CARTAN - HADAMARD'S Theorem

From last time: Taylor expansion of $f(t) = |J(t)|^2$, where $J(0) = 0$, $J'(0) =$

$$f'(0) = 0, \quad f''(0) = 2|w|^2, \quad f'''(0) = 0$$

$$f^{iv}(0) = -8 \langle R(w, \gamma'(0))\gamma'(0), w \rangle$$

If γ is normalized, and $J(t) \perp \gamma'(t)$, Then

$$\langle R(w,\gamma'(0))\gamma'(0), w\rangle = sec(w, \gamma'(0)) \cdot |w \wedge \gamma'(0)|$$

$$= sec(w, \gamma'(0)) \cdot \left(|w|^2 \cdot \underbrace{|\gamma'(0)|^2}_{=1} - \cancel{\langle \gamma'(0), w\rangle^2}^{\,0} \right)$$

$$= sec(w, \gamma'(0)) \cdot |w|^2$$

$\Downarrow$

$$f^{iv}(0) = -8|w|^2 \cdot sec(w, \gamma'(0)),$$ and

$$f(t) = \cancel{f(0)} + \cancel{f'(0)}\, t + \frac{1}{2} f''(0)\, t^2 + \cancel{\frac{1}{6} f'''(0) \cdot t^3} + \frac{1}{24} f^{iv}(0)\, t^4 + o(t^4)$$

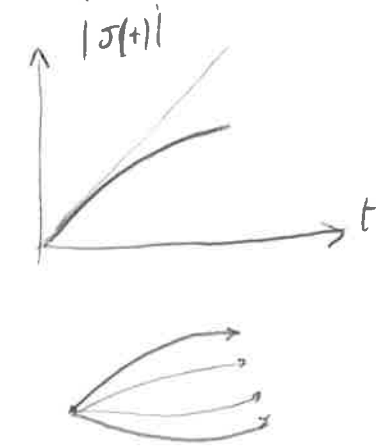
$$= |w|^2 t^2 - \frac{1}{3}|w|^2 sec(w, \gamma'(0))\, t^4 + o(t^4)$$

$$= |w|^2 t^2 \left(1 - \frac{1}{3} sec(\pi)\, t^2 + o(t^2)\right)$$

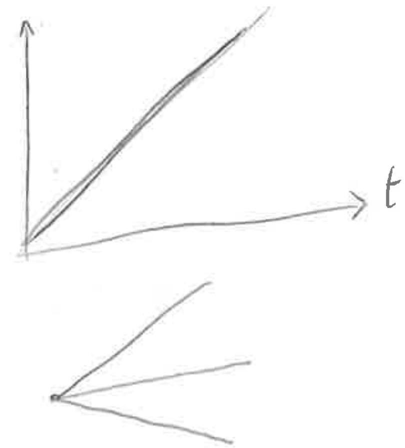
$$|J(t)| = \sqrt{f(t)} = |w|\, t \cdot \left(1 - \frac{1}{6} sec(\pi)\, t^2 + o(t^2)\right)$$

$$= |w|\, t - \frac{1}{6}|w| \cdot sec(\pi)\, t^3 + o(t^3)$$

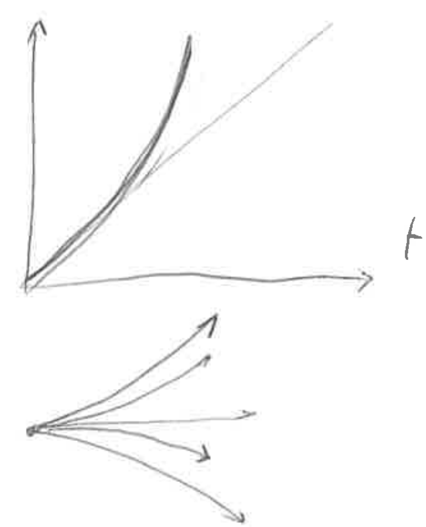
$sec(\pi) > 0$



$sec(\pi) = 0$

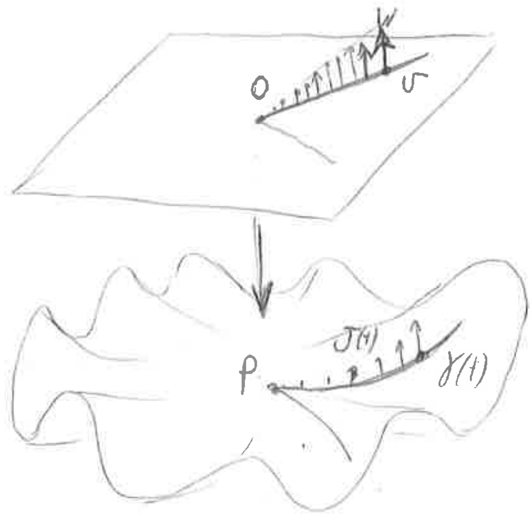


$sec(\pi) < 0$



<u>Thm</u> (Cartan-Hadamard)

