Jacobi fields, II

$\gamma : I \to M$ geodesic, $J(t)$ v.f. along γ is Jacobi field if either

1. $J(t)$ given by variation of γ through geodesics
2. $J'' + R_t J = 0$.

Also, if $J(0) = 0$ and $\gamma(t) = \exp_p tv$, then $J(t) = (d_{tv}\exp_p)\, tw$, $w = J'(0)$.

- $\mathcal{J}_\gamma = \{\text{Jacobi fields along } \gamma\} \xrightarrow{\ \cong\ } T_{\gamma(0)}M \oplus T_{\gamma(0)}M$, $J \mapsto (J(0), J'(0))$

<u>Example:</u>

- $J(t) = \gamma'(t)$. Then $J'(t) = 0$, $J''(t) = 0$, and $R_t J(t) = R(\gamma'(t), \gamma'(t))\gamma'(t) = 0$ ✓
  J given by $f(s,t) = \gamma(t+s)$ (shift)
- $J(t) = t\cdot\gamma'(t)$. Then $J'(t) = \gamma'(t)$, $J''(t) = 0$, $R_t J(t) = t \cdot R_t \gamma'(t) = 0$
  J given by $f(s,t) = \gamma(e^s \cdot t)$ (reparametriz.)

<u>Prop:</u> $f(t) = \langle J(t), \gamma'(t)\rangle = at + b$ $\quad a = \langle J'(0), \gamma'(0)\rangle$, $b = \langle J(0), \gamma'(0)\rangle$

<u>Pf:</u> $f'(t) = \langle J(t), \gamma'(t)\rangle' = \langle \frac{D}{dt}J(t), \gamma'(t)\rangle + \langle J(t), \frac{D}{dt}\gamma'(t)\rangle = \langle J'(t), \gamma'(t)\rangle$ (second term $= 0$)

$f''(t) = \langle J(t), \gamma'(t)\rangle'' = \langle J'(t), \gamma'(t)\rangle' = \langle J''(t), \gamma'(t)\rangle = -\langle R(J(t), \gamma'(t))\gamma'(t), \gamma'(t)\rangle = 0$

$\downarrow$

$\langle J(t), \gamma'(t)\rangle = f(0) + f'(0)\cdot t = \langle J(0), \gamma'(0)\rangle + t\langle J'(0), \gamma'(0)\rangle$

<u>Cor:</u> If $J(0), J'(0) \perp \gamma'(0) \Rightarrow J(t) \perp \gamma'(t)\ \forall t$

<u>Pf:</u> $\langle J(0), \gamma'(0)\rangle = \langle J'(0), \gamma'(0)\rangle = 0 \Rightarrow \langle J(t), \gamma'(t)\rangle \equiv 0$

<u>Cor 2:</u> $\mathcal{J}_\gamma = \{J \mid J \text{ Jac. field along } \gamma\}$. Then

$\mathcal{J}_\gamma = \mathcal{J}_\gamma^{\parallel} \oplus \mathcal{J}_\gamma^{\perp}$ $\quad \mathcal{J}_\gamma^{\parallel} = \operatorname{span}\{\gamma', t\cdot\gamma'\}$, $\mathcal{J}_\gamma^{\perp} = \{J \mid J(t) \perp \gamma'(t)\ \forall t\}$

<u>Pf:</u> Clearly $\mathcal{J}_\gamma^{\parallel} \cap \mathcal{J}_\gamma^{\perp} = 0$ since $J \in \mathcal{J}_\gamma^{\parallel}$ is parallel to γ', and $J \in \mathcal{J}_\gamma^{\perp}$ is per

$\forall J(t)$, $a = \langle J(0), \gamma'(0)\rangle$, $b = \langle J'(0), \gamma'(0)\rangle$, $\tilde{J}(t) = J(t) - a\gamma'(t) - b(t\cdot\gamma'(t))$

$\langle \tilde{J}(0), \gamma'(0)\rangle = \langle J(0), \gamma'(0)\rangle - a\cdot\langle\gamma'(0), \gamma'(0)\rangle = a - a = 0$

$\langle \tilde{J}'(0), \gamma'(0)\rangle = \langle J'(0) - b\gamma'(0), \gamma'(0)\rangle = \langle J'(0), \gamma'(0)\rangle - b\langle\gamma'(0), \gamma'(0)\rangle = b - b = 0$

