SYNGE - WEINSTEIN THEOREM.

Last time:

Thm (Bonnet-Myers):
M complete, $Ric \geq \frac{n-1}{r^2} > 0$ Then $\operatorname{diam} M < \pi r \Rightarrow$ ~~th~~ M cpt, $\pi_1 M$ finite

Rmk: Result is sharp: paraboloid has $Ric > 0$, but not cpt.

Thm: M^n cpt, oriented, $sec > 0$, $\varphi: M \to M$ isometry.
Suppose φ preserves/reverses orientation, if n even/odd.
Then φ has a fixed pt.

Lemma: $A \in O(n-1)$ with $\det A = (-1)^n$ has a fixed vector.

Pf of lemma: Need: 1 eval of A. $\mathbb{C}$-evals of A come in pairs $\lambda_i, \bar{\lambda}_i$,

$$\Rightarrow \det A = \prod_{\lambda \text{ evals}} \lambda = \left(\prod_i \lambda_i \bar{\lambda}_i\right) \cdot \prod_{\substack{\lambda \text{ real} \\ \text{evals}}} \lambda_i = \prod |\lambda_i|^2 \cdot \prod_{\text{real}} \lambda_i$$

n even $\Rightarrow \det(A - tI)$ odd degree $\Rightarrow \exists$ real eval λ (odd # of) $+ \det A = 1$

$\Rightarrow \exists$ ~~pos~~ λ real and > 0

$\Rightarrow \underbrace{\|Av\|^2}_{= \|v\|^2} = \lambda^2 \|v\| \Rightarrow \lambda = 1$

n odd $\Rightarrow \det A = -1 \Rightarrow \exists$ even # of real evals $\Rightarrow \exists$ positive real eval λ
$\Rightarrow \lambda = 1$ $\blacksquare$

Pf of Thm: Suppose not $\Rightarrow \forall q \in M$, $\varphi(q) \neq q$.
$\Rightarrow \exists p \in M$ st. $d(p, \varphi(p))$ minimum, and > 0.

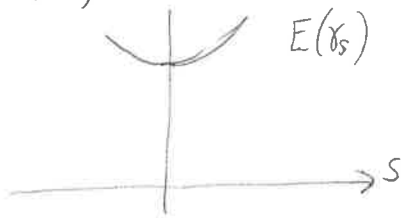
$\gamma: [0,1] \to M$ min'l geod $p \rightsquigarrow \varphi(p)$

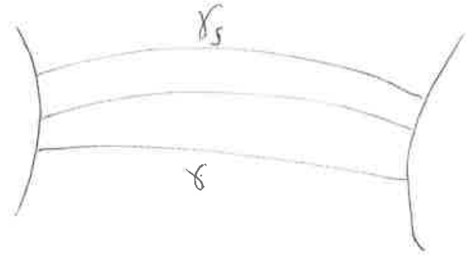
$\Downarrow$

$$\ell(\gamma_s) \geq d(\gamma_s(0), \gamma_s(1)) = d(\gamma_s(0), \varphi(\gamma_s(0))) \geq d(p, \varphi(p)) = \ell(\gamma)$$

$\Downarrow$

$$E(\gamma_s) \geq \ell^2(\gamma_s) \geq \ell^2(\gamma) = E(\gamma)$$





However:

$$\frac{1}{2} \frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_s) = \int_0^1 |\overset{0}{\cancel{V'(t)}}|^2 - \langle R(V(t), \gamma'(t))\gamma'(t), V(t)\rangle \, dt + \overset{0}{\cancel{\langle W, \gamma' \rangle}}\Big|_0^1$$

$$= - \int_0^1 sec(V(t), \gamma'(t)) \cdot |V(t) \wedge \gamma'(t)|^2 \, dt < 0$$

$*$

$\square$

<u>Cor</u> (Synge): M^n cpt, $sec > 0$

1. n odd $\Rightarrow$ M orientable
2. n even $\Rightarrow$ if M orientable is simply connected

<u>Pf:</u> $(\tilde{M}, \tilde{g})$ universal cover of (M, g) $\pi_1(M)$ acts on $\tilde{M}$ by isometries, ~~FREELY~~.

Fix $\gamma \neq e \in \pi_1 M \Rightarrow \gamma : \tilde{M} \to \tilde{M}$ does not fix points

$\Rightarrow$ n odd $\Rightarrow$ γ must preserve orientation $\Rightarrow \pi_1(M) \curvearrowright \tilde{M}$ preserves orientation

$\Rightarrow M = \tilde{M}/\pi_1(M)$ orientable

n even, M orientable $\Rightarrow \gamma$ must act preserving orientation

$\Rightarrow \gamma$ has fixed point $\Rightarrow \gamma$ acts trivially

(Free action) $\Rightarrow \gamma = e$. $\Rightarrow \pi_1(M) = e$.

