Mohr–Coulomb plasticity for sands incorporating density effects without parameter calibration

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Summary
A simple approach is proposed for enabling the conventional Mohr–Coulomb plasticity to capture the effects of relative density on the behavior of dilative sands. The approach exploits Bolton’s empirical equations to make friction and dilation angles state variables that depend on the current density and confining pressure. In doing so, the material parameters of Mohr–Coulomb plasticity become void ratios for calculating the initial relative density and the critical state friction angle, all of which are measurable without calibration. A Mohr–Coulomb model enhanced in this way shows good agreement with experimental data of different sands at various densities and confining pressures. In this regard, the proposed approach permits a significant improvement in the conventional Mohr–Coulomb plasticity for sands, without compromising its practical merits.

KEYWORDS
constitutive modeling, dilatancy, Mohr–Coulomb, plasticity, relative density, sands

1 | INTRODUCTION

Mohr–Coulomb plasticity* is perhaps the most widely used constitutive model by practitioners and academics alike for the simulation of sand behavior in geotechnical engineering problems. Such popularity of the Mohr–Coulomb model may be attributed to its unparalleled familiarity and simplicity. As mentioned by Muir Wood and Gajo,1 the Mohr–Coulomb failure criterion is understood by every undergraduate civil engineer, and its constitutive model version is available in almost all finite element programs for geotechnical analyses. As such, one can easily apply the model to practical problems using the familiar strength parameter(s) in the Mohr–Coulomb criterion. The Mohr–Coulomb plasticity model is also widely used in research work, since it allows one to capture the pressure-dependent strength of a geomaterial with a fraction of costs for sophisticated constitutive models. This exceptional efficiency makes the model a common choice for computationally intensive simulations in geomechanics, such as those involve complex domains, soil-structure interactions, contacts, large deformations, and/or coupled multiphysics.2–29

Despite its popularity and efficiency, Mohr–Coulomb plasticity is inherently limited as compared with advanced constitutive models. Some of the major limitations are as follows. First, the Mohr–Coulomb failure criterion is usually invalid for a contractive (loose) soil. Second, the model is unable to capture plastic deformations under cyclic loading conditions. These limitations are the primary motivations for advanced sand plasticity models that build on critical state soil

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*Here, we use this term to refer to a family of plasticity models taking the strength parameter(s) of the Mohr–Coulomb criterion, say the friction angle and the cohesion (if the soil is cohesive).
mechanics, bounding surface plasticity, and/or generalized plasticity, and/or others.30-44 Oftentimes, however, these limitations are not so problematic for routine geotechnical problems, such as the design and stability analyses of foundations, retaining walls, and slopes. In these cases, the Mohr–Coulomb model may be favored over advanced models, since it is much more efficient and simple.

Nevertheless, the conventional Mohr–Coulomb model has another important limitation: it does not take into account the effect of relative density on the strength and dilatancy of sands. This limitation is critical since the relative density exerts dominant control on the behavior of sands in most cases. See, for example, page 84 of Mitchell and Soga⁵⁵ that states “most important indicators of mechanical behavior are relative density, \( D_r \), and applied confining pressure” for granular soils. Furthermore, it has been shown that the spatial distribution of density—which is often highly heterogeneous—is a determining factor for the localized failure of sands.⁴⁶-⁴⁹ Whereas the density is completely ignored in the Mohr–Coulomb criterion, it is central to critical state soil mechanics since the density and the confining pressure are the two parameters that determine the state of a sand. In this regard, it may be thought that the density is the key variable that distinguishes the strengths predicted from Mohr–Coulomb and critical state soil mechanics theories.

A common way to incorporate the density into critical state plasticity is to use the state parameter proposed by Been and Jefferies,⁵⁰ which is defined as the difference between the current void ratio and the critical state void ratio. Examples of well-known constitutive models using the state parameter include the Nor-Sand model⁵¹ and the Manzari–Dafalias model.⁵² These models have shown that the use of the state parameter allows one to successfully capture the effects of density on the strength and dilatancy of sands. However, compared with the Mohr–Coulomb model, these critical state models demand non-trivial additional efforts in various stages, from calibration to implementation to computation. For this reason, it remains rather scarce to use a critical state sand model for computationally intensive simulation or geotechnical engineering practice.

The objective of this work is to enable the conventional Mohr–Coulomb plasticity to incorporate the density effects on sands, in a way as simple and efficient as possible. This objective is motivated by that the key advantage of the conventional Mohr–Coulomb model is its unmatched simplicity and efficacy. The most challenging task for achieving this objective may be to minimize the efforts for determining additional parameters. For example, while several hardening/softening laws have been proposed for Mohr–Coulomb plasticity,⁵¹-⁵³ they entail non-standard parameters that are uneasy to measure or calibrate. This is presumably the main reason why most practitioners and researchers often resort to the perfectly plastic Mohr–Coulomb model. In this regard, a successful enhancement of the Mohr–Coulomb model must minimize the number of additional parameters that need to be calibrated.

