MATH 60670: FINAL EXAM

Due: Thursday, May 10 (by midnight)

Possibly TeX up your exam, and send it by email at marco.radeschi@gmail.com.

Ground Rules. You may only use DoCarmo's and Lee's book, your notes from class, and the notes, homework and solutions from the course website – no other text books, and certainly not anything from the web.

Exercise 1 (Curvature of rotationally symmetric surfaces)**.** Recall that a *surface of revolution* is a surface in $\mathbb{R}^3$ given by the image of a map $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\phi(u,v) = (f(u)\cos(t), f(u)\sin(t), g(u))$, where $a(u) := f^2(u) > 0$ and $b(u) := (f'(u))^2 + (g'(u))^2 > 0$. These surfaces are always assumed to be equipped with the metric induced from $\mathbb{R}^3$.

(1) Compute the sectional curvature of a surface of revolution, as a function of a, b, and their derivatives.
(2) Prove that the paraboloid $z = x^2 + y^2$ has positive curvature.

Exercise 2 (Gradient and Hessian of a function)**.** Let f be a smooth function on $(M, \langle\, ,\, \rangle)$. Define the *gradient* of f, ∇f, as the vector field such that $\langle \nabla f, X\rangle = df(X)$ for every vector field X, and the *Hessian* of f, $\mathrm{Hess}(f)$, as the $(2,0)$ tensor field such that $\mathrm{Hess}(f)(X,Y) = \langle \nabla_X \nabla f, Y\rangle$ for all $X, Y \in \mathfrak{X}(M)$. Prove that

(1) In local coordinates $(x^1, \ldots x^n)$, $\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$.
(2) $\mathrm{Hess}(f)$ is symmetric.
(3) Given a smooth level set $N = f^{-1}(t_0)$, the second fundamental form of N is given by
$$II(x,y) = -\frac{1}{\|\nabla f\|^2} \mathrm{Hess}(f)(x,y) \nabla f$$
(4) If $\|\nabla f\| = 1$ (in this case f is called a *distance function*), then the integral curves of ∇f are (minimizing) geodesics.

Exercise 3 (Killing fields)**.** Let M be a complete Riemannian manifold. Recall that a vector field X is a Killing field, if its flow Φ_X^t is an isometry for every t.

(1) Deduce the Killing field equation: Given X Killing field, then
$$\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0 \qquad \forall Y, Z \in \mathfrak{X}(M).$$
(Ex. 5d in Chap. 3 of Do Carmo contains hints on how to do this).
(2) Prove that the only functions f such that ∇f is Killing, are *linear functions*, i.e. functions with $\mathrm{Hess}(f) = 0$.

(3) Prove that along a geodesic γ, a Killing field X satisfies
$$\langle X, \gamma' \rangle = const.$$

(4) Use the previous point to prove the following *Clairaut's formula*: given a geodesic $\gamma(t)$ on a surface of revolution $M \subset \mathbb{R}^3$, then
$$r(t)\cos(\beta(t)) = const.,$$
where $r(t)$ is the distance between $\gamma(t)$ and the z-axis, while $\beta(t)$ is the angle between $\gamma'(t)$ and the vector $\frac{\partial \phi}{\partial v}$.

Exercise 4 (Curvature)**.** Fill the following table according to whether each space admits a metric with the prescribed curvature (Y) or not (N). Of course, give an example of such a metric (if Y), or prove why such a metric cannot exist (if N).

	Metric with $sec > 0$	Metric with $sec \equiv 0$	Metric with $sec < 0$.
$\mathbb{RP}^2 \times \mathbb{RP}^2$			
$\mathbb{R}^2$			
$T^2 = \mathbb{R}^2/\mathbb{Z}^2$			

Kind-of hint: there is a square in this table which you might not be able to fill, and that is ok.

Exercise 5 (Lines, and ends)**.** Given a complete, non-compact manifold M, we say that M has *more than one end* if there exists a compact set K such that $M \setminus K$ contains at least two components with infinite diameter. Moreover, a *line* is a geodesic $\gamma : \mathbb{R} \to M$, parametrized by arc length, such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in \mathbb{R}$. Prove that if M has more than one end, then it contains a line.