SECOND VARIATION OF ENERGY

Recall: $\gamma : [0,1] \to M$ curve $\quad \gamma(0)=p \quad \gamma(1)=q \rightsquigarrow \gamma \in \Omega,\ \Omega_{p,q}$

(proper) variation $\gamma_s(t) = f(s,t)$ of $\gamma \rightsquigarrow$ path in $\Omega \left(/ \Omega_{p,q} \right)$

$V(t)$ v.f. along γ $\left(V(0) = V(1) = 0 \right)$ $\rightsquigarrow$ Vector $V \in T_\gamma \Omega \left(V \in T_\gamma \Omega_{p,q} \right.$

Energy: $E : \Omega \to \mathbb{R} \qquad E(\gamma) = \int_0^1 \|\gamma'\|^2 \, dt$

First variation: γ curve, f variation w/ $V = \left.\frac{\partial f}{\partial s}\right|_{s=0}$ Then

$$\frac{1}{2} \left.\frac{d\,E(\gamma_s)}{ds}\right|_{s=0} = -\int_0^1 \langle V, \tfrac{D}{dt}\gamma' \rangle \, dt + \sum_i \langle V(t_i), \gamma'(t_i^-) - \gamma'(t_i^+) \rangle + \langle V(t), \gamma'(t) \rangle \Big|_0^1$$

Crit pts for E on Ω_{pq} = geodesics $p \rightsquigarrow q$.

Today: γ geodesic, f variation $V = \left.\frac{\partial f}{\partial s}\right|_{s=0} \quad W = \left.\frac{D}{ds}\right|_{s=0} \frac{\partial f}{\partial s}$ Then

$$\frac{1}{2} \left.\frac{d^2 E(\gamma_s)}{ds^2}\right|_{s=0} = -\int_0^1 \langle V, V'' + R(V,\gamma')\gamma' \rangle \, dt + \sum_i \langle V_i, \left.\tfrac{D}{dt}\right|_{t_i^-} V_i - \left.\tfrac{D}{dt}\right|_{t_i^+} V \rangle + \langle V(t), \tfrac{D}{dt} V \rangle \Big|_0^1 + \langle W(t), \gamma'(t) \rangle \Big|_0^1$$

Pf: RECALL:

$$\frac{1}{2}\frac{dE}{ds} = -\int_0^1 \langle \frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle \, dt + \sum_i \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \Big|_{t_i}^{t_{i+1}}$$

$\downarrow \frac{d}{ds}$

$$\frac{1}{2}\frac{d^2E}{ds^2} = -\int_0^1 \langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} \rangle \, dt + \int_0^1 \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{D}{dt}\frac{\partial f}{\partial t} \rangle \, dt + \sum \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle \Big|_{t_i}^{t_{i+1}} + \sum_i \langle \frac{\partial f}{\partial s}, \frac{D}{ds}\frac{\partial f}{\partial t}$$

4 : $\frac{D}{ds}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{\partial f}{\partial s} \overset{@\, s=0}{\rightsquigarrow} \frac{D}{dt} V \quad \overset{@\, s=0}{\rightsquigarrow}$ 4 $= \sum_i \langle V, \frac{D}{dt} V \rangle \Big|_{t_i}^{t_{i+1}}$

3 : @ $s=0$ $= \sum_i \langle W, \gamma' \rangle \Big|_{t_i}^{t_{i+1}} = \langle W, \gamma' \rangle \Big|_0^1$ $\leftarrow$ γ geod $\Rightarrow \gamma'$ cont.

2 : @ $s=0$: $\int \langle \circ, \frac{D}{dt}\gamma' \rangle \, dt = 0$

1 : $\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \overset{@\, s=0}{=} V'' + R(V,\gamma')\gamma'$

1 $= -\int_0^1 \langle V, V'' + R(V,\gamma')\gamma' \rangle$

$\square$

Prop: Alternative formula

$$\frac{1}{2}\frac{d^2E}{ds^2}\Big|_{s=0} = \int_0^1 |V'|^2 - \langle R(V,\gamma')\gamma', V\rangle\, dt + \cancel{\langle V, V'\rangle\Big|_0^1} + \langle W,\gamma'\rangle\Big|_0^1$$

Pf: Need to check that $-\int_0^1 \langle V, V''\rangle\, dt + \sum_i \langle V,V'\rangle\Big|_{t_i}^{t_{i+1}} = \int_0^1 |V'|^2\, dt$

$$LHS = \sum_i \left(\int_{t_i}^{t_{i+1}} -\langle V,V''\rangle\, dt + \langle V,V'\rangle\Big|_{t_i}^{t_{i+1}}\right)$$

$$= \sum_i \left(\int_{t_i}^{t_{i+1}} -\frac{d}{dt}\langle V,V'\rangle\, dt + \int_{t_i}^{t_{i+1}}\langle V',V'\rangle\, dt + \langle V,V'\rangle\Big|_{t_i}^{t_{i+1}}\right)$$

$$= \sum_i \left(\cancel{-\langle V,V'\rangle\Big|_{t_i}^{t_{i+1}}} + \int_{t_i}^{t_{i+1}}\langle V',V'\rangle\, dt + \cancel{\langle V,V'\rangle\Big|_{t_i}^{t_{i+1}}}\right) = \int_0^1 |V'|^2\, dt \quad \blacksquare$$

