HOMEWORK #8 - DUE APR 6, AT NOON

Exercise 1. Let M be a Riemannian manifold, and R its Riemann curvature tensor. Let ∇R be the $(4,1)$ tensor

$$\nabla R(X,Y,Z,W) = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W.$$

(This tensor is usually referred to, as $(\nabla_X R)(Y,Z)W$). Prove that ∇R satisfies the same symmetries of R, namely

- Skew symmetry: $(\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,Y)W = 0$.
- Bianchi identity: $(\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,W)Y + (\nabla_X R)(W,Y)Z = 0$.

Solution 1. 1) Start with $R(Y,Z)W + R(Z,Y)W = 0$. Taking covariant derivative with respect to X one obtains

$$\begin{aligned}
0 =& \nabla_X(R(Y,Z)W) + \nabla_X(R(Z,Y)W) \\
=& (\nabla_X R)(Y,Z)W + R(\nabla_X Y, Z)W + R(Y,\nabla_X Z)W + R(Y,Z)\nabla_X W \\
&+ (\nabla_X R)(Z,Y)W + R(\nabla_X Z, Y)W + R(Z,\nabla_X Y)W + R(Z,Y)\nabla_X W \\
=& \big((\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,Y)W\big) + \big(R(\nabla_X Y, Z)W + R(Z,\nabla_X Y)W\big) \\
&+ \big(R(Y,\nabla_X Z)W + R(\nabla_X Z, Y)W\big) + \big(R(Y,Z)\nabla_X W + R(Z,Y)\nabla_X W\big) \\
=& (\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,Y)W
\end{aligned}$$

where the last equality follows from the skew-symmetry of R.

2) Start with the Bianchi identity $R(Y,Z)W + R(Z,W)Y + R(W,Y)Z = 0$. Taking covariant derivative with respect to X one obtains

$$\begin{aligned}
0 =& \nabla_X(R(Y,Z)W + R(Z,W)Y + R(W,Y)Z) \\
=& (\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,W)Y + (\nabla_X R)(W,Y)Z \\
&+ R(\nabla_X Y, Z)W + R(\nabla_X Z, W)Y + R(\nabla_X W, Y)Z \\
&+ R(Y,\nabla_X Z)W + R(Z,\nabla_X W)Y + R(W,\nabla_X Y)Z \\
&+ R(Y,Z)\nabla_X W + R(Z,W)\nabla_X Y + R(W,Y)\nabla_X Z.
\end{aligned}$$

Let us denote each term in the long sum above with coordinates, where (i,j) denotes the i-th term in the j-th row. By the Bianchi identity, we get $(2,1)+(3,3)+(4,2) = 0$, $(2,2)+(3,1)+(4,3) = 0$ and $(2,3)+(3,2)+(4,1) = 0$, so that the log sum above turns into

$$(\nabla_X R)(Y,Z)W + (\nabla_X R)(Z,W)Y + (\nabla_X R)(W,Y)Z = 0.$$

Exercise 2 (Taylor expansion of the metric in normal coordinates)**.** Let M be a Riemannian manifold, $p \in M$, and $\exp : B_\epsilon(0) \subset T_pM \to B_\epsilon(p) \subset M$ a normal neighbourhood. Let $e_1, \dots e_n$ be an orthonormal basis of T_pM, and consider the parametrization

$$\phi : B_\epsilon(0) \subset \mathbb{R}^n \to B_\epsilon(p) \subset M, \qquad \phi(x^1,\dots,x^n) = \exp_p(x^i e_i)$$

with corresponding chart $\mathbf{x} = \phi^{-1}$.

