0, 1]$ be a smooth function such that $\phi \equiv 0$ outside of U, and $\phi \equiv 1$ around p (ϕ is called *cut-off function*). Notice that $\phi^2 X = 0$ outside of U, and equal to $(\phi X^i)\phi \frac{\partial}{\partial x^i}$ on U. Define the smooth global functions and vector fields

$$\hat{X}^i = \begin{cases} \phi X^i & \text{in } U \\ 0 & \text{outside of } U \end{cases} \qquad \hat{V}_i = \begin{cases} \phi \frac{\partial}{\partial x^i} & \text{in } U \\ 0 & \text{outside of } U \end{cases}$$

Then $\phi^2 X = \hat{X}^i V_i$, and $\hat{X}^i(p) = 0$ for all i.

Since F is $C^\infty(M)$-multilinear, around p we have

$$\phi^2 F(X, Y, \omega) = F(\phi^2 X, Y, \omega) = F(\hat{X}^i V_i, Y, \omega) = \hat{X}^i F(V_i, Y, \omega).$$

Since $\phi(p) = 1$ and $\hat{X}^i(p) = 0$ for all i, evaluating the above identity at p gives

$$F(X, Y, \omega)(p) = 0,$$

as wanted.

2) First, it is clear from the definition that F_ϕ is $C^\infty(M)$-multilinear, and that for any two sections ϕ_1, ϕ_2 and functions f_1, f_2, one has

$$F_{f_1\phi_1 + f_2\phi_2} = f_1 F_{\phi_1} + f_2 F_{\phi_2}$$

Now, let us prove that if ϕ is a smooth section, then for any $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$, the function $F_\phi(X, Y, \omega)$ is smooth. This is a local statement, so it is enough to loot at a chart $(U, \mathbf{x})$. By the previous point, the value of $F_\phi(X, Y, \omega)$ in U only depends on the values of X, Y, ω in U. So let us write $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial x^j}$, $\omega = \omega_k dx^k$, $\phi = \phi_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$, where all functions $X^i, Y^j, \omega_k, \phi_{ij}^k$ are smooth. By multilinearity of F_ϕ we have

$$\begin{aligned} F_\phi(X, Y, \omega) &= X^i Y^j \omega_k \phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, dx^k\right) \\ &= X^i Y^j \omega_k \phi_{ij}^k \end{aligned}$$

which is smooth.

Finally, we need to prove that the map $\phi \mapsto F_\phi$ is a bijection. Injection follows from the computations above: in fact, if $F_\phi(X, Y, \omega) \equiv 0$ for every X, Y, ω, then on any chart $(U, \mathbf{x})$ one has $X^i Y^j \omega_k \phi_{ij}^k = 0$ for any X^i, Y^j, ω^k, which implies that $\phi_{ij}^k = 0$ thus $\phi = 0$.

As for surjectivity: given any F, define ϕ locally on any chart $(U, \mathbf{x})$ by $\phi_U := (\phi_U)_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$, where $(\phi_U)_{ij}^k = F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, dx^k\right)$. On overlapping charts $(U, \mathbf{x})$, $(V, \mathbf{y})$,one has to check (using the transformations for sections from last homework) that the two definitions of ϕ_U, ϕ_V coincide:

$$\begin{aligned} \phi_V =& F\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}, dy^c\right) dy^a \otimes dy^b \otimes \frac{\partial}{\partial y^c} \\ =& \left(\frac{\partial x^t}{\partial y^c}\frac{\partial y^a}{\partial x^r}\frac{\partial y^b}{\partial x^s} F\left(\frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, dx^t\right)\right)\left(\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}\right) \\ =& \left(\frac{\partial x^t}{\partial x^k}\frac{\partial x^i}{\partial x^r}\frac{\partial x^j}{\partial x^s} F\left(\frac{\partial}{\partial x^r}, \frac{\partial}{\partial x^s}, dx^t\right)\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \\ =& F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, dx^k\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} \\ =& \phi_U. \end{aligned}$$

Therefore, the local ϕ_U glue together to a global ϕ, and from the computations above it follows that $F = F_\phi$.

Exercise 2 (Lie groups)**.** Prove the following:

(1) Letting $G \times \mathfrak{g} \xrightarrow{\pi_1} G$ denote the trivial vector bundle over G, with fiber $\mathfrak{g}$, prove that the map $\psi : TG \to G \times \mathfrak{g}$ sending $(g, v) \in T_g G$ to $(g, dL_{g^{-1}})$ is a diffeomorphism, such that $\pi_1 \circ \psi = \pi$ and such that the restriction to each fiber is a linear isomorphism (TG and $G \times \mathfrak{g}$ are *isomorphic smooth vector bundles*).

(2) A vector field $X \in \mathcal{X}(G)$ is called *left* invariant if $L_g(X_h) = X_{gh}$ for every $g, h \in G$. Prove that the set of left invariant vector fields is a vector space, isomorphic to $\mathfrak{g}$.