With this objective in mind, here we propose a simple approach to incorporating the effects of density on sand behavior without any further calibration effort. It exploits Bolton’s empirical equations relating the relative density, strength, and dilatancy of sands,⁵⁴ in order to make the material parameters of the conventional Mohr–Coulomb model—friction and dilation angles—state variables that depend on the current density and confining pressure. The material parameters then become void ratios for calculating the initial relative density and the critical state friction angle, all of which are measurable without calibration. It is noted that here we use the empirical equations for capturing the evolving strength and dilatancy of sand during the course of loading, rather than for estimating constant parameters in the conventional Mohr–Coulomb model. To the best of our knowledge, while Bolton’s equations have been used to address a variety of geotechnical problems,⁵⁵-⁵⁷ they have not yet been interwoven with Mohr–Coulomb plasticity in the way proposed herein.

In the next section, we describe how these empirical relationships can be used to enhance the Mohr–Coulomb plasticity in a calibration-free and numerically efficient manner, such that the practical merit of the original model retains as much as possible. Subsequently, we show that a simple Mohr–Coulomb model enhanced by the proposed approach shows good agreement with experimental data of different sands at various densities and confining pressures.

As for notations, stresses and strains are positive in compression following the geomechanics sign convention. All stresses are understood as effective stresses without special notations. Bold-face letters denote tensors and vectors.

2 | MOHR–COULOMB PLASTICITY INCORPORATING RELATIVE DENSITY

This section proposes a simple approach to enabling the conventional Mohr–Coulomb plasticity to capture the effects of relative density on sand behavior. After brief review of the Mohr–Coulomb plasticity model, Bolton’s empirical equations are applied to make Mohr–Coulomb parameters dependent on the current density and confining pressure. Discussions on parameter determination and numerical implementation follow. For completeness, the modified Mohr–Coulomb plasticity is combined with the simplest elasticity model, namely, isotropic linear elasticity.
To develop a specific model, here we choose the yield and potential functions that directly emanate from the Mohr–Coulomb failure criterion. Note, however, that the proposed approach can also be applied to a family of related models that takes the same plasticity parameters, say the friction angle and the dilation angle. Examples include the Drucker–Prager model, which is also widely used in research and practice.

2.1 Yield and potential functions

To begin, we briefly review the conventional Mohr–Coulomb plasticity model for cohesionless sand. We denote by \( \sigma_1 \) and \( \sigma_3 \) the maximum and minimum principal stresses, respectively, with \( \Delta \sigma_1 = \sigma_1 - \sigma_3 \) being the (Cauchy) stress tensor, and denote by \( \phi \) the friction angle of the sand is denoted by \( \phi \). The yield function of the Mohr–Coulomb plasticity model can be written as follows:

\[
f = f(\sigma_1, \sigma_3, \phi) := \frac{\Delta \sigma_1}{2} - \sigma_1 + \sigma_3 \sin \phi \leq 0.
\]

The material undergoes elasto-plastic deformation when \( f = 0 \), whereas it undergoes purely elastic deformation when \( f < 0 \).

In addition to the yield function, a flow rule is necessary to determine the magnitude and the direction of plastic deformation. Without loss of generality, we consider infinitesimal deformation theory throughout. Then, as standard, it is postulated that the infinitesimal strain tensor \( \varepsilon \) is additively decomposed into an elastic part \( \varepsilon^e \) and a plastic part \( \varepsilon^p \), ie, \( \varepsilon = \varepsilon^e + \varepsilon^p \). When the material is yielding \( (f = 0) \), the rate of the plastic strain is given by the flow rule

\[
\dot{\varepsilon}^p = J \frac{\partial g}{\partial \sigma},
\]

where \( J \) is the plastic multiplier and \( g \) is the plastic potential function. The flow rule is associative if \( g = f \), and non-associative otherwise. For sands, a non-associative flow rule is supported by experimental evidence. Thus, we consider the following form of the potential function:

\[
g = g(\sigma_1, \sigma_3, \psi) := \frac{\Delta \sigma_1}{2} - \sigma_1 + \sigma_3 \sin \psi,
\]

where \( \psi \) is the dilatancy angle. The flow rule is non-associative when \( \psi \neq \phi \). Theoretically, the dilatancy angle must be smaller than the friction angle for positive energy dissipation during plastic deformation. Experimental results also suggest \( \psi < \phi \) for sands.

For geomaterials, it is often more convenient to express yield and potential functions in terms of the following three invariants of the stress tensor: the mean normal stress \( p \), the deviator (von Mises) stress \( q \), and Lode's angle \( \theta \). The three invariants are formally defined as follows:

\[
p := \frac{1}{3} \text{tr}(\sigma), \quad q := \sqrt{3} J_2, \quad \theta := \frac{1}{3} \sin^{-1} \left( \frac{3 \sqrt{3} J_3}{2 J_2^{3/2}} \right),
\]

where \( \text{tr}() \) is the trace operator, and \( J_2 \) and \( J_3 \) are the second and third invariants of the deviatoric stress tensor, \( s := \sigma - p \mathbf{1} \), respectively, with \( \mathbf{1} \) denoting the second-rank identity tensor. Here, Lode's angle is defined in the range of \(-30^\circ \leq \theta \leq 30^\circ \) such that \( \theta = -30^\circ \) under pure extension, \( \theta = 0^\circ \) under pure shear, and \( \theta = 30^\circ \) under pure compression. In terms of these three invariants, the yield and potential functions of the Mohr–Coulomb plasticity, respectively, can be written as follows:

\[
f = f(p, q, \theta, \phi) := \zeta \frac{q}{\sqrt{3}} - p \sin \phi \leq 0,
\]

\[
g = g(p, q, \theta, \psi) := \zeta \frac{q}{\sqrt{3}} - p \sin \psi,
\]

where

\[
\zeta(\theta, \phi) := \cos \theta - \frac{1}{\sqrt{3}} \sin \phi \sin \theta.
\]

