HOMEWORK #6 - DUE MAR 9 , AT NOON

Exercise 1 (Sequences of geodesics)**.** Let M be a complete manifold. Let $\{(p_n, v_n)\} \in T_pM$ be a sequence, converging to (p_0, v_0). Let $\gamma_n : \mathbb{R} \to M$ be the geodesic $\gamma_n(t) = \exp_{p_n}(tv_n)$. Prove that the sequence of geodesics $\{\gamma_n\}_n$ converges pointwise to $\gamma_0(t) := \exp_{p_0}(tv_0)$.

Exercise 2 (Rays)**.** A *ray* in a manifold M, is a geodesic $\gamma : [0, \infty) \to M$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \geq 0$. Prove that if M is complete and noncompact, there exists a ray from every point $p \in M$.

Exercise 3 (Divergent curves)**.** A curve $\alpha : [0, \infty) \to M$ is a *divergent curve* if it "escapes every compact set": more precisely, for every compact set K in M, there is a $T > 0$ such that, for all $t > T$, $\alpha(t) \notin K$. Define the *length* of a divergent curve as $\ell(\alpha) = \lim_{t\to\infty} \ell(\alpha|_{[0,t]})$. Prove that M is complete if and only if every divergent curve has infinite length.

Exercise 4 (Homogeneous manifolds)**.** A manifold M is called *homogeneous* if for every $p, q \in M$, there exists an isometry of M that takes p to q. Prove that every homogeneous manifold is complete.

Exercise 5 (Expanding maps)**.** Let M, N be Riemannian manifolds of the same dimension. A smooth map $\phi : M \to N$ is called *expanding* if $g_N(\phi_* v, \phi_* v)_{\phi(p)} \geq g_M(v, v)_p$ for all $p \in M$ and $v \in T_pM$. Prove that if M is complete and $\phi : M \to N$ is expanding, then:

- ϕ is a covering map, i.e. for every smooth curve $\gamma : [0, 1] \to N$ of finite length, there is a curve $\overline{\gamma} : [0, 1] \to M$ such that $\overline{\gamma} \circ \phi = \gamma$. (**Hint:** use an open-close argument to show that the set $J \subset [0, 1]$ of points s such that $\overline{\gamma}$ is defined up to s, is actually the whole $[0, 1]$. For openness, show that ϕ is a local diffeomorphism at every point. For closedness, prove that if $\overline{\gamma}(s_i)$ are defined and $s_i \to s_0$, then the closure of $S = \{\overline{\gamma}(s_i)\}$ is open and closed.)
- N is complete as well. (**Hint:** one option is to use ex. 3)