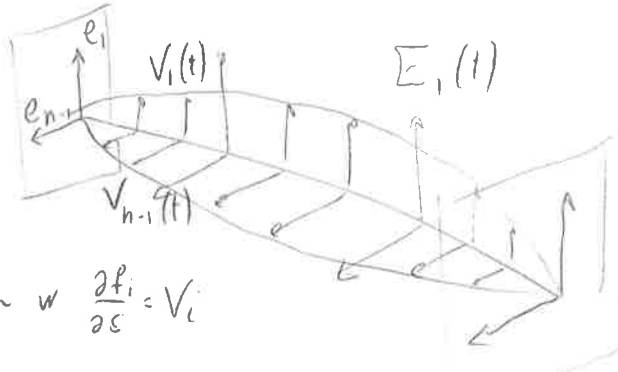
Thm (Bonnet - Myers): Suppose M complete has $Ric(x,x) \geq \frac{n-1}{r^2} > 0$ $\forall |x| =$
Then M is cpt, with $diam(M) \leq \pi r^2$

Pf: Take minimiz. geodesic γ in M. WTP: $\ell(\gamma) \leq \pi r$.

Suppose not: $\ell(\gamma) > \pi r$.

$e_1 \ldots e_{n-1}$ o.n.b. of $\gamma'(0)^\perp$,

$E_i(t)$ parallel w/ $E_i(0) = e_i$



$V_i(t) := \sin \pi t \cdot E_i$, f_i proper variation w $\frac{\partial f_i}{\partial s} = V_i$

$\downarrow$

$$\frac{1}{2}\frac{d^2E}{dV_i^2}\Big|_{s=0} = \int_0^1 |V_i'|^2 - \langle R(V_i,\gamma')\gamma', V_i\rangle\, dt$$

$$|V_i'|^2 = |\pi \cdot \cos \pi t\, E_i|^2 = \pi^2 \cos^2 \pi t$$

$$\langle R(V_i,\gamma')\gamma', V_i\rangle = \ell^2 \sin^2 \pi t \cdot \langle R(E_i,E_n)E_n, E_i\rangle$$ where $\ell = \ell(\gamma)$ [recall $\gamma' = \ell \cdot E_n$]

$\downarrow$

$$\sum_i \frac{1}{2}\frac{d^2E}{dV_i^2} = \sum_i \int_0^1 |V_i'|^2 - \langle R(V_i,\gamma')\gamma', V_i\rangle\, dt$$

$$= (n-1)\int_0^1 \pi^2 \cos^2 \pi t\, dt - \int_0^1 \ell^2 \sin^2 \pi t \cdot \underbrace{\sum_{i=1}^{n-1} \langle R(E_i,E_n)E_n, E_i\rangle}_{Ric(E_n,E_n) \geq \frac{n-1}{r^2}}\, dt$$

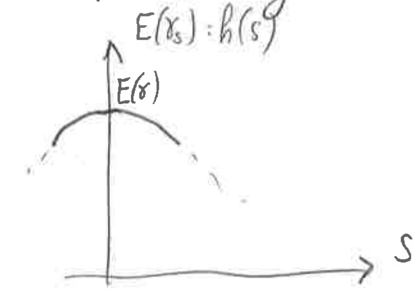
$$< \frac{(n-1)\pi^2}{2} - \frac{n-1}{\cancel{r^2}} \cdot \pi^2 \cancel{r^2} \cdot \frac{1}{2} = \frac{(n-1)\pi^2}{2} - \frac{(n-1)\pi^2}{2} = 0$$

$\downarrow$

$$\sum_i \frac{1}{2} \frac{d^2E}{dV_i^2} < 0 \Rightarrow \exists\, i \text{ st. } \frac{d^2E}{dV_i^2} < 0$$

- take corresponding variation γ_s, and $h(s) := E(\gamma_s)$ satisfies:
  - $h(0) = E(\gamma)$
  - $h'(0) = \frac{d}{ds}\Big|_{s=0} E(\gamma_s) = 0 \leftarrow (\gamma \text{ geod})$
  - $h''(0) = \frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_s) < 0 \leftarrow$ discussion above

$\rightarrow$



$\Rightarrow \exists\, \gamma_s$ near γ st. $E(\gamma_s) < E(\gamma)$. Contradiction! (γ minimizing geodesic)

$\blacksquare$

<u>Cor:</u> (M,g) Complete, $Ric \geq \delta > 0$. Then $\pi_1(M)$ is finite.

<u>Pf:</u> $\check{M}$ universal cover of M, $p : \check{M} \to (M,g)$. $\exists\, \check{g}$ metric in $\check{M}$, such that p is ~~a~~ (local) isometry.

$\Rightarrow Ric_{\check{M}} \geq \delta > 0 \Rightarrow \check{M}$ compact.

Take $\check{p} \in \check{M}$, and take the map
$$ev_{\check{p}} : \pi_1 M \longrightarrow \check{M}, \quad \gamma \longmapsto \gamma \cdot \check{p}$$

The map is injective (b/c the action of $\pi_1 M$ on $\check{M}$ is free) and the image is ~~discrete~~ finite (image discrete + $\check{M}$ compact)

$\Rightarrow \pi_1 M$ finite.

$\blacksquare$