An advantage of this type of representation is that one can adjust \( \zeta \) in order to smoothly approximate the compression/tension corners of the Mohr–Coulomb yield surface (at \( \theta = \pm 30^\circ \)). Here, we use a popular approximation function proposed by Abbo and Sloan\(^{59} \) for \( |\theta| \geq 28^\circ \), for simulating triaxial compression tests later on. Note, however, that a smooth approximation is not an essential component of the proposed approach.
As shown above, the Mohr–Coulomb plasticity for cohesionless soil requires two input parameters: (1) the friction angle, \( \phi \), and (2) the dilation angle, \( \psi \). It is well known that the values of the two parameters depend heavily on the current state of stress and density rather than being material constants. Nevertheless, they are often assumed to be material constants such that the model is perfectly plastic. This is mostly because there is lack of a widely accepted hardening/softening law for Mohr–Coulomb plasticity for sands, as well as because currently available hardening/softening laws entail significant additional efforts for parameter determination and calibration. This problem is tackled in the sequel.

### 2.2 Incorporation of relative density effects

In this section, we devise a way to incorporate the effect of relative density on the Mohr–Coulomb parameters, without any parameter that needs to be calibrated. For this purpose, we draw heavily on empirical relationships proposed in the soil mechanics literature. Let us first consider the peak friction angle, \( \phi \). Physically, the peak friction angle is determined by the combination of various particle-scale mechanisms: interparticle friction, particle rearrangement, crushing, and dilations, among others. In his seminar paper,\(^5\) Bolton analyzed triaxial and plane strain test data of 17 sands at various densities and confining pressures. Based on the data, he proposed that the peak friction angle of sand is well correlated with the following empirical equation (angles in degrees):

\[
\phi = \phi_{\text{crit}} + 0.8\psi,
\]

where \( \phi_{\text{crit}} \) is the critical state friction angle. Observe that this equation decomposes the peak friction angle into the constant volume part \( \phi_{\text{crit}} \) and the dilation part \( 0.8\psi \). This type of decomposition of soil strength, which dates back to the earlier works of Taylor\(^6\) and Rowe,\(^7\) is generally agreed in the soil mechanics community. Equation 8 is particularly useful for the purpose of Mohr–Coulomb modeling because it explicitly contains the dilation angle.

Experimental evidence suggests that, for a given sand, the dilation angle \( \psi \) is a function of the current state, whereas the critical state friction angle \( \phi_{\text{crit}} \) may be considered a constant material parameter at a strain rate typical in geotechnical engineering. Therefore, accurate modeling of sand plasticity must take into account the dependence of the dilatancy on the state of sand. In advanced constitutive models for sands, a common approach to incorporating the state-dependent dilatancy is to introduce the theory of critical state soil mechanics and the state parameter suggested by Been and Jeffries.\(^8\)

While it is possible to adopt the state parameter and critical state soil mechanics to the conventional Mohr–Coulomb plasticity, it would add non-standard parameters that require further calibration. This additional complexity may render the resulting Mohr–Coulomb plasticity far less practical than the conventional one.

For this reason, this work seeks to incorporate the state dependency of the dilatancy angle as phenomenologically as possible. In the same paper,\(^5\) Bolton proposed another empirical relationship of

\[
\phi - \phi_{\text{crit}} = \begin{cases} 
3I_R^\text{t} & \text{for triaxial compression conditions,} \\
5I_R^s & \text{for plane strain conditions,}
\end{cases}
\]

where \( I_R \) is the relative dilatancy index proposed in the paper, given by \( \psi \) (in kPa)

\[
I_R := D_r(10 - \ln \rho) - 1.
\]

Here, \( D_r \) is the relative density\(^7\) and defined as follows:

\[
D_r := \frac{e_{\text{max}} - e}{e_{\text{max}} - e_{\text{min}}},
\]

where \( e_{\text{max}} \) and \( e_{\text{min}} \) are the maximum and minimum void ratios, respectively. It is noted that this is the standard definition of relative density in geotechnical engineering, which may date back to Burmister.\(^6\)

Combining Equations 8-10, we arrive at the following simple equation for the dilation angle:

\[
\psi = kI_R = k[D_r(10 - \ln \rho) - 1],
\]

where \( k = 3/0.8 = 3.75 \) for triaxial compression conditions and \( k = 5/0.8 = 6.25 \) for plane strain conditions. One may use \( k = 3.75 \) for examining the performance of the model with triaxial test data, and use \( k = 6.25 \) for applying the model to practical geotechnical problems that are usually approximated to be in plane strain conditions. Note that Equation 12 agrees well with critical state soil mechanics theory in which the state of soil is determined by the confining pressure and the density (void ratio).

