CURV. OF RIEM. SUBM.

<u>Last time</u>: • $\varphi: M \to B$ subm. if $\varphi, d\varphi$ surj.
• Riem. subm, if $\forall \bar{x}, \bar{y} \in H_p$ $\quad g_M(\bar{x}, \bar{y}) = g_B(\varphi_* x, \varphi_* y)$

• vectors, vector fields, curves, geods in B admit <u>horiz</u> <u>lifts</u> to M.

<u>Ex</u>: • $(M_1 \times M_2, g_1 \times g_2) \xrightarrow{\pi} (M_2, g_2)$ Riem. subm.

• $(TM, \langle\langle\ \rangle\rangle) \longrightarrow (M, \langle , \rangle)$

• G cpt Lie group, $G \circlearrowright (M, \langle , \rangle)$ free, proper ($\rightsquigarrow M/G$ mfd) AND by isometries.
Then there exist a metric on M/G st. $M \xrightarrow{\pi} M/G$ is a Riem. sub

<u>Thm</u>: $\exists$ map $\{\gamma: [0,1] \to B$ piecewise immersed curves $q_1 \to q_2\} \longrightarrow$ Smooth maps $F_{q_1} \to F_{q_2}$

$$\gamma \longmapsto \Phi_\gamma$$

st. $\Phi_\gamma^{-1} = \Phi_{\gamma^{-1}}$, and $\Phi_{\gamma_1 * \gamma_2} = \Phi_{\gamma_1} \circ \Phi_{\gamma_2}$

<u>Pf</u>: Local: suppose $\gamma: [0,1] \to B$ integral curve of $X \in \mathfrak{X}(B)$.
Define

$$\Phi_{\bar{X}}: F_{q_1} \times [0,1] \to M \qquad \text{flow of } \bar{X} \text{ hor. lift of } X.$$

$$(p, t) \longmapsto \Phi^t_{\bar{X}}(p) = \bar{\gamma}_p(t) \longleftarrow \text{hor. lift of } \gamma, \text{ from } p$$

Notice:

$\varphi(\Phi^t_{\bar{X}}(p)) = \varphi(\bar{\gamma}_p(t)) = \gamma(t) \Rightarrow \Phi^t_{\bar{X}}(F_{q_1}) \subseteq F_{\gamma(t)}$

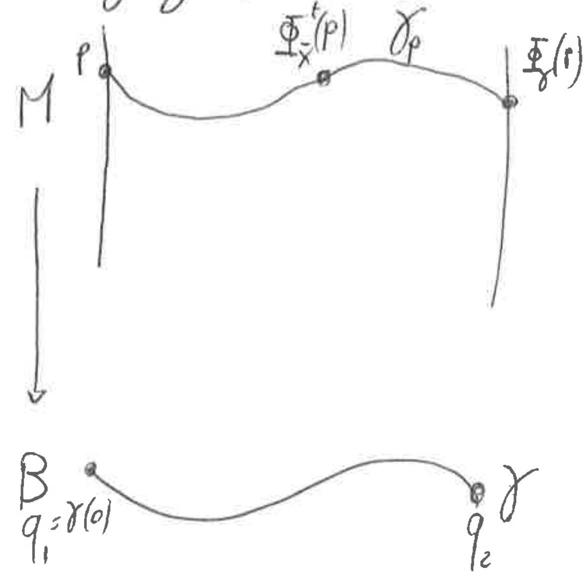
$\Downarrow$

Define $\Phi_\gamma := \Phi^1_{\bar{X}} : F_{q_1} \longrightarrow F_{\gamma(1)} = F_{q_2}$ smooth

Notice:

$$\Phi_{\gamma^{-1}} := \Phi^1_{-\bar{X}} = \left(\Phi^1_{\bar{X}}\right)^{-1} = \left(\Phi_\gamma\right)^{-1} \to \Phi_\gamma \text{ diffeo.}$$

γ^{-1} integral curve of $-X$



$\blacksquare$

$X, Y \in \mathfrak{X}(B)$, $\bar{X}, \bar{Y}$ hor lifts

Then $\bar{\nabla}_{\bar{X}} \bar{Y} = (\bar{\nabla}_{\bar{X}} \bar{Y})^h + (\bar{\nabla}_{\bar{X}} \bar{Y})^v$

Prop 1: $(\bar{\nabla}_{\bar{X}} \bar{Y})^h = \overline{(\nabla_X Y)}$

Pf: Need to check: $\bar{\nabla}^h_{\bar{X}} \bar{Y}$ horizontal ✓ $\quad \varphi_* \bar{\nabla}^h_{\bar{X}} \bar{Y} = \nabla_X Y$

$$\langle \varphi_* \bar{\nabla}^h_{\bar{X}} \bar{Y}, Z \rangle \overset{R.S.}{=} \langle \bar{\nabla}^h_{\bar{X}} \bar{Y}, \bar{Z} \rangle = \langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle$$

$$= \frac{1}{2}\left(\bar{Y} \langle \bar{X}, \bar{Z} \rangle + \dots + \langle [\bar{X}, \bar{Y}], \bar{Z} \rangle + \dots \right)$$

$\langle \bar{X}, \bar{Z} \rangle_p = \langle X, Z \rangle_{\varphi(p)}$

$$= \frac{1}{2}\left(\bar{Y}(\langle X, Z \rangle \circ \varphi) + \dots + \langle \varphi_* [\bar{X}, \bar{Y}], \varphi_* \bar{Z} \rangle + \dots \right)$$

$$= \frac{1}{2}\left((\varphi_* \bar{Y}) \langle X, Z \rangle + \dots + \langle [\varphi_* \bar{X}, \varphi_* \bar{Y}], \varphi_* \bar{Z} \rangle + \dots \right)$$

$$= \frac{1}{2}\left(Y \langle X, Z \rangle + \dots + \langle [X, Y], Z \rangle + \dots \right) = \langle \nabla_X Y, Z \rangle \quad \blacksquare$$