(1) Check the following:
- $\frac{\partial}{\partial x^i} = \exp_p *(e_i)$.
- For every $(x^1, \dots x^n) \in B_\epsilon(0)$, the curve $\gamma(t) := \phi(tx^1, \dots tx^n)$ is a geodesic.
- Along the geodesic γ in the previous point, the Jacobi field $J_i(t)$ with $J_i(0) = 0$, $J_i'(0) = e_i$ satisfies $J_i(t) = t\frac{\partial}{\partial x^i}|_{\gamma(t)}$, and any two such Jacobi fields satisfy $\langle J_i(t), J_i(t)\rangle = t^2 g_{ij}(tx^1, \dots tx^n)$.

(2) Prove that for i, j fixed, the Taylor expansion for $f(t) = \langle J_i(t), J_j(t)\rangle$ around $t = 0$ is

$$f(t) = \delta_{ij}t^2 - \frac{1}{3}R_{iklj}x^k x^l t^4 + o(t^4)$$

(3) Prove that for i, j fixed, the Taylor expansion for $h(t) = t^2 g_{ij}(tx^1, \dots tx^n)$ around $t = 0$ is

$$h(t) = g_{ij}(0)t^2 + g_{ij;k}(0)x^k t^3 + \frac{1}{2}g_{ij;kl}(0)x^k x^l t^4 + o(t^4)$$

(4) Conclude that $g_{ij}(0) = \delta_{ij}$ (we knew this already) $g_{ij;k}(0) = 0$ (we sort-of knew this already), $g_{ij;kl}(0) = -\frac{2}{3}R_{iklj}$ (this is new) and that the Taylor expansion for the metric $g_{ij}(x_1, \dots x_k)$ is

$$g_{ij}(x_1, \dots x_k) = \delta_{ij} - \frac{1}{3}R_{iklj}x^i x^j + o(|x|^2)$$

Solution 2. 1) Write ϕ as a composition $\phi = \exp_p \circ \psi$, where $\psi : \mathbb{R}^n \to T_pM$ is the map $\psi(x^1, \dots x^n) = x^i e_i$. Let $E_1, \dots E_n \in \mathbb{R}^n$ the standard vectors in $\mathbb{R}^n$, so that $\psi_* E_i = e_i$. We compute:

- Recall that the vector field $\frac{\partial}{\partial x^i}$ is defined for every $f \in C^\infty(B_\epsilon(p))$ by

$$\frac{\partial}{\partial x^i}(f) := E_i(f \circ \mathbf{x}^{-1}) = E_i(f \circ \phi) = (\phi_* E_i)(f)$$

  thus $\frac{\partial}{\partial x^i} = \phi_* E_i = (\exp_p)_*(\psi_* E_i) = (\exp_p)_* e_i$.
- $\gamma(t) = \phi(tx^1, \dots tx^n) = \exp_p\left(tx^i e_i\right)$ thus it is a geodesic.
- Recall that, from class, we know

$$\begin{aligned} J_i(t) =& (d_{tx^i e_i}\exp_p)(te_i) = (\exp_p)_*(\phi_*(tE_i)) \\ =& t(d_{(tx^1,\dots tx^n)}\phi)(E_i) = t\frac{\partial}{\partial x^i}\Big|_{\phi(tx^1,\dots tx^n)}. \end{aligned}$$

  In particular $\langle J_i, J_j\rangle = \langle t\frac{\partial}{\partial x^i}\Big|_{\phi(tx^1,\dots tx^n)}, t\frac{\partial}{\partial x^j}\Big|_{\phi(tx^1,\dots tx^n)}\rangle = t^2 g_{ij}(tx^1, \dots tx^n)$.

