HOMEWORK #3 - DUE FEB 16, AT NOON

Exercise 1 (One more thing on tensor fields)**.** Prove that:

(1) There is an isomorphism of $C^\infty(M)$-modules

$$\Gamma(T^{2,1}M) \to \{A : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \mid A \text{ is } C^\infty(M)\text{-multilinear}\}.$$

[**Hint:** use the characterization of $\Gamma(T^{2,1}M)$ from the last set of homeworks.]

(2) The torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is a (2,1)-tensor.

Exercise 2 (Parallel transport)**.** Let (M,g) be a Riemannian manifold, and let $\gamma : [0,1] \to M$ be a smooth curve. Prove that:

(1) Given a connection ∇ on TM, the parallel transport $P_\gamma^{0,1} : T_{\gamma(0)}M \to T_{\gamma(1)}M$ is a linear map.

(2) If the connection is compatible with the metric, then $P_\gamma^{0,1}$ is a linear isometry between $(T_{\gamma(0)}M, g_{\gamma(0)})$ and $(T_{\gamma(1)}M, g_{\gamma(1)})$.

Exercise 3 (Hyperbolic plane)**.** Define the *hyperbolic plane* as the upper half plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, with the metric $g = \frac{1}{y^2}(dx^2 + dy^2)$ (that is, $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$). Prove that:

(1) With respect to the frame $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$, the Christoffel symbols for the Levi Civita connection are $\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

(2) Given $\gamma(t) = (t,1)$, the parallel vector field $V(t)$ along γ with $V(0) = \frac{\partial}{\partial x}$ is given by $V(t) = \cos(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} - \sin(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)}$.

Exercise 4 (Difference tensor)**.** Given two connections ∇^0, ∇^1 on TM, show that:

(1) The sum $\nabla^0 + \nabla^1$ is not a connection, while the sum $a\nabla^0 + b\nabla^1$ is a connection, provided that $a + b = 1$.

(2) The map $A(X,Y) = \nabla^1_X Y - \nabla^0_X Y$ defines a $(2,1)$-tensor field (called *difference tensor*).

(3) The set of all connections on TM is precisely $\nabla^0 + \Gamma(T^{2,1}M)$.

(4) The connections have the same geodesics if and only if the difference tensor is skew symmetric ($A(X,Y) = -A(Y,X)$).

(5) The connections have the same torsion if an only if the difference tensor is symmetric ($A(X,Y) = A(Y,X)$).

Exercise 5 (Naturality of Levi Civita connection)**.** Let $(M, \langle,\rangle_M)$, $(N, \langle,\rangle_N)$ be Riemannian manifolds, with Levi Civita connections ∇^M, ∇^N respectively. Suppose there is a diffeomorphism $\phi : M \to N$.

(1) Check that for every vector field $X \in \mathcal{X}(M)$, one can define a *push-forward* vector field $\phi_* X \in \mathcal{X}(N)$ by $(\phi_* X)_p := \phi_*(X_{\phi^{-1}(p)})$. **Prove** that $\phi_*[X,Y] = [\phi_* X, \phi_* Y]$.

(2) Prove that if ϕ is an isometry, then $\phi_*(\nabla^M_X Y) := \nabla^N_{\phi_* X} \phi_* Y$ for every $X, Y \in \mathcal{X}(M)$.