COMPLETENESS.

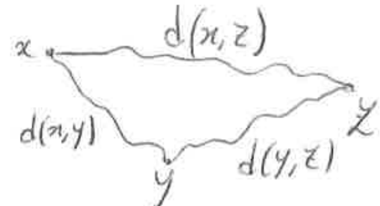
(M,g) Riem mfd, $p,q \in M$

$$d(p,q) = \inf\left\{ \ell(\gamma) \;\middle|\; \gamma : I \to M \text{ piecewise smooth curve } p \rightsquigarrow q \right\}$$

<u>Prop:</u>

1. d is a dist funct.
2. The topology induced by d is the same as the top of M.

<u>Pf:</u>



1. Identity, symmetry, transitivity are clear.
   - Only need to prove: $p \neq q \Rightarrow d(p,q) > 0$

   Take $B_\varepsilon(p)$ normal ball. If $q \notin B_\varepsilon(p)$, $\forall \gamma : p \rightsquigarrow q$ has $\ell(\gamma) > \varepsilon$

   $\Rightarrow d(p,q) \geq \varepsilon > 0$ If $q \in B_\varepsilon(p)$ $\exists!$ curve of minimal length:

   if $q = \exp_p v \Rightarrow \ell(\exp_p tv) = |v| = d(p,q)$.

2. By 1, $\cancel{B_\varepsilon \text{ for } \varepsilon}$ $\forall p \in M$ $\exists \varepsilon_0$ st. $B_\varepsilon(p)$ normal ball $\forall \varepsilon < \varepsilon_0$

   $\Rightarrow B_\varepsilon(p) = \{ q \in M \mid d(p,q) < \varepsilon \}$ $\Rightarrow$ All these normal ball are
   - A basis of open sets for the topology of M (coordinate nbds)
   - A basis of open sets for the metric topology (distance balls)

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<u>Cor:</u> $\forall p \in M$, $f = \text{dist}(p, \cdot)$ is continuous on M.

<u>Def:</u> M is called:

1. <u>Metrically complete</u> if $\forall$ Cauchy sequence has a limit
2. <u>Geodesically complete</u> if $\forall$ geodesic can be extended to $\gamma : \mathbb{R} \to M$

Theorem (Hopf-Rinow): ~~TFAE~~

M Riem mfd, $p \in M$. TFAE:

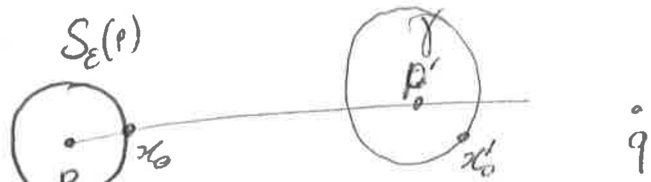
1) $\exp_p$ is defined for all $v \in T_pM$
2) Closed and bded sets are compact
3) M metrically complete
4) M geodesically complete
5) $\exists$ cpt. sets $\{K_i\}_i$, $K_i \subseteq K_{i+1}$, $\bigcup_i K_i = M$, st. $\forall$ seq $q_i \in K_i^c$, $d(p, q_i) \to \infty$

Moreover, these are all equivalent to

6) $\forall\, p, q\ \exists$ geodesic $\gamma: p \rightsquigarrow q$, with $d(p,q) = \ell(\gamma)$.

Pf:

$1 \Rightarrow 6)$



$d(p,q) = r$

$B_\varepsilon(p)$ normal ball, $S_\varepsilon(p) = \partial B_\varepsilon(p)$. $x_0 \in S_\varepsilon(p)$ st. $d(x_0, q)$ minimal in $S_\varepsilon(p)$. $x_0 = \exp_p \varepsilon v$, $|v| = 1$.

Define $\gamma(t) = \exp_p tv$. <u>Claim: $\gamma(r) = q$</u> $\Rightarrow d(p,q) = r = \ell(\gamma)$

<u>Pf of claim</u>: define $J = \{t \in [0, r]$ st. $d(\gamma(t), q) = r - t\}$

$0 \in J$, J closed $\Rightarrow$ enough to prove that whenever $[0, t_0] \subseteq J$, then $\forall \delta$ small enough, $t_0 + \delta \in J$.