2) Given $f(t) = \langle J_i, J_j\rangle$, we get

$$\begin{aligned} f'(t) =& \langle J_i', J_j\rangle + \langle J_i, J_j'\rangle \\ f''(t) =& \langle J_i'', J_j\rangle + 2\langle J_i', J_j'\rangle + \langle J_i, J_j''\rangle \\ =& -2\langle R(J_i, \gamma')\gamma', J_j\rangle + 2\langle J_i', J_j'\rangle \\ f'''(t) =& -2\langle R'(J_i, \gamma')\gamma', J_j\rangle - 2\langle R(J_i', \gamma')\gamma', J_j\rangle \\ & - 2\langle R(J_i, \gamma')\gamma', J_j'\rangle + 2\langle J_i'', J_j'\rangle + 2\langle J_i', J_j''\rangle \\ =& -2\langle R'(J_i, \gamma')\gamma', J_j\rangle - 4\langle R(J_i', \gamma')\gamma', J_j\rangle - 4\langle R(J_i, \gamma')\gamma', J_j'\rangle \\ f''''(t) =& -8\langle R(J_i', \gamma')\gamma', J_j'\rangle + \text{ stuff depending linearly on } J_i, J_j \end{aligned}$$

Evaluatig at $t = 0$ we get $f(0) = f'(0) = f'''(0) = 0$, and

$$f''(0) = 2\langle e_i, e_j\rangle = 2\delta_{ij}, \quad f''''(0) = -8\langle R(e_i, \gamma')\gamma', e_j\rangle$$

since $\gamma'(0) = x^k e_k$, we get

$$f''''(0) = -8x^k x^l \langle R(e_i, e_k)e_l, e_j\rangle = -8R_{iklj}x^k x^l.$$

The Taylor series for f is then:

$$\begin{aligned} f(t) &= f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \frac{1}{6}f'''(0)t^3 + \frac{1}{24}f''''(0)t^4 + o(t^4) \\ &= \delta_{ij}t^2 - \frac{1}{3}R_{iklj}x^k x^l t^4 + o(t^4) \end{aligned}$$

3) Since $(x^1, \dots x^n)$ is fixed, let us write $g_{ij}(tx^1, \dots tx^n)$ simply as g_{ij}, and notice that $g'_{ij} = g_{ij;k}x^k$, where $g_{ij;k}$ denotes $\frac{\partial}{\partial x^k} g_{ij}$, evaluated at $(tx^1, \dots tx^n)$. Notice that $g'_{ij;k} = g_{ij;kl}x^l$, and so on.

We can now compute the derivatives of $h(t) = t^2 g_{ij}$:

$$\begin{aligned} h'(t) =& 2tg_{ij} + t^2 g_{ij;k}x^k \\ h''(t) =& 2g_{ij} + 4tg_{ij;k}x^k + t^2 g_{ij;kl}x^k x^l \\ h'''(t) =& 6g_{ij;k}x^k + 6tg_{ij;kl}x^k x^l + t^2(\text{something}) \\ h''''(t) =& 12g_{ij;kl}x^k x^l + t(\text{something}) \end{aligned}$$

At $t = 0$, we then have $h(0) = h'(0) = 0$, $h''(0) = 2g_{ij}(0)$, $h'''(0) = 6g_{ij;k}(0)x^k$, and $h''''(0) = 12g_{ij;kl}(0)x^k x^l$. The Taylor series for h is then:

$$\begin{aligned} h(t) &= h(0) + h'(0)t + \frac{1}{2}h''(0)t^2 + \frac{1}{6}h'''(0)t^3 + \frac{1}{24}h''''(0)t^4 + o(t^4) \\ &= g_{ij}(0)t^2 + g_{ij;k}(0)x^k t^3 + \frac{1}{2}g_{ij;kl}(0)x^k x^l t^4 + o(t^4) \end{aligned}$$

4) Comparing the Taylor series for $f(t) = h(t)$, we get $g_{ij}(0) = \delta_{ij}$, $g_{ij;k}(0)x^k = 0$ (thus $g_{ij;k}(0)$ for all i, j, k since the coordinate functions x^k are independent) and $\frac{1}{2}g_{ij;kl}(0)x^k x^l = -\frac{1}{3}R_{iklj}x^k x^l$ (and thus $g_{ij;kl}(0) = -\frac{2}{3}R_{iklj}$ for all i, j, k, l). Finally, since the Taylor expansion og the metric g_{ij} around p is given by

$$g_{ij} = g_{ij}(0) + g_{ij;k}(0)x^k + \frac{1}{2}g_{ij;kl}(0)x^k x^l + o(|x|^3)$$

we get the desired series.