\(^\dagger\)This corresponds to \( I_p \) in Bolton.\(^5\) Here, it is denoted by \( D_r \), which is a more common notation in geotechnical engineering.
To summarize, we have proposed to replace the friction angle and the dilation angle in the conventional Mohr–Coulomb plasticity with Equations 8 and 12, respectively. Both the friction and dilation angles then become state variables depending on the confining pressure and the void ratio (relative density), which are the major two variables controlling the mechanical behavior of granular soils. This means that the effect of density has been incorporated to the yielding hardening/softening, and plastic flow of the model. The input parameters are \( \phi_{\text{crit}}, e_{\text{max}}, e_{\text{min}}, \) and \( e_0 \) (the initial value of \( e \)), all of which are readily measurable by standard laboratory tests. No parameter calibration is necessary. For implementation, the current void ratio \( e \) is the only variable that needs to be newly calculated for \( IR \), since \( p \) is already calculated in the conventional Mohr–Coulomb plasticity. It is noted that changes in the void ratio \( e \) during the course of loading can be easily calculated from the volumetric strain, \( tr \epsilon \), and the initial void ratio, \( e_0 \).

Several remarks on the proposed approach are made in the following:

- **Applicability and limitations.** For obvious reason, the applicability of the proposed approach is tied to that of Bolton’s empirical equations. Since proposed in 1986, these empirical equations have been extensively used for the analysis and prediction of a variety of problems in geotechnical engineering. Remarkably, it has also been demonstrated that critical state lines constructed by setting \( IR = 0 \) in Equation 10 show good agreement with experimental data (see other works \( ^{15,45,57} \) for example). This suggests that the empirical equations could work well for all values of \( \phi \geq \phi_{\text{crit}} \) mobilized after the peak stress, as stated by Bolton.\(^{54}\) These previous studies may justify the application of Bolton’s equations to Mohr–Coulomb plasticity modeling of various problems in geotechnical engineering. Nevertheless, one must aware the limitations of the empirical equations. For example, the equations do not consider some important properties that can greatly affect the strength and dilatancy of sands, such as fabric and particle shapes (see, eg, other works \( ^{53-67} \) for the significance of these properties). Also importantly, the equations have been derived from experimental data under axisymmetric and plane strain conditions, so their validity becomes increasingly questionable as the stress/strain condition deviates more from these two conditions. See the original paper of Bolton\(^{54}\) for more detailed and extensive discussions on the limitations of these equations. At the same time, however, it is also noted that many of these limitations already exist in the conventional Mohr–Coulomb model.

- **Upper and lower bounds of \( IR \).** Bolton’s empirical equations have been suggested to be valid for \( IR \) in between 0 and 4. Since \( IR = 0 \) means that the sand is at the critical state, the sand is loose (contractive) if \( IR \) is negative. Loose sands are inappropriate to be modeled by Mohr–Coulomb plasticity anyway, as mentioned in Introduction. Also, when the confining pressure is very low, \( IR \) may become too large. For this reason, it has been suggested to limit the value of \( IR \) to be smaller than 4. Usually, \( 0 < IR < 4 \) is satisfied without special treatment.

- **Unit of \( IR \).** As typical in empirical parameters in geotechnical engineering, the unit of \( IR \) in Equations 9 and 10 are inconsistent. While Bolton’s original equations may be slightly revised to make it consistent, this is not done for the reader’s familiarity as well as for simplicity. One just needs to be aware of two units: degrees for the angles, and kPa for the confining pressure.

- **Alternative expression for \( IR \).** Although Equation 10 is commonly referred to in the literature, an alternative expression for \( IR \) has also been put forward by Bolton in his response to the discussion raised by Tatsuoka.\(^{68}\) The alternative expression is given by

\[
IR := \begin{cases} 
D_r[5 - \ln(p/150)] - 1 & \text{for } p > 150 \text{ kPa}, \\
5D_r - 1 & \text{for } p \leq 150 \text{ kPa}.
\end{cases}
\]  

(13)

It seems that Equations 10 and 13 have not been directly compared in the literature. Figure 1 juxtaposes the two expressions in the range of \( 0 \leq p \leq 2000 \text{ kPa} \) (for \( D_r = 0.6 \)). It can be seen that the revised Equation 13 gives lower values of \( IR \) throughout. We have tested the two equations, and found that the original Equation 10 gives results closer to experimental data.

