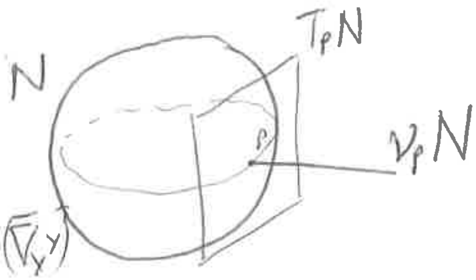
THE SECOND FUNDAMENTAL FORM.

Last time: $N \to M$ isometric immersion. $T_pM = T_pN \oplus \nu_pN$

$X, Y \in \mathcal{X}(N) \rightsquigarrow \overline{\nabla}_X Y \in \mathcal{X}(M)|_N$, i.e.

$\forall p \in N$, $(\overline{\nabla}_X Y)_p \in T_pM$



$(\overline{\nabla}_X Y)_p^T = (\nabla_X Y)_p$ connection $(\overline{\nabla}_X fY)^T = f \cdot (\overline{\nabla}_X Y) + X(f) \cdot Y$

$(\overline{\nabla}_X Y)_p^\perp = \mathbb{II}(X,Y)$ tensor $(\overline{\nabla}_X fY)^\perp = f \cdot (\overline{\nabla}_X Y)^\perp$

$\mathbb{II}(X, fY) = f \cdot \mathbb{II}(X,Y)$

$\gamma : I \to N$ curve $\nabla_{\gamma'}\gamma'$ acceleration of γ

γ geodesic $\iff \nabla_{\gamma'}\gamma' = 0$ no acceleration (no forces act upon γ)

Supp. $\gamma : I \to N$ geodesic, $\gamma : I \to N \hookrightarrow M$

$\Rightarrow \overline{\nabla}_{\gamma'}\gamma' = (\overline{\nabla}_{\gamma'}\gamma')^T + (\overline{\nabla}_{\gamma'}\gamma')^\perp = \nabla_{\gamma'}\gamma' + \mathbb{II}(\gamma',\gamma')$

$\Rightarrow \mathbb{II}(\gamma',\gamma')$ measures the obstruction of γ from being a geod. in M as well.

Prop: $\varphi : N \hookrightarrow M$ isometric immersion. Then the geods in N are geods of M iff. $\mathbb{II} \equiv 0$

Pf. - if $\mathbb{II} = 0 \Rightarrow \forall \gamma : I \to N$ geodesic, $\overline{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + \mathbb{II}(\gamma',\gamma') = 0$ (γ geod: $\nabla_{\gamma'}\gamma' = 0$, $\mathbb{II}(\gamma',\gamma') = 0$)

$\Rightarrow \gamma$ is a geod in M.

- Suppose every geod. of N is a geodesic of M as well. $\forall v \in T_pN$, let $\gamma : I \to N$ geod. s.t. $\gamma'(0) = v$. Then

$0 = \overline{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' + \mathbb{II}(v,v) = \mathbb{II}(v,v) \Rightarrow \mathbb{II}(v,v) = 0 \quad \forall v \in T_pN.$

$\forall v, w \in T_pN \quad 0 = \mathbb{II}(v+w, v+w) = \mathbb{II}(v,v) + \mathbb{II}(w,w) + \mathbb{II}(v,w) + \mathbb{II}(w,v)$

$= 2\,\mathbb{II}(v,w) \Rightarrow \mathbb{II}(v,w) = 0 \quad \forall v, w.$ ☒

Def: If N satisfies conditions of proposition, it is called TOTALLY GEODESIC.