Exercise 3 (Killing fields)**.** Let M be a Riemannian manifold. Given a vector field X, assume for simplicity that the flow of X, Φ_X^t, is defined in the whole of M, $\Phi_X^t : M \to M$. We say that X is a *Killing vector field* if Φ_X^t is an isometry for every t.

(1) Prove that if X is a Killing field in $\mathbb{R}^n$ with $X(0) = 0$, then there exists a skew symmetric n-by-n matrix A such that $X(v) = A \cdot v$ for every $v \in \mathbb{R}^n$. (**Hint:** Prove first that the map $t \mapsto \Phi_X^t$ defines a group homomorphism $\mathbb{R} \to \mathsf{O}(n)$. Show that for every v, $\langle \Phi_X^t(v), \Phi_X^t(v)\rangle$ is constant in t, thus its t-derivative is 0).

(2) Back to a general manifold M. Let X be a Killing field with $X(p) = 0$ for some $p \in M$. Prove that for every $q \in B_\epsilon(p)$, $\Phi^t_X(q)$ is contained in a geodesic sphere around p (i.e. the image of a round sphere, under the normal exponential map).
(3) Given a Killing field X in M and γ a geodesic in M, $J(t) = X(\gamma(t))$ is a Jacobi field along γ.
(4) Use the previous result to prove that if M is connected, and X is a Killing field such that $X(p) = 0$ and $\nabla_y X(p) = 0$ for all $y \in T_pM$, then $X \equiv 0$.

Solution 3. 1) Notice that since $X(0) = 0$, the flow Φ^t_X fixes 0: in fact, the curve $\gamma(t) = 0$ satisfies $\gamma'(t) = 0 = X_0 = X_{\gamma(t)}$, thus it is the integral curve for X starting at 0 and $\Phi^t_X(0) = \gamma(t) = 0$ for all t.

Since X is a Killing field, for every t the map $\Phi^t_X : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ which fixes 0. These are precisely the elements in $\mathsf{O}(n)$, thus Φ_X defines a map $\mathbb{R} \to \mathsf{O}(n)$. Since $\Phi^0_X = Id$ and $\Phi^{s+t}_X = \Phi^s_X \cdot \Phi^t_X$, the map Φ_X is a group homomorphism, and thus we can write $\Phi^t_X = e^{tA}$ for some matrix A.

Since $\Phi^t_X \in \mathsf{O}(n)$ we have that for every $x \in \mathbb{R}^n$, $\langle \Phi^t_X v, \Phi^t_X v\rangle = \langle v, v\rangle$. Taking the derivative, we get

$$0 = \frac{d}{dt}\Big|_{t=0} \langle \Phi^t_X v, \Phi^t_X v\rangle = \frac{d}{dt}\Big|_{t=0} \langle e^{tA} v, e^{tA} v\rangle = 2\langle Av, v\rangle$$

the fact that $\langle Av, v\rangle = 0$ for all x implies that A is skew symmetric. Finally, for every $v \in \mathbb{R}^n$,

$$X(v) = \frac{d}{dt}\Big|_{t=0} \Phi^t_X(v) = \frac{d}{dt}\Big|_{t=0} e^{tA}(v) = Av.$$

2) As before, if $X(p) = 0$, then $\Phi^t_X(p) \equiv p$. For every $q \in B_\epsilon(p)$, let $r = d(p, q)$ and let $S_r = \{q' \mid d(q', p) = r\}$ the geodesic sphere containing q. Then for every t, Φ^t_X is an isometry, in particular it is a distance-preserving map, and thus

$$d(\Phi^t_X(q), p) = d(\Phi^t_X(q), \Phi^t_X(p)) = d(q, p) = r \quad \Rightarrow \quad \Phi^t_X(q) \in S_r.$$

