Stabilized mixed continuous/enriched Galerkin formulations for locally mass conservative poromechanics

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Abstract

Local (element-wise) mass conservation is often highly desired for numerical solutions to coupled poromechanical problems. As an efficient numerical method featuring this property, mixed continuous Galerkin (CG)/enriched Galerkin (EG) finite elements have recently been proposed in which piecewise constant functions are enriched to the pore pressure interpolation functions of the conventional mixed CG/CG elements. While this enrichment of the pressure space provides local mass conservation, it unavoidably alters the stability condition for mixed finite elements. Because no stabilization method has been available for the new stability condition, high-order displacement interpolation has been required for mixed CG/EG elements if undrained condition is expected. To circumvent this requirement, here we develop stabilized formulations for the mixed CG/EG elements that permit equal-order interpolation functions even in the undrained limit. We then derive two types of stabilized formulations, one based on the polynomial pressure projection (PPP) method and the other based on the fluid pressure Laplacian (FPL) method. A key finding of this work is that both methods lead to stabilization terms that should be augmented only to the CG part of the pore pressure field, not to the enrichment part. The two stabilized formulations are verified and investigated through numerical examples that involve various conditions spanning 1D to 3D, isotropy to anisotropy, and homogeneous to heterogeneous domains. The methodology presented in this work may also help stabilize other types of mixed finite elements in which the constraint field is enriched by additional functions.

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1. Introduction

Coupled poromechanical simulations serve as a useful tool to address many problems that involve strong interactions between fluid flow and solid deformation in porous materials. Examples include landslides [1,2], soil liquefaction [3,4], ground subsidence [5,6], induced seismicity [7,8], geologic carbon sequestration [9,10], and hydraulic fracturing [11–15].

Numerical approximations of coupled poromechanical problems require spatial discretization of two sub-problems, namely, the solid deformation problem and the fluid flow problem. The two problems may be discretized
by the same type of numerical method, or by different types of methods. In the literature, the most common choice might be two-field mixed continuous Galerkin (CG) finite elements (FE), in which both the solid deformation and fluid flow problems are approximated by the CG method. See [16–21] for some of the recent works that have used mixed CG/CG elements for poromechanical problems and their extensions. In this combination, the use of the CG method for the solid deformation problem is a natural and well-justified choice.

However, for numerical modeling of flow in porous media, the CG method is often not preferred because it lacks the property of local (element-wise) mass conservation. This property is particularly important when the flow problem involves a highly heterogeneous permeability field and/or a transport phenomenon. For this reason, numerical simulations of flow in porous media often employ locally conservative methods, such as the mixed velocity/pressure finite element method [22], the finite volume (FV) method [23], the discontinuous Galerkin (DG) method [24], and the enriched Galerkin method (EG) [25–28]. These methods have also been extended to coupled poromechanical problems by being combined with the CG method for solid deformation problems. These combinations have led to three-field mixed finite elements [29–32], mixed FE/FV [33–35], mixed CG/DG [36,37], and mixed CG/EG elements [38–40]. Among them, the mixed CG/EG elements is the focus of this work.

The mixed CG/EG finite element method for poromechanics has recently been proposed and used for providing local mass conservation to mixed CG/CG elements with relatively little additional cost [38–40]. The EG method is a locally conservative method that solves the same bilinear form as the DG method by augmenting piecewise constant functions to the CG finite element space. The EG and DG methods give very similar numerical solutions for a given mesh and polynomial order, but the EG method entails substantially fewer degrees of freedom than the DG method [25,26]. As such, the mixed CG/EG method is one of the most efficient locally mass conservative method for coupled poromechanical problems. Furthermore, the method can handle arbitrary grid and permeability configurations, which is a significant advantage over mixed FE/FV methods employing a standard linear two-point flux approximation scheme. Details of the formulation and performance of the mixed CG/EG formulations are presented in recent papers [38,39].

Despite its contribution to local mass conservation, the pressure space enrichment in the mixed CG/EG elements unavoidably gives rise to a new inf–sup stability condition that is different from the standard inf–sup condition for mixed CG/CG elements [41]. In coupled poromechanics, inf–sup stability is critical to undrained problems wherein the pore pressure field acts as an incompressibility constraint for solid deformation [42,43]. Such undrained condition arises when the solid and fluid phases are nearly incompressible and the permeability or time step is not large enough to induce relative flow between the solid and fluid phases. Undrained condition is common in most engineering problems dealing with low-permeability materials such as clays and shales. Also, even for materials with relatively high permeability, undrained condition may emerge especially when the time domain is finely discretized to accurately capture a rather rapid process. Notable examples include fluid-induced fracture and fault activations. For example, Santillán et al. [44] have reported that very small time steps (smaller than a microsecond) are typically required to simulate unstable growth of hydraulic fracture. Prévost and Sukumar [45] have also concluded that undrained condition is relevant for fault activation problems in both high and low permeable rocks.