(3) Prove that the Lie bracket of two left invariant vector fields is again left invariant. Use this to prove that $\mathfrak{g}$ can be given the structure of a finite-dimensional Lie algebra.

(4) Prove that every inner product $\langle , \rangle_e : \mathfrak{g} \times \mathfrak{g}$ can be extended to a Riemannian metric $\langle , \rangle$ on G that is *left-invariant*, that is, such that every left multiplication $L_g : G \to G$ is an isometry.

Solution 2. 1) The map ψ is a bijection, with inverse $\psi^{-1}(g,x) = (g,(L_g)_*x)$. Clearly $\pi(g,v) = \pi_1 \circ \psi(g,v) = g$. Finally, for every $g \in G$, ψ restricts to a linear isomorphism $\psi_g = (L_{g^{-1}})_* : T_gG \to \mathfrak{g}$, with inverse $(L_g)_*$. It is left to prove that it is a diffeomorphism. We will prove that ψ is smooth, a similar proof will work for ψ^{-1}. This is a local statement, to let us consider a coordinate neighbourhood $(U,\mathbf{x})$ in G, and prove that $\psi : \pi^{-1}U \to U \times \mathfrak{g}$ is smooth. First, we prove the following

Lemma 1. *Let $(V,\underline{y})$ be a coordinate neighbourhood of G around e. Then for any $i,j = 1,\dots,n$, the functions $\phi_{ij} : U \to \mathbb{R}$ such that $(L_{g^{-1}})_* \frac{\partial}{\partial x^i}|_g = \phi_{ij}(g)\frac{\partial}{\partial y^j}|_e$ are smooth.*

Proof. Let $\mu : G\times G \to G$ be the (smooth) map $\mu(g,h) = g^{-1}h$. Possibly shrinking U, we can suppose that $\mu(U \times U) \subseteq V$. The function ϕ_{ij} can be rewritten as $\left((L_{g^{-1}})_*\frac{\partial}{\partial x^i}|_g\right)(y^j)$. Consider first the function $\hat{\phi}_{ij} : U \times U \to \mathbb{R}$ given by $(g,h) \mapsto \left((L_{g^{-1}})_*\frac{\partial}{\partial x^i}|_h\right)(y^j)$. Define the coordinates on $U \times U$ given by $x_1^i = x^i \times 0$ and $x_2^i = 0 \times x^i$. Then we can rewrite $\hat{\phi}_{ij}$ as

$$\hat{\phi}_{ij}(g,h) = \left((L_{g^{-1}})_*\frac{\partial}{\partial x^i}\Big|_h\right)(y^j) = \frac{\partial}{\partial x_2^i}\Big|_{(g,h)}(y^j \circ \mu)$$

Thus $\hat{\phi}_{ij} = \frac{\partial}{\partial x_2^i}(y^j \circ \mu)$ is a smooth function, and therefore so is $\phi_{ij}(g) = \hat{\phi}_{ij}(g,g)$. ∎

Composing $\psi : \pi^{-1}U \to U \times \mathfrak{g}$ with the trivialization $\varphi^{-1} : U \times \mathbb{R}^n \to \pi^{-1}(U)$, we get

$$\begin{aligned}\psi \circ \varphi^{-1}(g,(a_1,\dots a_n)) &= \psi\left(g, a_i\frac{\partial}{\partial x_i}\Big|_g\right) = \left(g,(L_{g^{-1}})_*\left(a_i\frac{\partial}{\partial x_i}\Big|_g\right)\right)\\ &= \left(g, a_i\phi_{ij}(g)\frac{\partial}{\partial y^j}\Big|_e\right).\end{aligned}$$

Since ϕ_{ij} are smooth, $f = \psi \circ \varphi^{-1}$ is smooth, thus $\psi = f \circ \varphi$ is smooth as well.

2) Clearly if X, Y are left invariant, so is $X + Y$:

$$(L_g)_*(X+Y)_h = (L_g)_*(X_h+Y_h) = (L_g)_*X_h + (L_g)_*Y_h = X_{gh} + Y_{gh} = (X+Y)_{gh}.$$

Define the map {Left invariant vector fields}$\to \mathfrak{g}$ sending X to X_e. This is a linear map. It is injective, since if $X_e = 0$ then $X_g = (L_g)_*X_e = 0$ for all $g \in G$. It is surjective, since for any $x \in \mathfrak{g}$ we can define X by $X_g = (L_g)_*x$. This is left invariant, because

$$(L_g)_*X_h = (L_g)_*(L_h)_*x = (L_gL_h)_*x = (L_{gh})_*x = X_{gh}.$$

3) Notice that a vector field $X \in \mathcal{X}(G)$ is left invariant if and only if $(L_g)_*X = X \circ L_g$ or equivalenty, $X(f \circ L_g) = X(f) \circ L_g$ for every smooth function f on G. Thus, for left invariant X, Y, it is enough to prove that $[X,Y](f \circ L_g) = [X,Y](f) \circ L_g$ for every function f. This is true because

$$\begin{aligned}[X,Y](f \circ L_g) &= X(Y(f \circ L_g)) - Y(X(f \circ L_g)) = X(Y(f) \circ L_g) - Y(X(f) \circ L_g)\\ &= XY(f) \circ L_g - YX(f) \circ L_g = (XY - YX)(f) \circ L_g\\ &= [X,Y](f) \circ L_g.\end{aligned}$$