~~Prop~~ Def: $A : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{V}_p$

$x, y \longmapsto A_x y = \bar{\nabla}^v_{\bar{X}} \bar{Y}$ $\qquad \bar{X}, \bar{Y}$ horiz. ext. of x, y

Prop: A is a skew symm. tensor

Pf: $\bar{\nabla}^v_{f\bar{X}} \bar{Y} = f \cdot \bar{\nabla}^v_{\bar{X}} \bar{Y}$, $\quad \bar{\nabla}^v_{\bar{X}} f\bar{Y} = \left(\bar{X}(f) \cdot \bar{Y} + f \cdot \bar{\nabla}_{\bar{X}} \bar{Y} \right)^v = f \cdot \bar{\nabla}^v_{\bar{X}} \bar{Y}$

→ only depends on $y = \bar{Y}_p$, $\bar{X}_p = x$ → A well defined, tensor.

Claim: $A_x y = \frac{1}{2} [\bar{X}, \bar{Y}]^v$: choose $\bar{X}, \bar{Y}$ horiz. lifts of $X, Y \in \mathfrak{X}(B)$

$$2 \langle \bar{\nabla}_{\bar{X}} \bar{Y}, \underset{\text{vertical v.f.}}{V} \rangle = \bar{X} \underbrace{\cancel{\langle \bar{Y}, V \rangle}}_{0} + \bar{Y} \underbrace{\cancel{\langle \bar{X}, V \rangle}}_{0} - V \langle \bar{X}, \bar{Y} \rangle - \langle [\bar{X}, V], \bar{Y} \rangle$$

$$- \cancel{\langle [\bar{Y}, V], \bar{X} \rangle} + \langle [\bar{X}, \bar{Y}], V \rangle \quad (*)$$

$\langle \bar{X}, \bar{Y} \rangle = \langle X, Y \rangle \circ \varphi \Rightarrow \langle \bar{X}, \bar{Y} \rangle$ constant on fibers $\Rightarrow V \langle \bar{X}, \bar{Y} \rangle = 0$

$\varphi_* [\bar{X}, V] = [\varphi_* \bar{X}, \varphi_* V] = [X, 0] = 0 \Rightarrow [\bar{X}, V]$ vertical $\to \langle [\bar{X}, V], \bar{Y} \rangle = 0$

$\longrightarrow (*) = \langle [\bar{X}, \bar{Y}], V \rangle \longrightarrow A_x y = \bar{\nabla}^v_{\bar{X}} \bar{Y} = \frac{1}{2} [\bar{X}, \bar{Y}]^v$

skew sym. $\blacksquare$

$\varphi: M \to B$ R. subm $\to$ $TM = \mathcal{H}_p \oplus \mathcal{V}_p$ and $\forall \cancel{x,y} \in \mathcal{H}_p$, $\overline{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + A_x y$

$\varphi: N \to M$ isometric immersion $TM = T_pN \oplus \nu_p N$, and $\forall X, Y \in \mathfrak{X}(N)$, $\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(x,y)$

Gauss eqn: $R^N(xyzw) - R^M(xyzw) = \langle \mathrm{II}(x,y), \mathrm{II}(\cancel{x}y, z)\rangle - \langle \mathrm{II}(x,z), \mathrm{II}(y,$

O'Neill eqn: $R^B(xyzw) - R^M(\bar{x}\bar{y}\bar{z}\bar{w}) = \langle A_{\bar{x}}\bar{w}, A_{\bar{y}}\bar{z}\rangle - \langle A_{\bar{x}}\bar{z}, A_{\bar{y}}\bar{w}\rangle + 2\langle A_{\bar{x}}\bar{y}, A_{\bar{z}}\bar{w}$

$\Downarrow$

<u>Cor:</u> $M \xrightarrow{\varphi} B$ Riem subm, $\forall \cancel{z}\; x, y \in T_qB$ o.normal,

$$\sec_B(x,y) = \sec_M(\bar{x},\bar{y}) + 3|A_{\bar{x}}\bar{y}|^2$$

<u>Pf:</u> $\sec_B(x,y) = R^B(xyyx) = R^M(\bar{x}\bar{y}\bar{y}\bar{x}) + \langle A_{\bar{x}}\overset{0}{\cancel{\bar{x}}}, A_{\bar{y}}\bar{y}\rangle - \langle A_{\bar{x}}\bar{y}, A_{\bar{y}}\bar{x}\rangle - 2\langle A_{\bar{x}}\bar{y}, A_{\bar{y}}$

$\bar{x}, \bar{y}$ still o.n.

$$= \sec_M(\bar{x},\bar{y}) + 3|A_{\bar{x}}\bar{y}|^2$$

<u>Cor:</u> If M has $\sec \geq 0$ (resp > 0) and $M \to B$ Riem. Subm, then B has $\sec \geq 0$ (resp > 0)

<u>Cor:</u> $\forall$ space G/H, G cpt Lie group, H closed subgroup, admits a metric w/ $\sec \geq 0$

<u>Pf:</u> G cpt $\Rightarrow$ G admits bi-inv. metric $\langle , \rangle$ $\to$ $(G, \langle , \rangle)$ has $\sec \geq 0$.

Also, $H \times G \to G$ $\Rightarrow$ H acts by isometries (because $\langle , \rangle$ is, in particular, left invariant)

$h \cdot g \mapsto hg$

$\downarrow$

$\exists$ metric on G/H such that $G \to G/H$ is a Riem subm.

$\Rightarrow$ for this metric, G/H has $\sec \geq 0$

$\boxtimes$