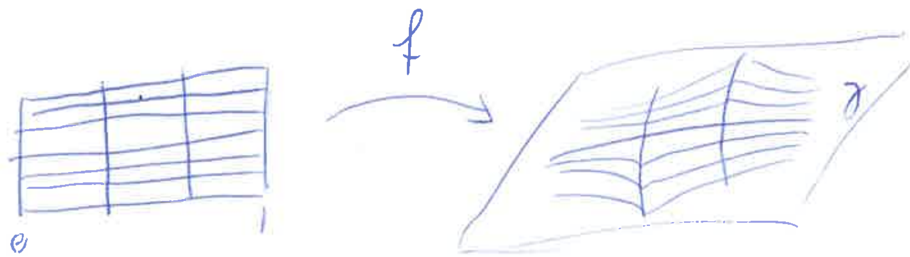
# FIRST VARIATION OF ENERGY

$\gamma : [0,1] \to M$ curve

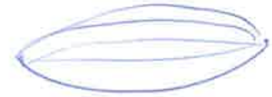
$\rightsquigarrow \gamma \in \Omega = \{$pw smooth curves in $M\}$ , $\Omega_{pq} = \{$pw smooth curves $p \rightsquigarrow q\}$

<u>Def</u>: <u>variation of $\gamma$</u> = parametrized surface $f : (-\varepsilon, \varepsilon) \times [0,1] \to M$ s.t.

- $\gamma_0(t) := f(0,t) = \gamma(t)$
- ~~$\forall s$~~ $\exists$ partition $0 = t_0 < t_1 < \dots t_N = 1$ s.t. $f|_{(-\varepsilon,\varepsilon)\times[t_{i-1},t_i]}$ smooth




<u>Def</u>: a variation is PROPER if $f(s,0) \equiv \gamma(0)$ $f(s,1) \equiv \gamma(1)$



<u>Remk</u>: $\gamma_s(t)$ variation $\rightsquigarrow$ curve $(-\varepsilon,\varepsilon) \to \Omega$
$s \mapsto \gamma_s$

$\gamma_s(t)$ proper variation $\rightsquigarrow$ curve $(-\varepsilon,\varepsilon) \longrightarrow \Omega_{\gamma(0),\gamma(1)}$

<u>Def</u>: the VARIATIONAL v.f. of $f$ is $V(t) := \left.\frac{\partial f}{\partial s}\right|_{s=0}$

<u>Prop</u>: $V(t)$ piecewise v.f. along $\gamma \Rightarrow \exists$ variation $f$ with variational v.f. $V(t)$. Moreover $f$ can be chosen to be proper, iff $V(0) = V(1) = 0$

<u>Pf</u>: define $f(s,t) = \exp_{\gamma(t)} s \cdot V(t)$. Choose $\varepsilon$ s.t. $\exp_{\gamma(t)} sV(t)$ is defined for $|s| < \varepsilon$ $\blacksquare$

$\{V(t)$ pw smooth v.f. along $\gamma\} \cong T_\gamma \Omega$

$\{V(t)$ pw smooth v.f. along $\gamma \mid V(0) = V(1) = 0\} \cong T_\gamma \Omega_{\gamma(0)\gamma(1)}$

Want to study $\ell : \Omega \longrightarrow \mathbb{R}$
$\gamma \longmapsto \ell(\gamma)$
length: Turns out, it is not the best functional.

Def: Energy $E : \Omega \to \mathbb{R}$

$$E(\gamma) = \int_0^1 |\gamma'(t)|^2 dt.$$

$E(\gamma)$ DEPENDS ON PARAMETRIZATION OF $\gamma$, unlike $\ell$.

Lemma: Let $\gamma : [0,1] \to M$ minimizing geod. $p \rightsquigarrow q$. Then for any other $c : [0,1] \to M$ $p \rightsquigarrow q$, $E(\gamma) \leq E(c)$, with "=" iff $c$ is also a minimizing geodesic.

Proof: $\ell(c) = \int_0^1 |c'(t)| \cdot 1 \, dt$. By C-S inequality,

$$\ell(c)^2 = \underbrace{\left(\int_0^1 |c'(t)| \cdot 1 \, dt\right)^2}_{(|c'(t)|,\, 1)^2_{L^2([0,1])}} \leq \underbrace{\int_0^1 |c'(t)|^2 dt}_{\| |c'(t)| \|^2_{L^2([0,1])}} \cdot \underbrace{\int_0^1 1 \, dt}_{\|1\|^2_{L^2[0,1]}} = E(c)$$

with "=" iff $|c'(t)| = \lambda \cdot 1 = \lambda$ iff $c(t)$ parametr. by arc l.