3) Given a geodesic γ, take the variation $f(s, t) = \Phi^s_X(\gamma(t))$. Since Φ^s_X is an isometry for every s, we have $\gamma_s(t) := f(s, t) = \Phi^s_X \circ \gamma(t)$ is a geodesic, with $\gamma_0 = \gamma$. Thus $f(s, t)$ is a variation of γ though geodesics, and

$$X(\gamma(t)) = \frac{d}{ds}\Big|_{t=0} \Phi^s_X(\gamma(t)) = \frac{\partial f}{\partial s}\Big|_{s=0}.$$

Therefore $X(\gamma(t))$ is a Jacobi field.

4) Let S be the subset of points q in M such that $X(q) = 0$ and $(\nabla X)(q) = 0$. Clearly the set is closed, and nonempty since $p \in S$. We claim that it is open as well. To prove this, fix a point $q \in S$ and take $B_\epsilon(q)$ a normal neighbourhood of q. For every $q' \in B_\epsilon(q)$, let γ be a geodesic from q to q', and let $J(t)$ be the Jacobi field $X(\gamma(t))$. Then $J(0) = X(q) = 0$, and $J'(0) = (\nabla_{\gamma'(0)} X)(q) = 0$, which means $J(t) \equiv 0$ and $X \equiv 0$ along γ. Doing so for all points in $B_\epsilon(q)$, it follows that $X \equiv 0$ on $B_\epsilon(q)$, and therefore ∇X vanishes on $B_\epsilon(q)$ as well. Thus $B_\epsilon(q) \subset S$ and S is open. Since S is nonempty, open and closed, and M is connected, we obtain that $S = M$ hence the result.

Exercise 4 (Killing fields in surfaces of revolution)**.** Recall that a surface of revolution S is the image of a parametrized surface $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

where $f > 0$ and $f'(u)^2 + g'(u)^2 \neq 0$. Prove that $\frac{\partial \phi}{\partial v}$ is a Killing field on S. (**Hint:** observe that the map $\Psi^t : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\Psi^t(x, y, z) = (\cos(t)x - \sin(t)y, \sin(t)x + \cos(t)y, z)$$

is a one-parameter group of isometries of $\mathbb{R}^3$ which preserves S, thus a one-parameter group of isometries of S).

Solution 4. As in the hint, the map $\Psi^t(x, y, z) = (\cos(t)x - \sin(t)y, \sin(t)x + \cos(t)y, z)$ is a 1-parameter group of linear maps, with associated matrix

$$\Psi^t = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow (\Psi^t)(\Psi^t)^T = I_3$$

Thus $\Psi^t \in \mathsf{O}(n)$ for all t. Moreover,

$$\begin{aligned} \Psi^t(\varphi(u, v)) =& \Psi^t(f(u)\cos(v), f(u)\sin(v), g(u)) \\ =& (f(u)\cos(v)\cos(t) - f(u)\sin(v)\sin(t), f(u)\cos(v)\sin(t) + f(u)\sin(v)\cos(t), g(u)) \\ =& (f(u)\cos(v + t), f(u)\sin(v + t), g(u)) \\ =& \varphi(u, v + t) \end{aligned}$$

It follows that Ψ^t preserves S, and its restriction to S defines a 1-parameter group of isometries. Notice the general fact the flow Φ^t of $\frac{\partial \varphi}{\partial v}$ is given by $\Phi^t(\varphi(u, v)) = \varphi(u, v + t)$: in fact, for every $(u, v) \in U$, we have

$$\frac{d}{dt}\Big|_{t=0} \Phi^t(\varphi(u, v)) = \frac{d}{dt}\Big|_{t=0} \varphi(u, v + t) = \frac{\partial \varphi}{\partial v}.$$

From the computation above it shows that $\Psi^t(\varphi(u, v)) = \varphi(u, v + t) = \Phi^t(\varphi(u, v))$, and therefore the flow of $\frac{\partial \varphi}{\partial v}$ is by isometries.