HOMEWORK #6 - DUE MAR 9 , AT NOON

Exercise 1 (Sequences of geodesics)**.** Let M be a complete manifold. Let $\{(p_n, v_n)\} \in T_pM$ be a sequence, converging to (p_0, v_0). Let $\gamma_n : \mathbb{R} \to M$ be the geodesic $\gamma_n(t) = \exp_{p_n}(tv_n)$. Prove that the sequence of geodesics $\{\gamma_n\}_n$ converges pointwise to $\gamma_0(t) := \exp_{p_0}(tv_0)$ (i.e. $\gamma_n(t) \to \gamma_0(t)$ for all t). Is this convergence uniform (in the sense that for n large enough, $d(\gamma_n(t), \gamma_0(t)), C_n$, for $C_n \to 0$)?

Solution 1. For any $t \in \mathbb{R}$, the rescaling $r_t : TM \to TM$, $r_t(p, v) = (p, tv)$ is continuous. Since the exponential map $\exp : TM \to M$ is continuous as well, we have

$$\gamma_n(t) = \exp(p_n, tv_n) = \exp r_t(p_n, v_n) \to \exp r_t(p_0, v_0) = \exp_{p_0} tv_0 = \gamma_0(t).$$

Exercise 2 (Rays)**.** A *ray* in a manifold M, is a geodesic $\gamma : [0, \infty) \to M$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \geq 0$. Prove that if M is complete and noncompact, there exists a ray from every point $p \in M$.

Solution 2. Take a sequence of points $q_i \in M$ such that $d_i := d(p, q_i) \to \infty$. Let $\gamma_i(t) = \exp_p tv_i$ be minimizing geodesics between p and q_i, where $|v_i| = 1$. Since the sequence $\{v_i\}$ is contained in the unit sphere of T_pM which is compact, there exists a converging subsequence, which we still call $\{v_i\}$, converging to some v_0. By the previous exercise, $\gamma_i(t) \to \gamma_0(t) := \exp_p tv_0$. We claim that $\gamma_0(t)$ is a ray. In fact, for every $s > t \in \mathbb{R}$, pick n large enough that $d_i > s$ for all $i > n$. Then for all $i > n$, $\gamma_i|_{[0,s]}$ is minimizing and thus $d(\gamma_i(s), \gamma_i(t)) = |s - t|$. Taking the limit as $i \to \infty$, one gets $d(\gamma_0(s), \gamma_0(t)) = |s - t|$, as wanted.

Exercise 3 (Divergent curves)**.** A curve $\alpha : [0, \infty) \to M$ is a *divergent curve* if it "escapes every compact set": more precisely, for every compact set K in M, there is a $T > 0$ such that, for all $t > T$, $\alpha(t) \notin K$. Define the *length* of a divergent curve as $\ell(\alpha) = \lim_{t\to\infty} \ell(\alpha|_{[0,t]})$. Prove that M is complete if and only if every divergent curve has infinite length.

Solution 3. Suppose first that M is complete. Let $p = \alpha(0)$ and let $\{K_n = B_n(p)\}_n$ (metric balls of radius n). Since α is a divergent curve, for every n there is a t_n such that $\alpha|_{[t_n,\infty]} \subset K_n^c$. In particular, $\ell(\alpha|_{[0,t_n]}) \geq d(\alpha(0), \alpha_n) \geq n$, and therefore

$$\ell(\alpha) = \lim_{t\to\infty} \ell(\alpha|_{[0,t]}) \geq \lim_{n\to\infty} n = \infty.$$

On the other hand, suppose that α is a divergent curve of finite length L, and let $p_n = \alpha(n)$. Then $\{p_n\}_n$ is a Cauchy sequence: in fact, for every $\epsilon > 0$, take N big enough that $\ell(\alpha|_{[0,N]}) > L - \epsilon$. Then for every $i, j > N$,

$$d(p_i, p_j) = d(\alpha(i), \alpha(j)) \leq \ell(\alpha|_{[i,j]}) \leq \lim_{t\to\infty} \ell(\alpha|_{[N,t]}) = L - \ell(\alpha|_{[0,N]}), \epsilon.$$

If M was complete, then the sequence$\{p_n\}$ would have a limit p_0. Taking the (compact) metric ball $B = B_L(p_0)$, we would have for all $t > 0$

$$d(\alpha(t), p_0) = \lim_{n\to\infty} d(\alpha(t), p_n) = \lim_{n\to\infty} d(\alpha(t), \alpha(n)) \leq \lim_{n\to\infty} L = L.$$

Thus α would never escape the compact set B, contradicting the fact that α is divergent.

Exercise 4 (Homogeneous manifolds)**.** A manifold M is called *homogeneous* if for every $p, q \in M$, there exists an isometry of M that takes p to q. Prove that every homogeneous manifold is complete.

