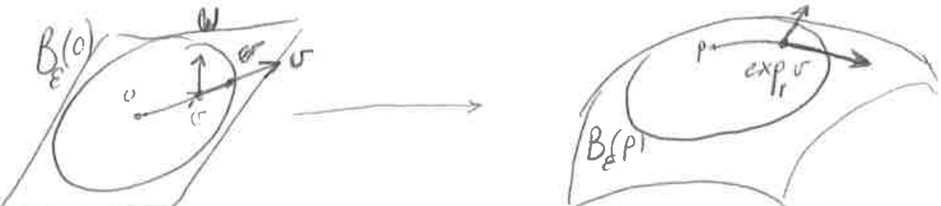
Geods are locally minimizing, I

<u>Last time</u>: par. surfaces $f: A \subseteq \mathbb{R}^2_{(s,t)} \longrightarrow M$ $\quad (df \neq 0)$

symmetry lemma: $\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{ds}\frac{\partial f}{\partial t}$

<u>Gauss' Lemma</u>: $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p) \subseteq M$ normal ball



For $v \in B_\varepsilon(0)$, $w \in T_v B_\varepsilon(0) = T_p M$, then

$$g_{\exp_p v}\left(d_v \exp_p w,\ d_v \exp_p v\right) = g_p(w, v)$$

<u>Pf</u>: statement is linear in w. $w = w_1 + w_2$, $\quad w_1 = \lambda v$, $w_2 \perp v$

Enough to prove the statement for w_1 and w_2 separately.

- $w_1 = \lambda v$, enough to prove it for $w_1 = v$

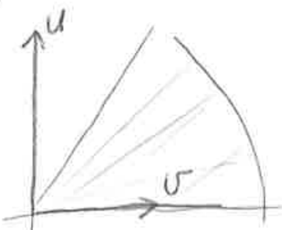
Recall: $\gamma(t) = \exp_p tv$ geod, and $\gamma'(t) = (d_{tv} \exp_p)\, v$

$$\Rightarrow g_{\exp_p v}\left(d_v \exp_p v,\ d_v \exp_p v\right) = \|\gamma'(1)\|^2 = \|\gamma'(0)\|^2 = g_p(v, v) \quad \checkmark$$

- $w_2 \perp v$, $\quad w_2 = \lambda u$, $\quad |u| = |v|$. Enough to prove statement for u:

$$g_{\exp_p v}\left(d_v \exp_p u,\ d_v \exp_p v\right) = 0$$

Take



$f(s,t) = \exp_p\left(t(\cos s\, v + \sin s\, u)\right)$

$\to f(s_0, t)$ are all geods

$d_v \exp_p u = \frac{\partial f}{\partial s}(0,1)$, $\quad d_v \exp_p v = \frac{\partial f}{\partial t}(0,1)$

Consider

$$\frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle_{(0,t)} = \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \underbrace{\frac{D}{dt}\frac{\partial f}{\partial t}}_{=\nabla_{\gamma'}\gamma' = 0}\right\rangle$$

$$= \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle = \frac{1}{2}\frac{d}{ds}\underbrace{\left\|\frac{\partial f}{\partial t}\right\|^2}_{=\|v\|} = 0$$

$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle (0,t) \equiv cost.$ However

$$\frac{\partial f}{\partial s}\Big|_{(0,t)} = d_{(0,t)} \exp_p tu \Rightarrow \left|\frac{\partial f}{\partial s}\right| \xrightarrow{t=0} 0 \Rightarrow \overset{C}{\overset{|||}{\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle}} \xrightarrow{t=0} 0 \Rightarrow cost = 0.$$

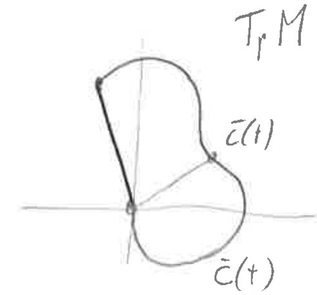
∎

Prop: (geods locally minimize length)

B: $B_\varepsilon(p)$ geod ball $\gamma = \exp_p tu : [0,1] \to B$ geod btw $\gamma(0)=p$ and $\gamma(1)=$

$c : [0,1] \to M$ curve w. $c(0) = p$, $c(1) = q$.

Then $\ell(\gamma) \le \ell(c)$, with "=" iff $Im(c) = Im(\gamma)$.



Pf: Supp. first that $Im(c) \subseteq B$. $\bar{c}(t) = \exp_p^{-1} c(t)$

Can use "polar coordinates": $\bar{c}(t) = r(t) \cdot v(t)$,

$r(t) \in \mathbb{R}_+$, $v(t) : [0,1] \to S^{n-1}$ $(|v| = 1)$

$$c(t) = \exp_p r \cdot v \Rightarrow c'(t) = d_{\bar{c}(t)}\exp(r' \cdot v + r \cdot v') = d_{r\cdot v}\exp(r'v + rv')$$

$$|c'(t)|^2 = \langle d_{rv}\exp r'v, d_{rv}\exp r'v \rangle + 2\langle d_{rv}\exp r'v, d_{rv}\exp rv' \rangle + |d\exp_p rv'|^2$$

$$= (r')^2 \cdot \underset{1}{\underset{\shortparallel}{\langle v, v \rangle}} + 2 \underset{0 \text{ bc. } v \perp v'}{\underset{\shortparallel}{\langle r'v, rv' \rangle}} + |d\exp_p rv'|^2$$

Gauss

$\ge (r'(t))^2$ with "=" iff. $v'(t) = 0$

$$\ell(c) = \int_0^1 |c'(t)|\, dt \ge \int_0^1 r'(t)\, dt = r(1) - \overset{0}{\overset{\shortparallel}{r(0)}} = r(1) = \ell(\gamma)$$

"=" iff $v'(t) \equiv 0 \Rightarrow v(t) \equiv v_0 \Rightarrow c(t) = \exp r(t) \cdot v_0$ and $r'(t) \ge 0 \ \forall t$

$\Rightarrow c(t) = \gamma(r(t)) \Rightarrow Im(c) = Im(\gamma)$.

If $c(t)$ escapes B at $t = t_1 < 1 \Rightarrow \ell(c) > \ell(c|_{[0,t_1]}) \ge \ell(\gamma)$.

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Recall:

Def: $U \ni p$ normal nbd for p, if $\exp_p : \exp_p^{-1}(U) \to U$ is a diffeo.