- **Alternative relationship between \( IR \) and \( \psi \).** In 1996, Schanz and Vermeer\(^{69}\) proposed an equation to unify the two expressions in Equation 9. The equation is

\[
\psi = \sin^{-1} \left( \frac{0.3IR}{2 + 0.3IR} \right).
\]  

(14)

Note that the definition of \( IR \) here corresponds to the original one, Equation 10. They, however, did not show how this equation is different from the two expressions in Equation 9 for triaxial and plane strain conditions. This difference is illustrated in Figure 2 in case \( \phi_{\text{crit}} = 32^\circ \). The figure shows that Equation 14 leads to somewhat a high dilation angle rather than the average of the two expressions in Equation 9. We have also observed that Equation 14
FIGURE 1  Comparison between two relationships between $p$ and $I_R$: Bolton’s original equation in the 1986 paper, Equation 10, and revised equation in the 1987 discussion paper, Equation 13 [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2  $\psi$ versus $I_R$ from the equation proposed by Schanz and Vermeer, in comparison with Bolton’s original equations under triaxial (TX) and plane strain (PS) conditions [Colour figure can be viewed at wileyonlinelibrary.com]

often leads to an overestimation of the soil strength under triaxial compression tests. Therefore, for validation of the approach proposed in this work, here we use Bolton’s original expressions. Regardless, we note that Equation 14 and/or Equation 13 can readily substitute Equation 12 and/or Equation 10 if justified by experimental results of a specific sand, and that this replacement does not entail any significant additional effort. This is true for other modified versions of Bolton’s equations as well.

2.3 Parameter determination

As described above, the proposed model takes four material parameters: (1) the maximum void ratio, $e_{\text{max}}$, (2) the minimum void ratio, $e_{\text{min}}$, (3) the initial void ratio, $e_0$, and (4) the critical state friction angle, $\phi_{\text{crit}}$. Among them, $e_{\text{max}}$ and $e_{\text{min}}$ can be readily measured by standard test methods, eg, ASTM D4253 and D4254 respectively. Notably, variations in the limiting void ratios measured by different methods are usually smaller than 0.02. The initial void ratio can also be determined in a standard manner. Also, if the model is used in coupled deformation-flow simulation, this is not an additional parameter since porosity (equivalently, void ratio) is already an input parameter for the coupled simulation. Furthermore, because the relative density is very widely used in geotechnical engineering, several empirical relations are available for estimating the relative density in practice. An example is the empirical relation between the relative density and the standard penetration test value. The relative density has also been related to qualitative descriptions of sand density. Such a
TABLE 1  Classification of sand density according to the value of relative density. From figure 4.1 of Mitchell and Soga.45

<table>
<thead>
<tr>
<th>Relative Density, $D_r$, %</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-20</td>
<td>Very loose</td>
</tr>
<tr>
<td>20-40</td>
<td>Loose</td>
</tr>
<tr>
<td>40-60</td>
<td>Medium dense</td>
</tr>
<tr>
<td>60-80</td>
<td>Dense</td>
</tr>
<tr>
<td>80-100</td>
<td>Very dense</td>
</tr>
</tbody>
</table>

relation, which seems to be first proposed by Burmister,62 is also widely used in geotechnical engineering. Table 1 is an example from Mitchell and Soga.45 Note that this classification is subjective in that it differs slightly by people.

The critical state friction angle, $\phi_{\text{crit}}$, can be obtained by performing a series of shear tests on soil samples of various densities, as explained in Bolton.54 This way uses the same type of experiment for the conventional Mohr–Coulomb model. Alternatively, one can use a simple test suggested by Santamarina and Cho73 that measures the angle of repose. In either case, it has been reported that a measurement error for $\phi_{\text{crit}}$ usually does not exceed 1°. Interestingly, if $\phi_{\text{crit}}$ is determined from the angle of repose, all the parameters of the proposed Mohr–Coulomb model can be obtained without a single shear test.

In some sense, the parameters in the proposed model can be determined more objectively than those in the conventional Mohr–Coulomb model. Firstly, the dilation angle in the conventional model is often roughly approximated because it indeed evolves continuously throughout plastic deformation. The peak friction angle in the conventional model can be measured by performing multiple shearing tests. However, it is well known that the measured value of the peak friction angle can differ by the specific test method used. For example, it is not uncommon that the peak friction angles measured by triaxial, plane strain, and direct shear tests are all significantly different. On the other hand, the four parameters in the proposed model are less sensitive to the measurement method, as described above. In this regard, although the proposed approach increases the number of parameters from two to four, it may allow a more objective determination of the model parameters.

2.4  | Elasticity

The approach described above is completely independent of elasticity. Regardless, it should be combined with an elasticity model to complete an elasto-plastic model. Since soils show highly non-linear and complex elasticity responses,74–77 advanced constitutive models often employ pressure-dependent non-linear elasticity models. Although such non-linear elasticity is fully compatible with Mohr–Coulomb plasticity, it would be rather inconsistent to use a complex elasticity model in conjunction with Mohr–Coulomb plasticity. Thus, here we just use the simplest elasticity model, namely, isotropic linear elasticity. As is well known, it requires two independent material parameters, such as Young’s modulus and Poisson’s ratio.

2.5  | Numerical implementation

Since the proposed approach introduces a hardening/softening law derived from Bolton’s empirical equations, it requires one to modify an existing implementation of the conventional Mohr–Coulomb model. The modification is also simple, however. In what follows, it is described in the context of implicit stress integration.

