HOMEWORK #10 - DUE APR 27, AT NOON

Exercise 1 (Fixed points of isometries)**.** Let $\phi : M^m \to M^m$ an isometry, and let $Fix(\phi) = \{q \in M \mid \phi(q) = q\}$. Let $p \in Fix(\phi)$.

(1) Let $V \subset T_pM$ be the subset of T_pM fixed by $d_p\phi : T_pM \to T_pM$. Prove that V is a linear subspace of T_pM.
(2) Given $v \in T_pM$, prove that v is fixed by $d_p\phi$ if and only if $\exp_p(tv) \in Fix(\phi)$.
(3) Given a normal ball $B_\epsilon(p)$ around p, prove that $N := Fix(\phi) \cap B_\epsilon(p)$ is a manifold, and the injection $N \to M$ is an injective immersion.
(4) Prove that for every $v, w \in T_pN$, $II(v, w)$ is fixed by $d_p\phi$. Deduce that N is totally geodesic.

Exercise 2 (Product manifold)**.** Let $M = M_1 \times M_2$ be a product manifold, with product metric $g_1 \times g_2$.

(1) Given $(p_1, p_2) \in M_1 \times M_2$, prove that $T_{(p_1,p_2)}(M_1 \times M_2) = T_{p_1}M_1 \oplus T_{p_2}M_2$. In particular, vector fields of M can be written as $X = (X_1, X_2)$, where $X_i(p_1, p_2) \in T_{p_i}M_i$, $i = 1, 2$.
(2) Given ∇^i the Levi Civita connection of (M_i, g_i), $i = 1, 2$, and vector fields $X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathfrak{X}(M)$, prove that $\nabla^M_X Y = (\nabla^1_{X_1} Y_1, \nabla^2_{X_2} Y_2)$.
(3) Prove that if $\gamma_i : [0, 1] \to M_i$ is a geodesic, $i = 1, 2$, then $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic in $M_1 \times M_2$. Deduce that *every* geodesic in $M_1 \times M_2$ has this form.
(4) Prove that, fixing $p_1 \in M_1$ and $p_2 \in M_2$, then $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally geodesic submanifolds of $M_1 \times M_2$.
(5) Prove that, letting R_i denote the Riemann curvature operator of M_i, $i = 1, 2$, then the curvature operator R of $M_1 \times M_2$ satisfies
$$R\big((x_1, x_2), (y_1, y_2)\big)(z_1, z_2) = (R^1(x_1, y_1)z_1, R^2(x_2, y_2)z_2)$$

Exercise 3 (Tori in S^3)**.** Let $M = S^3 \subset \mathbb{R}^4$, and let $S^1 = \mathbb{R}/\mathbb{Z}$ (the circle). Fixing a number $a \in (0, 1)$, let $b := \sqrt{1 - a^2}$ and define

$$\phi_a : S^1 \times S^1 \to S^3, \quad \phi_a(s, t) : (a \cos(2\pi s), a \sin(2\pi s), b \cos(2\pi t), b \sin(2\pi t)).$$

(1) Prove that ϕ_a is an immersion. In particular, $N = S^1 \times S^1$ inherits a metric $g_a = \phi_a^*(g_{S^3})$, where g_{S^3} is the round metric of S^3.
(2) Identify N with its image in S^3. Show that for every point $p = \phi_a(s, t) \in N$, the vectors
$$E_1 = (-\sin(2\pi s), \cos(2\pi s), 0, 0), \qquad E_2 = (0, 0, -\sin(2\pi t), \cos(2\pi t))$$
form an orthonormal basis of T_pN, and
$$\eta = (-b \cos(2\pi s), -b \sin(2\pi s), a \cos(2\pi t), a \sin(2\pi t))$$
is a unit vector spanning ν_pN. (Recall that for $p \in S^3$, T_pS^3 can be identified with $p^\perp \subseteq \mathbb{R}^4$).

(3) Prove that $S_\eta E_1 = \frac{b}{a} E_1$ and $S_\eta E_2 = -\frac{a}{b} E_2$. (**Hint:** Think of N as a submanifold of $\mathbb{R}^4$; extend E_1, E_2 and η to vector fields $\bar{E}_1 = -\frac{1}{a} x_2 \frac{\partial}{\partial x_1} + \frac{1}{a} x_1 \frac{\partial}{\partial x_2}$, $\bar{E}_2 = \cdots$ in $\mathbb{R}^4$; and use the fact that, in the connection $\bar{\nabla}$ of $\mathbb{R}^4$, $\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$).

(4) Is there a value of a such that ϕ_a is a totally geodesic immersion? Is there a value of a such that ϕ_a is a minimal immersion? (**Hint:** No. Yes.)

Exercise 4 (Totally geodesic submanifolds)**.** 1. Let $N_1, N_2 \subset M$ be two complete, totally geodesic submanifolds of M, such that $p \in N_1 \cap N_2$ and $T_p N_1 = T_p N_2$. Prove that $N_1 = N_2$.

2. Use the previous point to prove that a submanifold of $\mathbb{R}^n$ is totally geodesic if and only if it is an affine plane (you can feel free to assume that affine planes are totally geodesic in $\mathbb{R}^n$, but if you want to go the extra mile, you can prove that as well!)

Exercise 5 (Surfaces containing lines)**.** Let N be a complete surface in $\mathbb{R}^3$. Suppose that through every point in N there is a line of $\mathbb{R}^3$ contained in N. Must N be a flat plane? What if through every point N contains 2 lines? What if through every point N contains 3 lines?