Ex:
- $\gamma : \mathbb{R} \to M$ geodesic is a Totally geodesic immersion.
- $M = \mathbb{R}^{m}$, $\mathbb{R}^n = N \hookrightarrow \mathbb{R}^m$ affine subspace is Totally geodesic.
  $v \longmapsto A\cdot v + b$
- $(M_1 \times M_2, g_1 \times g_2)$ product mfd. $\forall q \in M_2$,
  $M_1 \longrightarrow M_1 \times \{q\} \subseteq M_1 \times M_2$ is an isometric immersion.
  $p \longmapsto (p, q)$

PROP: $N \longrightarrow M$ Totally geodesic immersion. Then $\forall p \in N$, ~~$\exists B_\varepsilon$~~
$\exists\, U \ni p$ nbd in N, st.
$\forall q_1, q_2 \in U$, $d_N(q_1, q_2) = d_M(q_1, q_2)$.

$$\begin{array}{ccc} T_pN & \xrightarrow{\exp_p^N} & N \\ \downarrow \varphi_* & \circlearrowleft & \downarrow \varphi \\ T_pM & \xrightarrow{\exp_p^M} & M \end{array} \qquad M \supseteq U$$

Pf: Take ε small enough, that $B^M_\varepsilon(p) = \exp_p(B_\varepsilon(0) \subseteq T_pM)$ and $B^N_\varepsilon(p) = \exp_p(B_\varepsilon(0) \subseteq T_pN)$ are ~~round~~ strongly convex.

Notice $B^N_\varepsilon(p) \hookrightarrow B^M_\varepsilon(p)$. Take $q_1, q_2 \in B^N_\varepsilon(p) =: U$

Then:
- $B^N_\varepsilon(p)$ strongly convex $\Rightarrow \exists!$ (min.) geodesic γ in $B^N_\varepsilon(p)$ $q_1 \rightsquigarrow q_2$
- $B^M_\varepsilon(p)$ strongly convex $\Rightarrow \exists!$ (min) geodesic $\bar\gamma$ in $B^M_\varepsilon(p)$ $q_1 \rightsquigarrow q_2$

N Totally geodesic + $B^N_\varepsilon(p) \hookrightarrow B^M_\varepsilon(p)$ $\Rightarrow \gamma$ is a geodesic in B^M_ε $q_1 \rightsquigarrow q_2$
$\Rightarrow \gamma = \bar\gamma$

$$\Rightarrow d_N(q_1, q_2) = \ell(\gamma) = \ell(\bar\gamma) = d_M(q_1, q_2)$$

☑

Rmk: not global in general!

N geod of $M \Rightarrow N$ Totally geodesic.
$d_N(p, q) > d_M(p, q)$

Stronger conditions: $N \to M$ (complete mfds)

1. Convex if $\forall\, p,q \in N\ \exists!$ M-minimizing geod. $p \rightsquigarrow q$ in N
2. Totally convex if $\forall\, p,q \in N,\ \forall \gamma$ M-geod. $p \rightsquigarrow q,\ \gamma \subseteq N$

Ex:			
Tot geod.	✓	✓	✓
convex	✗	✓	✓
Tot. convex	✗	✗	✓

RMK: If $N \hookrightarrow M$ convex, then $\forall\, p,q \in N\quad d_N(p,q) = d_M(p,q)$

Def: $N \to M,\ p \in N \Rightarrow$ def: mean curvature operator

$$K_p : \nu_p N \to \mathbb{R}$$
$$\eta \mapsto \operatorname{tr} S_\eta = \sum_i \langle S_\eta e_i, e_i \rangle = \sum_i \langle \eta, \mathrm{II}(e_i,e_i) \rangle \quad e_i \text{ onb of } T_pN$$

Def: mean curvature vector

$H_p \in \nu_p N$ st. $\forall \eta \in \nu_p N,\quad K_p(\eta) = \langle \eta, H_p \rangle$

$$K_p(\eta) = \langle \eta, \sum_i \mathrm{II}(e_i,e_i) \rangle$$

$\Downarrow$

$$H_p = \sum_i \mathrm{II}(e_i,e_i) \qquad e_i \text{ onb of } T_pN$$

Def: mean curvature vector field $H \in \Gamma(\nu N)$ $\quad H : N \to \nu N,\ p \mapsto H_p$

Fact: H points in the direction of maximal decrease of area.

Def: $N \subseteq M$ is called Minimal if $H \equiv 0$

N minimal is a "critical point for the area"