Fix t_0, $p' := \gamma(t_0)$, δ small enough that $S_\delta(p') = \partial B_\delta(p')$ contained in a normal nbd. Take $x_0' \in S_\delta(p')$ minimizing $d(\cdot, q)$.

a. $d(p',q) = \inf(\ell(\gamma) \mid \gamma : p' \rightsquigarrow q)$. For δ small enough, $q \in B_\delta(p')^c$

$\Rightarrow$ any $\gamma : p' \rightsquigarrow q$ crosses $S_\delta(p')$ at some x

$$d(p',q) = \inf\{\ell(\gamma_1) \mid \gamma_1 \, p' \rightsquigarrow x \in S_\delta(p')\} + \inf\{\ell(\gamma_2) \mid \gamma_2 : x \to q\}$$

$$= \delta + d(x_0', q)$$

$$\Rightarrow d(x_0', q) = d(p', q) - \delta = r - t_0 - \delta$$

$$\Rightarrow d(p, x_0') \geqslant d(p,q) - d(x_0', q) = r - r + \delta + t_0 = t_0 + \delta$$

$$\begin{array}{c} \wedge\!| \\ \ell(\gamma|_{[0,t_0]} * \gamma_2|_{[0,\delta]}) = t_0 + \delta \end{array}$$

γ_2 geod ray $p' \rightsquigarrow x_0'$

all equivalences $\Rightarrow$ $\gamma|_{[0,t_0]} * \gamma_2$ is minimizing curve

$\Rightarrow$ γ, γ_2 are pieces of the same geodesic $\Rightarrow \gamma_2(t) = \gamma(t_0 + t)$

$\Rightarrow x_0' = \gamma(t_0 + \delta)$

$\Downarrow$

$d(\gamma(t_0+\delta), q) = d(x_0', q) = r - (t_0 + \delta) \Rightarrow t_0 + \delta \in J$

$\Rightarrow$ (open closed) $r \in J \Rightarrow d(\gamma(r), q) = 0$ ✓.

$(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1)$ $\quad (2 \Leftrightarrow 5)$

$1 \Rightarrow 2)$ A closed & bded

$A \subseteq B_R(p) \Rightarrow A \subseteq \exp \overline{B_R(0)}$

$\Rightarrow A$ bded $\Rightarrow$

$\overline{B_R(0)}$ cpt in $\mathbb{R}^n \Rightarrow \exp \overline{B_R(0)}$ cpt, A closed $\Rightarrow A$ cpt.

$2 \Rightarrow 3)$ $\{p_n\}$ Cauchy sequence $\Rightarrow S = \cup\{p_n\}$, S bded

$\Rightarrow \bar{S}$ bded & closed $\Rightarrow \bar{S}$ cpt $\Rightarrow \{p_n\} \in \bar{S}$ has converging

subsequence $\underset{p_n \text{ Cauchy}}{\Longrightarrow} \{p_n\}$ converges.

3 ⇒ 4) Supp. M not geodesically complete

$\Rightarrow \exists\, \gamma : (a,b) \longrightarrow M$ normalized geod., st $b < \infty$ and γ cannot be extended

Take $\{t_i\}$ Cauchy sequence in $\mathbb{R}$, st $t_i \to b$

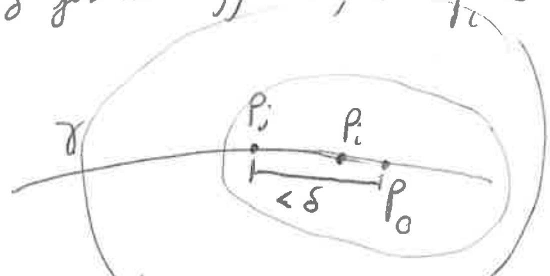
$\Rightarrow d(\gamma(t_i), \gamma(t_j)) \leq \ell(\gamma|_{[t_i,t_j]}) = |t_j - t_i| \Rightarrow p_i = \gamma(t_i)$ are

a Cauchy seq. in M $\Rightarrow \exists$ limit p_0.

Take totally normal nbd W of p_0 ($\forall\, p,q \in W\ \exists!$ min geod. γ w/ $\ell(\gamma) < \delta$)

Take N st. $d(p_i,p_j) < \delta$ for all $i,j > N$, and $p_i \in W$

$\Downarrow$



$\gamma|_{[p_i,p_j]}(t) = \exp_{p_j}(t-t_j)v$

however, $\exp_{p_j}(t-t_j)v$ is defined for $t - t_j$ up to δ

$\Downarrow$

if $d(p_j,p_0) < \frac{\delta}{2}$ $\exp_{p_j}(t-t_j)v$ extends γ past p_0.

4 ⇒ 1) Obvious (4 is stronger than 1)

2 ⇒ 5) Take $K_i = \overline{B_i(p)}$

5 ⇒ 2) C closed and bded $\Rightarrow \exists\, K_n$ st. $C \subseteq K_n$.

If not, C contains a sequence q_i st. $q_i \in K_i^c \Rightarrow d(p,q_i) \to \infty$

Contradicts bdedness.

Since C is closed and contained in K_n cpt $\Rightarrow C$ cpt. ∎