Among several possible choices for implicit integration, here we use the technique of return mapping in principal strain space described in chapter 6 of Borja.78 The technique is chosen since it is general enough for accommodating non-linear elasticity and finite deformation, if necessary. For brevity, we do not repeat the description of the return mapping
technique explained in Borja. Instead, specific points that need to be modified for the present Mohr–Coulomb model are concisely described in the following. In the process of return mapping in principal strain space \((\varepsilon_1, \varepsilon_2, \text{and} \varepsilon_3)\), the (local) residual vector \(r\) and the (local) unknown vector \(x\) are given by

\[
\begin{align*}
    r(x) &= \left\{ \begin{array}{l}
    \varepsilon_1^e - \varepsilon_1^{e\text{tr}} + \Delta \lambda \partial g_1 \\
    \varepsilon_2^e - \varepsilon_2^{e\text{tr}} + \Delta \lambda \partial g_2 \\
    \varepsilon_3^e - \varepsilon_3^{e\text{tr}} + \Delta \lambda \partial g_3 \\
    \bar{f}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \psi) \\
    \bar{h}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \psi)
    \end{array} \right\}, \\
    x &= \left\{ \begin{array}{l}
    \varepsilon_1^e \\
    \varepsilon_2^e \\
    \varepsilon_3^e \\
    \Delta \lambda \\
    \psi
    \end{array} \right\},
\end{align*}
\]

where \(\partial g_A := \partial g / \partial \sigma_A\) for \(A = 1, 2, 3\), the superscript \((\cdot)^{\text{tr}}\) denotes the trial value, and \(\Delta \lambda\) denotes the increment in the plastic multiplier. In the residual vector, the first three components come from the predictor–corrector forms of the three principal strains. The fourth component, \(\bar{f}\), is the discrete consistency condition, and it takes \(\psi\) as an argument since \(\phi = \phi_{\text{crit}} + 0.8 \psi\). The fifth component, \(\bar{h}\), is the discrete hardening law, given by

\[
\bar{h} := \psi - k \left( \frac{e_{\text{max}} - e}{e_{\text{max}} - e_{\text{min}}} (10 - \ln p) - 1 \right).
\]

It is noted that \(p\) is a function of the elastic strains, while \(e\) is a function of the trial strains.

The return mapping technique uses Newton’s method to find an \(x\) making \(r(x) \approx 0\). At the \(k\)th Newton iteration, the tangent (Jacobian) matrix is given by

\[
A^k = r'(x^k) =
\begin{bmatrix}
    1 + \Delta \lambda \partial g_1 / \partial \varepsilon_1^e & \Delta \lambda \partial g_1 / \partial \varepsilon_2^e & \Delta \lambda \partial g_1 / \partial \varepsilon_3^e & \partial g_1 & \Delta \lambda \partial g_1 / \partial \psi \\
    \Delta \lambda \partial g_2 / \partial \varepsilon_1^e & 1 + \Delta \lambda \partial g_2 / \partial \varepsilon_2^e & \Delta \lambda \partial g_2 / \partial \varepsilon_3^e & \partial g_2 & \Delta \lambda \partial g_2 / \partial \psi \\
    \Delta \lambda \partial g_3 / \partial \varepsilon_1^e & \Delta \lambda \partial g_3 / \partial \varepsilon_2^e & 1 + \Delta \lambda \partial g_3 / \partial \varepsilon_3^e & \partial g_3 & \Delta \lambda \partial g_3 / \partial \psi \\
    \partial \bar{f} / \partial \varepsilon_1^e & \partial \bar{f} / \partial \varepsilon_2^e & \partial \bar{f} / \partial \varepsilon_3^e & 0 & \partial \bar{f} / \partial \psi \\
    \partial \bar{h} / \partial \varepsilon_1^e & \partial \bar{h} / \partial \varepsilon_2^e & \partial \bar{h} / \partial \varepsilon_3^e & 0 & 1
\end{bmatrix}.
\]

Here, the partial derivatives with respect to the elastic principal strains can be conveniently obtained by the chain rule. Note that whenever \(\psi\) is updated, \(\phi\) is also updated as \(\phi = \phi_{\text{crit}} + 0.8 \psi\).

Once the local residual is converged, one can obtain the (global) algorithmic tangent operator which is essential for optimal convergence in boundary value problems. The only necessary change is to calculate the derivatives of the discrete hardening law with respect to the trial principal strains, because the void ratio \(e\) is a function of the trial volumetric strain. This change is trivial, and the reader is referred to Borja for a complete derivation of the algorithmic tangent operator.