(M,g) complete mfd w/ $\sec \leq 0$. Then the universal cover $\tilde{M}$ is diff to $\mathbb{R}^n$



$$\begin{array}{c} T_pM \\ \big\downarrow \exp_p \\ M \end{array}$$

<u>Pf</u>: $p \in M$

$\exp_p : T_pM \longrightarrow M$

<u>Step 1</u>: $\exp_p$ local diffeo:

Take $v \in T_pM$, $w \in T_v(T_pM) \cong T_pM$

Want to prove $(d_v \exp_p) w \neq 0$.

Define $\gamma(t) = \exp_p tv$, $J(t) = (d_{tv} \exp_p)\, tw$

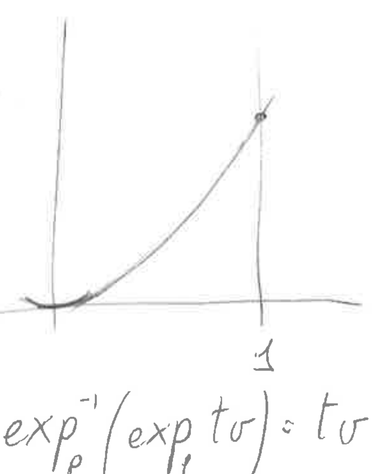
$\Rightarrow$ $J(t)$ is a J field w/ $J(0) = 0$, $J'(0) = w$, $J(1) = (d_v \exp_p) w$

$$(|J|^2)' = 2\langle J, J' \rangle \qquad (|J|^2)'' = 2\langle J', J' \rangle + 2\langle J, J'' \rangle$$
$$= 2\langle J', J' \rangle - 2\langle R(J,\gamma')\gamma', J \rangle$$
$$= 2|J'|^2 - 2\sec(J(t), \gamma'(t)) \cdot |J \wedge \gamma'|^2$$
$$\geq 2|J'|^2 \geq 0 \quad (>0 \text{ at } t=0)$$

$\rightarrow$ $f(t) = |J(t)|^2$ is convex, and strongly convex around $t=0$, and

$f(0) = f'(0) = 0 \Rightarrow f(1) > 0$

$\Rightarrow |J(1)|^2 = |d_v \exp_p w|^2 \neq 0$



<u>Step 2</u>: define the metric $(T_pM, \overbrace{\exp_p^* g}^{\bar{g}})$, so that $\exp_p : (T_pM, \bar{g}) \longrightarrow (M,g)$ is a (local) isometry,

$\Rightarrow \bar{g}$ geodesics of T_pM from 0 = $\exp_p^{-1}$(geodesic in M from p) = $\exp_p^{-1}(\exp_p tv) = tv$

$\Rightarrow (T_pM, \bar{g})$ is complete, and

$\exp_p : T_pM \longrightarrow M$ satisfies

$$\left|(d_v \exp_p)\,w\right|^2_{g} = |w|^2_{\bar{g}}$$

<u>$\exp_p$ expanding map</u>

$\Rightarrow \exp_p$ is a covering space + T_pM simply connected

$\Rightarrow T_pM \underset{\text{diffeo}}{\cong} \mathbb{R}^n$ is the universal cover of M. ∎

<u>Cor:</u> If M has $\sec \leq 0$, then

1. $\pi_i(M) = 0 \quad i \geq 2$
2. $\pi_1(M)$ ~~is~~ has elements of infinite order
3. If $\pi_1(M) = 0$, $M \underset{\text{diffeo}}{\cong} \mathbb{R}^n$

<u>Pf:</u>

1. Follows from facts that $\pi_i(\mathbb{R}^n) = 0$ $(\mathbb{R}^n \simeq p)$, and for any coverings, $\tilde{M} \to M$, $\pi_i(\tilde{M}) \cong \pi_i(M) \quad \forall\, i \geq 2$
2. If $\gamma \in \pi_1(M)$ has finite order, $\langle\gamma\rangle = \Gamma \leq \pi_1 M$, and $\Gamma \curvearrowright \tilde{M} = \mathbb{R}^n$ acts freely, contradiction (finite cyclic groups cannot act freely on $\mathbb{R}^n$)
3. Clear.

<u>Cor:</u> S^n cannot admit metrics of $\sec \leq 0$