Solution 4. We will show that, fixing $p \in M$, the exponential map $\exp_p$ is defined for all T_pM. Fix ϵ such that $\exp_p tv$ is defined for every unit v, and every $|t| < 2\epsilon$. Let $s \in (0, \infty]$ be the maximum value such that $\exp_p tv$ is defined for every unit v, and every $|t| < s$. We will show by contradiction that $s = \infty$. in fact, if s was finite, then take v such that the geodesic $\gamma(t) = \exp_p tv$ does not exist for $t \geq s$, and let $q = \gamma(s-\epsilon)$, and let ϕ be an isometry taking p to q. Then $\hat{\gamma}(t) = (\phi^{-1} \circ \gamma)(t-s+\epsilon)$ is a normalized geodesic such that $\hat{\gamma}(0) = p$, and which does not exist for $t \geq \epsilon$. This, however, contradicts the fact that all normalized geodesics starting at p, exist for at least time 2ϵ.

Exercise 5 (Expanding maps)**.** Let M, N be Riemannian manifolds of the same dimension. A smooth map $\phi : M \to N$ is called *expanding* if $g_N(\phi_* v, \phi_* w)_{\phi(p)} > g_M(v, w)_p$ for all $p \in M$ and $v, w \in T_pM$. Prove that if M is complete and $\phi : M \to N$ is expanding, then:

- ϕ is a covering map, i.e. for every smooth curve $\gamma : [0, 1] \to N$ of finite length, there is a curve $\overline{\gamma} : [0, 1] \to M$ such that $\overline{\gamma} \circ \phi = \gamma$. (**Hint:** use an open-close argument to show that the set $J \subset [0, 1]$ of points s such that $\overline{\gamma}$ is defined up to s, is actually the whole $[0, 1]$. For openness, show that ϕ is a local diffeomorphism at every point. For closedness, prove that if $\overline{\gamma}(s_i)$ are defined and $s_i \to s_0$, then the closure of $S = \{\overline{\gamma}(s_i)\}$ is open and closed.)
- N is complete as well. (**Hint:** one option is to use ex. 3)

Solution 5. 1)Let $\gamma : [0, 1] \to N$ be a curve in N, let $p \in M$ be a point such that $\phi(p) = \gamma(0)$, and let $J = \{s \in [0, 1] \mid \exists \overline{\gamma} : [0, s] \to M : \phi(\overline{\gamma}(t)) = \gamma(t)\}$. Clearly $0 \in J$, and we want to prove that $1 \in J$. We prove that J is open and closed. For the openness, suppose $s \in J$, and let $p = \overline{\gamma}(s)$. By the expanding property of ϕ, we have that $d_p\phi$ is injective: in fact, if $d_p\phi(v) = 0$ for some v, then $\|v\|^2 \leq \|d_p\phi(v)\|^2 = 0$ which implies $v = 0$. Since M, N have the same dimension, it follows that $d_p\phi$ is an isomorphism and, by the inverse function theorem, it is a local diffeomorphism. In particular, there are open sets $U \subset M$ around p, and $V \subset N$ around $\phi(p)$, such that $\phi^{-1} : V \to U$ makes sense. Then one can extend $\overline{\gamma}$ past p, by $\overline{\gamma}(t) := \phi^{-1}(\gamma(t))$.

To prove closedness, let $\{s_i\}_i \subset J$ be a sequence converging to $s_0 \in [0, 1]$. We claim that the set $S = \{\overline{\gamma}(s_i)\}$ is bounded. In fact, otherwise we would have

$d(\overline{\gamma}(0), \overline{\gamma}(s_i)) \to \infty$, and then

$$\begin{aligned}\ell(\gamma|_{[0,s_0]}) &= \int_0^{s_0} \|\gamma'(s)\| ds = \lim_{i\to\infty} \int_0^{s_i} \|\gamma'(s)\| ds \\ &= \lim_{i\to\infty} \int_0^{s_i} \|d_{\overline{\gamma}(s)}\phi \overline{\gamma}'(s)\| ds \\ &\geq \lim_{i\to\infty} \int_0^{s_i} \|\overline{\gamma}'(s)\| ds \\ &\geq \lim_{i\to\infty} d(\overline{\gamma}(0), \overline{\gamma}(s_i)) = \infty\end{aligned}$$

which contradicts the fact that γ has finite length. Since $S = \{\overline{\gamma}(s_i)\}$ is bounded, its closure is compact (because M is complete) and therefore $\gamma(s_i)$ has a subsequence converging to q. Defining $\overline{\gamma}(s_0) = q$, we have $\phi(\overline{\gamma}(s_0)) = \gamma(s_0)$ and thus, using a local diffeomorphism around q, we get that $\overline{\gamma}$ can be extended to (and past) s_0.

2) To prove that N is complete, let $\gamma : [0, \infty) \to N$ be a divergent curve. Then the lift $\overline{\gamma} : [0, \infty) \to M$ is a divergent curve as well: In fact, if $\overline{\gamma}$ could not escape some compact set K, then $\gamma(t)$ would not be able to escape the compact set $\phi(K)$. Since M is complete, from ex. 3 we have that for any sequence $t_i \to \infty$ we have $\ell(\overline{\gamma}|_{[0,t_i]} \to \infty$ and, since ϕ is expanding,

$$\ell(\gamma|_{[0,t_i]}) > \ell(\overline{\gamma}|_{[0,t_i]}) \to \infty$$

Again by ex. 3, N is then complete.