### 3 MODEL EVALUATION

In this section, we evaluate the proposed Mohr–Coulomb model with respect to two aspects: (1) whether it can capture experimental results of different sands at various densities and confining pressures, and (2) whether it can well predict experimental results of different sands at various densities and confining pressures. For an objective assessment of the model, we purposely select three different sands that were not included in the database of Bolton’s work. They are Toyura, Erksak, and Fuji River sands. The material parameters for the three sands are directly adopted from the literature, and their specific values are summarized in Table 2. It is noted at the outset that only dilative sands subjected to drained and monotonic loading are considered, since they are the target of Mohr–Coulomb modeling. We also exclude experimental data that appear to be significantly affected by strain localization, because they involve heterogeneous deformation. For

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Toyura\textsuperscript{79}</th>
<th>Erksak\textsuperscript{34,80}</th>
<th>Fuji River\textsuperscript{42,81}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum void ratio, (e_{\text{max}})</td>
<td>0.977</td>
<td>0.963</td>
<td>1.123</td>
</tr>
<tr>
<td>Minimum void ratio, (e_{\text{min}})</td>
<td>0.597</td>
<td>0.525</td>
<td>0.489</td>
</tr>
<tr>
<td>Critical state friction angle, (\phi_{\text{crit}})</td>
<td>31.15°</td>
<td>30°</td>
<td>36.64°</td>
</tr>
</tbody>
</table>
this reason, we focus on data of triaxial compression tests, since strain localization is far more pronounced in plane strain compression tests. All the simulation results in the following are obtained using the algorithm described in Bardet and Choucair, with the strain rate of 0.05% per load step. The implicit integration algorithm described in the previous section is used in doing so.

### 3.1 Model behavior at different relative densities

Firstly, to investigate the basic behavior of the model, we simulate triaxial compression tests for sands at three different relative densities: (1) $D_r = 40 \%$, (2) $D_r = 70 \%$, and (3) $D_r = 100 \%$. According to Table 1, they correspond to medium dense, dense, and very dense sands, respectively. The material is assumed to be Toyura sand, and it is subjected to an isotropic confining pressure of 100 kPa. The elasticity parameters are set as $E = 90$ MPa and $\nu = 0.25$. The triaxial tests are conducted until the axial strain reaches 10%.

Figure 3 presents the simulation results of triaxial compression tests for the three different densities. The model produces a higher peak strength and dilatancy for a denser soil, confirming that it has correctly incorporated the effect of relative density on sand behavior. The densest one ($D_r = 100\%$) shows softening throughout, whereas the loosest one ($D_r = 40\%$) is more or less perfectly plastic. All the three cases show shear-induced dilatancy after yielding, and dilatancy becomes more pronounced as the material becomes denser. The kinks at the yield points are very sharp, which is typical in Mohr–Coulomb plasticity models combined with linear elasticity.

To demonstrate the efficiency and correctness of the computational implementation, Figure 4 shows the convergence profiles during Newton iterations for the local return mapping and global triaxial loading processes at the last load step. It can be seen that both the local and global iterations converge optimally, i.e., asymptotically quadratic. In both of the local

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**FIGURE 3** Simulation of the drained triaxial tests of Toyura sand at different relative densities. The confining pressure is 100 kPa. A, Deviator stress versus axial strain. B, Volumetric strain versus axial strain [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 4** Convergence profiles during A, local Newton iterations for return mapping and B, global Newton iterations for triaxial loading, at the last load step [Colour figure can be viewed at wileyonlinelibrary.com]
and global iterations, the convergence rate is slightly slower for a denser sand because it manifests more softening. Nevertheless, all the local iterations converge at the third iteration (for tolerance $10^{-9}$), and all the global iterations converge at the fifth iteration (for tolerance $10^{-10}$). The results thus show that the model can be efficiently used for computationally intensive simulations.

At this point, we remark that the conventional Mohr–Coulomb model—which does not consider relative density—would produce the same results for the three density cases as they are under the same confining pressure. In other words, if the three cases were simulated by the conventional Mohr–Coulomb model, the three curves in each plot of Figure 3 would collapse into a single curve. Therefore, the foregoing results have demonstrated that the proposed approach can provide the ability to capture changes in the stress–strain–strength response of a sand solely due to density differences. In what follows, we examine whether this model can reasonably predict experimental results of different sands at various densities and confining pressures.

### 3.2 Comparison with experimental data

Now, we investigate the predictive capabilities of the model by simulating laboratory experiments in the literature. Toyura sand is considered first among the three sands in Table 2. We specifically simulate the triaxial tests of Verdugo and Ishihara,79 of which results have been commonly used for validation of several advanced constitutive models.38,40,42 From these results, we select two tests that exhibit dilation under monotonic loading. The first test is conducted at the initial void ratio $e_0 = 0.831$ and the isotropic confining pressure $p_c = 100$ kPa, and the second one is conducted at $e_0 = 0.810$ and $p_c = 500$ kPa. Both of the tests are performed up to an axial strain around 25%.

Figure 5 compares the model predictions of the two tests with the experimental data of Verdugo and Ishihara.79 It can be seen that, for both tests, the proposed Mohr–Coulomb model can predict the peak stresses fairly closely. In terms of the deviator stress-axial strain response, the most significant difference between the model prediction and the experimental data is that the transition from elasticity to plasticity is more abrupt in the model. As discussed previously, this is rather unavoidable due to the use of linear elasticity and Mohr–Coulomb plasticity. Other than that, however, the predicted stress-strain curves match well with the experimental data at two different combinations of densities and confining pressures. As for the volumetric behavior, the model slightly underpredicts the dilation in the case of $e_0 = 0.831$ and $p_c = 100$ kPa, but it predicts the dilation of the other case ($e_0 = 0.810$ and $p_c = 500$ kPa) quite well.