<u>Ex:</u> • $\mathbb{RP}^n = S^n/_{\langle -1 \rangle}$ has $sec > 0$

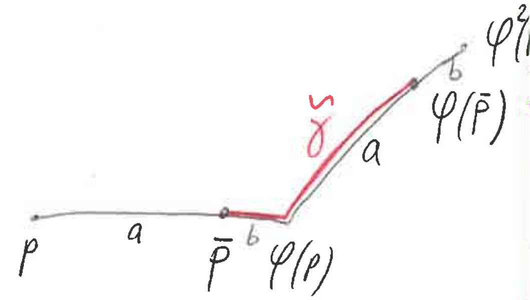
$\rightarrow \mathbb{RP}^{even}$ is orientable.

• M^n non orientable $sec > 0$. $\Rightarrow$ 1. n even

2. Ex: $\exists$ orientable manifold $\hat{M}$ with $\hat{M} \xrightarrow{\text{2-cover}} M$

$\Rightarrow \hat{M}$ simply connected $\Rightarrow \pi_1(M) = \mathbb{Z}_2$

· $\varphi\circ\gamma$ geod $\quad \varphi(p) \rightsquigarrow \varphi\varphi(p)$



Claim: $\varphi\circ\gamma$ geod continuation of γ.

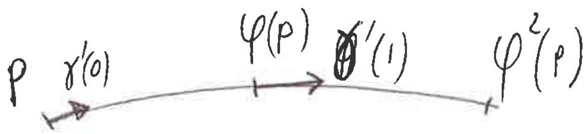
$\bar p \in \gamma[0,1]$

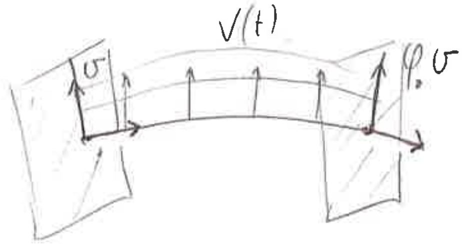
$$d(\bar p, \varphi(\bar p)) \leq b + a = d(p, \varphi(p))$$

by minimality of $d(p,\varphi(p))$, $d(\bar p, \varphi(\bar p)) = a + b = \ell(\tilde\gamma) \Rightarrow \tilde\gamma$ minimizing curve

$\Downarrow$

$\tilde\gamma$ geod.





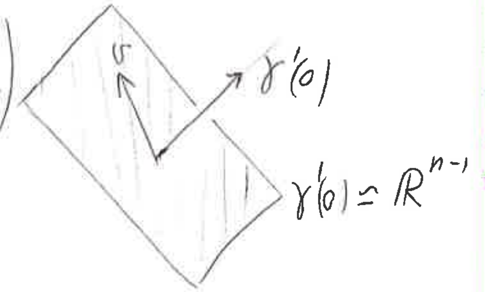
$$\Rightarrow \varphi_*(\gamma'(0)) = \gamma'(1)$$

Take $\hat A : T_pM \xrightarrow{P_\gamma^{0,1}} T_{\varphi(p)}M \xrightarrow{\varphi_*^{-1}} T_pM$. Then:

1. $\hat A$ is an isometry
2. $\hat A(\gamma'(0)) = \varphi_*^{-1}\left(P_\gamma^{0,1}(\gamma'(0))\right) = \varphi_*^{-1}(\gamma'(1)) = \gamma'(0)$
3. $\hat A$ preserves orientation if n even / reverses if n odd $\Rightarrow \det \hat A = (-1)^n$

$\Downarrow$

$\hat A$ takes $\gamma'(0)^\perp$ to $\gamma'(0)^\perp$. Def. $A = \hat A|_{\gamma'(0)^\perp}$ $\quad \hat A = \left(\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right)$



$\Rightarrow A \in O(n-1)$, and $\det A = \det \hat A = (-1)^n$

$\rightarrow$ Lemma: $\exists\, v$ st. $Av = v \Rightarrow \varphi_*^{-1} P_\gamma^{0,1} v = v$

$\Rightarrow P_\gamma^{0,1} v = \varphi_* v$

Define $V(t)$ parallel st. $V(0) = v \Rightarrow V(1) = \varphi_* v$

Define $\gamma_s(t) = f(s,t) = \exp_{\gamma(t)} sV(t) \rightsquigarrow \gamma_0(t) = \gamma(t)$, $\left.\frac{\partial f}{\partial s}\right|_{s=0} = V(t)$, $\left.\frac{D}{ds}\right|_{s=0} \frac{\partial f}{\partial s} \equiv 0$

$$\gamma_s(1) = \exp_{\gamma(1)} s\cdot V(1) = \exp_{\varphi(\gamma(0))} \varphi_* sV(0) = \varphi\left(\exp_{\gamma(0)} sV(0)\right)$$
$$= \varphi(\gamma_s(0))$$