One can the define a Lie bracket $[,]$ on $\mathfrak{g}$ by $[x, y] := [X, Y]_e$ where X, Y are the unique left invariant vector fields with $X_e = x$, $Y_e = y$. This bracket is clearly bilinear and skew symmetric, Moreover, the Jacobi identity is satisfied because the Lie bracket of left invariant vector fields is again left invariant, therefore $(\mathfrak{g}, [,])$ is a Lie algebra.

4) Given $\langle , \rangle_e : \mathfrak{g} \times \mathfrak{g}$, define $\langle , \rangle_g : T_g G \times T_g G$ by $\langle v, w \rangle_g = \left\langle (L_{g^{-1}})_* v, (L_{g^{-1}})_* w \right\rangle_e$. Via the identification ψ in point 3, this corresponds to the constant section $\sigma \in \Gamma(G \times Sym^2(\mathfrak{g}^*))$ given by $\sigma(g) = (g, \langle , \rangle_e)$, which is smooth. Left translations are isometries since for every $v, w \in T_h G$, $(L_g)_* v, (L_g)_* w \in T_{gh} G$ and

$$\begin{aligned} \langle (L_g)_* v, (L_g)_* w \rangle_{gh} &= \left\langle (L_{(gh)^{-1}})_* (L_g)_* v, (L_{(gh)^{-1}})_* (L_g)_* w \right\rangle_e \\ &= \langle (L_{h^{-1}})_* v, (L_{h^{-1}})_* w \rangle_e \\ &= \langle v, w \rangle_h . \end{aligned}$$

Exercise 3 (Real projective space)**.** Let $a : S^n \to S^n$ denote the antipodal map, sending p to $-p$.

(1) Prove that $\{id_{S^n}, a\}$ is a group of diffeomorphisms of S^n, isomorphic to $\mathbb{Z}_2$, and that the action of $\mathbb{Z}_2$ on S^n induced by a is properly discontinuous. Therefore, the quotient $S^n/\mathbb{Z}_2$ is a manifold, called *real projective space* and denoted $\mathbb{RP}^n$.

(2) Prove that a is an isometry of S^n with its round metric g_0. Use this fact to produce a metric g on $\mathbb{RP}^n$ such that the quotient $\pi : S^n \to \mathbb{RP}^n$ satisfies $g_0 = \pi^* g$.

Solution 3. 1) Clearly both id and the antipodal map are smooth maps (they are restriction of smooth maps on $\mathbb{R}^n$). Moreover, $a^2(p) = p$ for every p, thus $a^2 = id$, which means that the inverse of a, which is a itself, is smooth, thus a is a diffeomorphism. Finally, the action is properly discontinuous because every for every point p, any small enough distance ball U around p (say, of radius 0.1) satisfies $U \cap a(U) = \emptyset$.

2) Any tangent space $T_p S^n$ in the sphere can be identified with the subspace V_p of $\mathbb{R}^n$ orthogonal to p. With this identification, $\langle , \rangle_p$ is simply the restriction of the standard inner product $\langle , \rangle_0$ of $\mathbb{R}^n$, to V_p. Moreover, $T_p S^n$, $T_{a(p)} S^n$ are both identified with V_p, and under this identifications, the map $a_* T_p S^n \to T_{a(p)} S^n$ is simply multiplication by -1. Therefore

$$\langle a_* v, a_* w \rangle_{a(p)} = \langle -v, -w \rangle_0 = \langle v, w \rangle_0 = \langle v, w \rangle_p .$$

Given a point $[p] \in S^n/\mathbb{Z}_2$, define the inner product $\langle \bar{v}, \bar{w} \rangle_{[p]}$ by choosing a representative $p \in S^n$ for $[p]$, vectors $v, w \in T_p S^n$ such that $\pi_* v = \bar{v}$, $\pi_* w = \bar{w}$, and defining $\langle \bar{v}, \bar{w} \rangle_{[p]} = \langle v, w \rangle_p$. By the discussion above, this is well defined: in fact, given another p' representing $[p]$ and vectors $v', w' \in T_{p'} S^n$ with the same properties as v, w, it follows that $p' = a(p)$ and $v' = a_*(v)$, $w' = a_*(w)$, and so $\langle v, w \rangle_p = \langle v', w' \rangle_{p'}$, therefore the definition of $\langle \bar{v}, \bar{w} \rangle_{[p]}$ does not depend on the choice of representative for $[p]$.