$\Rightarrow E(\gamma) = \ell(\gamma)^2 \leq \ell(c)^2 \leq E(c)$, and "=" iff

- $\ell(\gamma) = \ell(c) \Rightarrow c$ is a min. geod up to reparametriz.
- $\ell(c)^2 = E(c) \Rightarrow c$ parametrized proport. to arclength

$\Big] \to c$ geod.

Thm (First variation of energy): $\gamma : [0,1] \to M$ curve, $f : (-\varepsilon, \varepsilon) \times [0,1] \to M$ variation, $V(t) = \frac{\partial f}{\partial s}\Big|_{s=0}$ variation. Let $E(s) := E(\gamma_s)$ $\quad \gamma_s(t) = f(s,t)$. Then

$$\frac{1}{2} E'(0) = -\int_0^1 \left\langle V(t), \frac{D}{dt}\gamma'(t) \right\rangle dt - \sum_i \langle V(t_i), \gamma'(t_i^+) - \gamma'(t_i^-) \rangle + \langle V(1), \gamma'(1) \rangle - \langle V(0), \gamma'(0) \rangle$$

Pf: $E(s) = \int_0^1 \langle \gamma_s'(t), \gamma_s'(t) \rangle dt = \int_0^1 \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt$

$\Downarrow$

$$\frac{1}{2}\frac{d}{ds}E(s) = \frac{1}{2}\int_0^1 \frac{d}{ds}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle dt = \int_0^1 \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle dt$$

$$= \int_0^1 \left\langle \frac{D}{dt}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt =$$

$$= \int_0^1 \frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt - \int_0^1 \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt$$

$$= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle dt - \int_0^1 \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t}\right\rangle dt$$

$$\frac{1}{2}\frac{dE}{dV}\cancel{(s) = \left\langle \frac{\partial f}{\partial s}\right|_{s=0}} \quad \frac{1}{2}\frac{d}{ds}\Big|_{s=0} E(s) = \sum_i \int_{t_{i-1}}^{t_i} \frac{d}{dt}\langle V, \gamma'\rangle dt - \int_0^1 \left\langle V, \frac{D}{dt}\gamma'\right\rangle dt$$

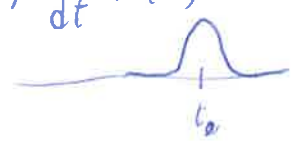
$$= \sum_i \langle V(t_i^-), \gamma'(t_i^-)\rangle - \langle V(t_{i-1}^+), \gamma'(t_{i-1}^+)\rangle - \int_0^1 \left\langle V, \frac{D}{dt}\gamma'\right\rangle dt$$

$$= \langle V(1), \gamma'(1)\rangle - \langle V(t_{N-1}), \gamma'(t_{N-1}^+)\rangle + \langle V(t_{N-1}), \gamma'(t_{N-1}^-)\rangle \cdots$$
$$+ \langle V(t_1), \gamma'(t_1^-)\rangle - \langle V(0), \gamma'(0)\rangle - \int_0^1 \left\langle V, \frac{D}{dt}\gamma'\right\rangle dt$$

$\boxtimes$

COR: if $\gamma$ satisfies $\frac{1}{2}\frac{dE}{dV} = 0 \ \forall V$, then $\gamma$ is a geodesic:

Pf: $\cdot$ if $\frac{D}{dt}\gamma'(t_0) \neq 0$, choose $\cancel{V(t_0) = \frac{D}{dt}\gamma'(t_0)}$, $V(t) = \varphi(t)\cdot\frac{D}{dt}\gamma'(t)$

$\varphi(t)$ 


$\Downarrow$

$\frac{1}{2}\frac{dE}{dV} = \int \varphi(t)\cdot\left|\frac{D}{dt}\gamma'(t)\right|^2 > 0$ $\ast$ $\Rightarrow$ $\gamma'$ is pw geodesic.

If $\gamma'(t_i^+) \neq \gamma'(t_i^-)$ choose $V$ st. $V(t_i) = \gamma'(t_i^+) - \gamma'(t_i^-)$, times a bump function

$$\Rightarrow \frac{1}{2}\frac{dE}{dV} = 0 + \langle V(t_i), \gamma'(t_i^+) - \gamma'(t_i^-)\rangle \neq 0 \quad \ast$$

$\boxtimes$

"geodesics are critical points for the energy functional.