HOMEWORK #5 - DUE MAR 2 , AT NOON

Exercise 1 (A natural metric on the tangent bundle)**.** Let $(M, \langle,\rangle)$ be a Riemannian manifold, and $\pi : TM \to M$ the tangent bundle. Recall that a curve $\alpha : (0,1) \to TM$ has the form $\alpha(t) = (p(t), v(t))$. In other words, you can think of α as a curve $p(t) = \pi(\alpha(t))$ on M, together with a vector field $v(t)$ along $p(t)$. Given vectors $V, W \in T_{(p,v)}(TM)$, take curves $\alpha(t) = (p(t), v(t))$, $\beta(t) = (q(t), w(t))$ with $\alpha(0) = \beta(0) = (p, v)$ and $V = \alpha'(0)$, $W = \beta'(0)$, and define

$$\ll V, W \gg_{(p,v)} := \langle \pi_* V, \pi_* W \rangle_p + \left\langle \frac{D}{dt}\Big|_{t=0} v(t), \frac{D}{dt}\Big|_{t=0} w(t) \right\rangle_p .$$

(1) Prove that the definition does not depend on the choice of curves α, β, and it defines a Riemannian metric on TM.
(2) Given $(p, v) \in TM$, the *horizontal space at* (p, v) is the subspace $\mathcal{H}_{(p,v)} \subset T_{(p,v)}(TM)$ perpendicular to $\pi^{-1}(p) = T_pM$ in the metric above. Prove that $d_{(p,v)}\pi$ defines a linear isomorphism from $\mathcal{H}_{(p,v)}$ to T_pM.
(3) A curve $\alpha(t) = (p(t), v(t))$ is called *horizontal* if for every t, $\alpha'(t) \in \mathcal{H}_{\alpha(t)}$. Prove that α is horizontal iff $v(t)$ is parallel along $p(t)$.
(4) Prove that the geodesic vector field G is everywhere horizontal (i.e. $G_{(p,v)} \in \mathcal{H}_{(p,v)}$).
(5) Suppose that $\alpha, \beta : [0, 1] \to TM$, $\alpha(t) = (p(t), v(t))$, $\beta(t) = (p(t), w(t))$ are curves projecting to the same curve $p(t)$. Prove that if $v(t)$ is parallel, then $\ell(\beta) \leq \ell(\alpha)$ with equality if and only if $w(t)$ is parallel as well.
(6) Prove that the integral curves of G are geodesics in the metric above. (**Hint:** if $\overline{\gamma} : [0, 1] \to TM$ is an integral curve of G, prove that $\ell(\overline{\gamma}) = \ell(\pi \circ \overline{\gamma})$ and that around any $t_0 \in (0, 1)$, any curve $\alpha(t)$ between $\overline{\gamma}(t_0 - \epsilon)$ and $\overline{\gamma}(t_0 - \epsilon)$ satisfies $\ell(\alpha) \geq \ell(\pi \circ \alpha)$).

Exercise 2 (Metrics of Lie groups with bi-invariant metric)**.** Let G be a Lie group, $\mathfrak{g} = T_eL$ its *Lie algebra*. Recall that, for any inner product $\langle,\rangle_e$ on $\mathfrak{g}$, there is a left-invariant metric on G. A metric $\langle,\rangle$ on G is called *bi-invariant* if any of the following conditions is satisfied:

- $\langle,\rangle$ is invariant under left translations *and* right translations.
- the inner product $\langle,\rangle_e : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfies $\langle [x, y], z\rangle_e = -\langle y, [x, z]\rangle_e$ for any $x, y, z \in \mathfrak{g}$, where $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{g}$ defined last time.

(We will give for granted that the two conditions are equivalent).

(1) Prove that for any left invariant vector field X, the integral curve $c : (a, b) \to \mathbb{R}$ of X with $c(0) = e$ satisfies $c(t+s) = c(t)\cdot c(s)$ whenever $s, t, s+t \in (a, b)$.
(2) Prove that the integral curve c in the previous point is actually defined for all $t \in \mathbb{R}$. (we call $c(t) = \exp(tx)$, $x = X_e \in \mathfrak{g}$).
(3) Prove that if G carries a bi-invariant metric, the integral curves $c(t)$ in the point before are in fact geodesics (and in fact $\exp(tx) = \exp_e(tx)$). (**Hint:** Use the formula that defines the Levi Civita connection, and apply it to left-invariant vector fields).