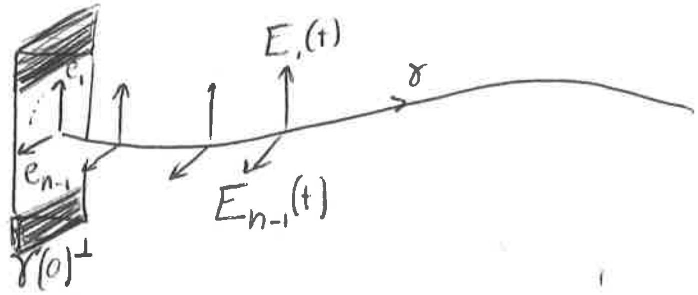
$\tilde{J}^b(t) \in \mathcal{J}_\gamma^\perp \Rightarrow J(t) = (a\gamma'(t) + b\,t\cdot\gamma'(t)) + \tilde{J}(t) \in \mathcal{J}_\gamma^{\parallel} \oplus \mathcal{J}_\gamma^\perp$

□

Ex: M mfd w sec $\equiv K$, γ normalized geodesic. $J(t)$ Jacobi field in $\mathcal{J}_\gamma^\perp$ w/ $J(0) = v$, $J'(0) = W$

take onb. e_i of $T_{\gamma(0)}M$, w/ $\gamma'(t) = e_n$, and take $E_i(t)$ parallel v.f. such that $E_i(0) = e_i$.

Then $E_1(t) \dots E_{n-1}(t)$ span $\gamma'(t)^\perp$ $\forall t$.



$J(t) = \sum f_i(t) E_i(t)$.

Then:

1. $J''(t) = \sum f_i''(t) \cdot E_i(t)$
2. since sec $\equiv K$, $R_t(J(t)) = R(J(t), \gamma'(t))\gamma'(t) = K(\langle \gamma'(t), \gamma'(t)\rangle \cdot J(t) - \underbrace{\langle J(t), \gamma'(t)\rangle}_{=0} \gamma'(t))$

$= K\cdot J(t) = K \cdot \sum f_i(t) E_i(t)$

$\Downarrow$

(JE) becomes $J''(t) = -K J(t)$

$\rightsquigarrow$ system of ODE's $f_i''(t) = -K\cdot f_i(t) \Rightarrow f_i(t) = \begin{cases} a\cdot\cos\sqrt{K}t + b\sin\sqrt{K}t & K>0 \\ a + bt & K=0 \\ a\cosh(\sqrt{-K}t) + b\sinh(\sqrt{-K}t) & K<0 \end{cases}$

If $J(0) = v$, $J'(0) = w$, and $V(t)$, $W(t)$ are par. ext. of v, w, then

$$J(t) = \begin{cases} \sin\sqrt{K}t \cdot V(t) + \cos\sqrt{K}t\, W(t) & K>0 \\ V(t) + t\cdot W(t) & K=0 \\ \cosh(\sqrt{-K}t)V(t) + \sinh(\sqrt{-K}t)W(t) & K<0 \end{cases}$$

~~Expan~~ Taylor expansion of $|J(t)|^2$, for $J \in \mathcal{J}_\gamma^\perp$, $J(0) = 0$

$f(t) = \langle J(t), J(t)\rangle$

$f'(t) = 2\langle J'(t), J(t)\rangle$

$$f''(t) = 2 < J''(t), J(t) > + 2 < J'(t), J'(t) >$$
$$= -2 < R(J,\gamma')\gamma', J > + 2 < J', J' >$$

$$R' = \left(\nabla_{\gamma'(t)} R\right)$$

$$f'''(t) = -2 < R'(J,\gamma')\gamma', J > - 2 < R(J',\gamma')\gamma', J >$$
$$-2 < R(J,\gamma')\gamma', J' > + 4 < J'', J' >$$
$$= -2 < R'(J,\gamma')\gamma', J > - 4 < R(J',\gamma')\gamma', J > - 4 < R(J,\gamma')\gamma', J' >$$
$$= -2 < R'(J,\gamma')\gamma', J > - 8 < R(J',\gamma')\gamma', J >$$

$$f''''(t) = |J(t)| \cdot \text{smth.} \; - 8 < R(J',\gamma')\gamma', J' >$$

$$f(0) = |J(0)|^2 = 0, \; f'(0) = 0, \; f''(0) = |J'(t)| = 2|w|^2, \; f'''(0) = 0, \; f''''(0) = -8 \, Rm(w\gamma'\gamma'$$

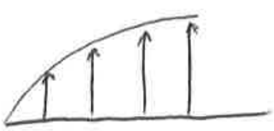
$\Rightarrow$ If $|w| = 1$, then

$$|J(t)|^2 = f(t) = \frac{4|w|^2}{2} t^2 - \frac{8}{4!} Rm(w\gamma'\gamma'w) \cdot t^4 + o(t^4)$$
$$= t^2 - \frac{1}{3} \sec(w,\gamma') \cdot t^4 + o(t^4)$$

$\Downarrow$

$$|J(t)| = t - \frac{1}{6} \sec(w,\gamma') \cdot t^2 + O(t^2)$$

K>0



K=0

K<0