Next, we consider the triaxial test results of Erksak sand that were used to validate the Nor-Sand model by Jefferies.34 The critical state friction angle is determined from the slope of the critical state line of the Nor-Sand model in that paper. The maximum and minimum void ratio values of Erksak sand are not presented in the same paper, so they are taken from Jefferies and Been.80 The elasticity parameters are assigned as $E = 60$ MPa and $\nu = 0.25$. Same as the Toyura sand case, we use two sets of experimental results in which the sand shows dilative behavior. The initial void ratio and the confining pressure are $e_0 = 0.590$ and $p_c = 130$ kPa for the first set, and $e_0 = 0.675$ and $p_c = 300$ kPa for the second set.

**FIGURE 5** Model prediction of the triaxial compression tests of Vergudo and Ishihara79 for Toyura sands. A, Deviator stress versus axial strain. B, Deviator stress versus void ratio [Colour figure can be viewed at wileyonlinelibrary.com]
The model predictions and the experimental results of Erksak sand are compared in Figure 6. As shown, the model again predicts the sand’s strengths fairly well for both combinations of densities and confining pressures. Although some differences are observed between the predicted volumetric responses and the experimental data, they are not very significant from a practical point of view, and they are also due to elastic compression prior to plastic dilation. Importantly, the model predicts higher dilation for the case of $e_0 = 0.590$ and $p_c = 130$ kPa than the other, being consistent with the experimental results. Such change in dilatancy cannot be captured by the conventional Mohr–Coulomb model assuming a constant dilation angle.

Lastly, we consider the experimental data of Tatsuoka for Fuji River sand, which have been used for validating advanced constitutive models of Ling and Yang, among others. The critical state friction angle is calculated from the slope of the critical state line reported in Ling and Yang. The maximum and minimum void ratios are computed from two values of relative density mentioned in the same paper, $D_r = 51\%$ for $e = 0.80$ and $D_r = 92\%$ for $e = 0.54$. For simplicity, we use the same elasticity parameters used for Erksak sand, because they reasonably reproduce the initial portions of the stress-strain curves. This time, we simulate three triaxial tests conducted on dense sands, whose initial void ratios and confining pressures are $e_0 = 0.496$ and $p_c = 98$ kPa, $e = 0.519$ and $p_c = 196$ kPa, and $e_0 = 0.515$ and $p_c = 294$ kPa, respectively.

Figure 7 shows the proposed model’s prediction results compared with the experimental data of Fuji River sand. The model again provides decent predictions of the strengths of Fuji River sands at three different densities and confining pressures. The model predictions of the volumetric responses are also good, except the most dilative case after about 4% of axial strain. Given the shape of the experimental curves, this difference may be in part due to the inception of shear band in the experimental specimen, although the exact point of strain localization is unknown from the literature. Other than this, the predictions of the proposed Mohr–Coulomb model for Fuji River sand are as good as those for the other two sands.
The foregoing comparisons indicate that the proposed Mohr–Coulomb model can give reliable predictions of the strengths of different sands at various densities and confining pressures. Remarkably, all these predictions have been made without calibration of a single parameter of the Mohr–Coulomb model. Given that the primary goal of most Mohr–Coulomb modeling is to predict the strength, it can be concluded that the proposed Mohr–Coulomb model has the ability to capture the effects of relative density on dilative sands.

4 | CLOSURE

This paper has introduced a simple approach to incorporating density effects into Mohr–Coulomb plasticity for dilative sand. Exploiting Bolton’s empirical equations, the approach changes the friction and dilation angles to be state variables that depend on the current density and confining pressure, from constant material parameters as in the conventional Mohr–Coulomb model. As a result, void ratios for calculating the initial relative density and the critical state friction angle become material parameters of the proposed model. These parameters can be readily measurable through standard procedures without calibration. A Mohr–Coulomb model enhanced in this way well predicted the strengths of different sands at various densities and confining pressures. Thus, the proposed Mohr–Coulomb plasticity model allows very efficient capture of density effects on sand behavior, which would be useful for various problems from practical geotechnical analyses to computationally intensive simulations.

Future work of this study includes the application of the proposed model to boundary-value problems. It would reveal whether Bolton’s equations could remain reasonably valid for general loading conditions, which is an important question not only for the present model but also for other classes of geotechnical problems to which Bolton’s equations have been applied. Another, and related, question is whether the Mohr–Coulomb model could also capture persistent shear band in heterogeneously dense sand, as the Nor-Sand model did in previous studies. Investigation of this question would allow us to further validate the model for problems involving strain localization.

Moreover, the proposed approach may be generalized to a wider class of sandy soils by using an extended version of Bolton’s equations. For example, Xiao et al. have modified the strength-dilatancy equation to accommodate the effect of nonplastic fines. Fern and Soga have shown that the relative density index can also be correlated with the dilatancy of sands with different water contents. The findings of these studies suggest that it may be possible to develop another simple Mohr–Coulomb model for silty or unsaturated sand. Studies on these topics will be reported in future publications.

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REFERENCES


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