When the inf–sup stability condition is not satisfied, the discrete system is not well-posed and may admit non-physical pressure modes which typically cause checkerboard oscillations. A straightforward way to avoid this problem is to employ high-order interpolation of the displacement field. However, high-order displacement interpolation may require prohibitively large computational cost, particularly for 3D, large-scale, and/or nonlinear problems. For this reason, it is often compelling to use equal-order (linear) mixed finite elements by circumventing the inf–sup condition. This has motivated the development of several stabilization methods that enable the use of equal-order interpolation for the mixed CG/CG elements for poromechanics [9,46–56]. Unfortunately, the existing stabilized formulations cannot be applied to the mixed CG/EG elements, because they are subjected to a new form of inf–sup stability condition for which no stabilization method has been proposed in the literature. Due to this lack of a stabilized formulation, so far mixed CG/EG elements have resorted to high-order displacement interpolation when undrained condition is expected [38,39]. Therefore, it is desired to develop a new stabilization method for enriched mixed finite elements to minimize the computational cost of the mixed CG/EG method.

In this paper, we formulate stabilized versions of mixed CG/EG elements that permit equal-order interpolation functions for undrained poromechanical problems. To tackle the new stability problem arising from the pressure enrichment, we first phrase the mixed CG/EG formulation as a twofold saddle point problem. This way allows us to identify that the inf–sup condition for mixed CG/EG elements is form-identical to the standard inf–sup condition for...
mixed CG/CG elements, the difference being that the pressure function in the stability condition is now an addition of the CG-approximated part and the piecewise constant enrichment part. Building on this insight, we draw on two methods originally developed for stabilizing mixed CG/CG formulations, namely the polynomial pressure projection (PPP) method [50,57] and the fluid pressure Laplacian (FPL) method [9,49]. These methods are particularly chosen as they do not require change in existing algorithms, unlike operator splitting methods such as [46,47,56]. Using these methods we derive two types of stabilization terms and find that both stabilization terms should be augmented only to the CG part of the pressure field, not to the enrichment part. This means that these stabilization terms can be easily implemented without any additional shape functions, preserve the symmetry structure of the matrix system, and do not depend on the choice of a specific interior penalty method. We then verify and investigate the two stabilized formulations through numerical examples that involve various conditions spanning 1D to 3D, isotropy to anisotropy, and homogeneous to heterogeneous domains.

2. Coupled poromechanical formulation and discretization

This section briefly reviews a coupled poromechanical formulation for deformable porous materials and its mixed CG/EG discretization described in Choo and Lee [38]. As the focus of this work is on numerical instability arising in undrained condition, we specialize the formulation to two-phase porous materials composed of incompressible solid and fluid phases. Furthermore, to avoid non-essential complications, we assume the following: quasi-static in undrained condition, we specialize the formulation to two-phase porous materials composed of incompressible solid and fluid phases. Furthermore, to avoid non-essential complications, we assume the following: quasi-static condition, infinitesimal deformation, isothermal condition, and absence of mass exchange between the solid and fluid phases. It is noted that these assumptions are common in previous works that proposed stabilized finite elements for poromechanics (e.g., [9,50,52]).

2.1. Governing equations

Let \( \Omega \in \mathbb{R}^d \) denote a \( d \)-dimensional domain \( (d = 2, 3) \) occupied by a porous material and \( \partial \Omega \) denote the domain’s boundary with unit outward normal vectors \( n \). The boundary is decomposed into displacement (Dirichlet) and traction (Neumann) boundaries, \( \partial \Omega_u \) and \( \partial \Omega_t \) respectively, for the solid deformation problem, and pressure (Dirichlet) and flux (Neumann) boundaries, \( \partial \Omega_p \) and \( \partial \Omega_q \) respectively, for the fluid flow problem. The decomposed boundaries satisfy \( \partial \Omega = \partial \Omega_u \cup \partial \Omega_t = \partial \Omega_p \cup \partial \Omega_q \) and \( \emptyset = \partial \Omega_u \cap \partial \Omega_t = \partial \Omega_p \cap \partial \Omega_q \). The time interval of the problem is denoted by \( \mathbb{T} := (0, t_{\text{final}}) \) with \( t_{\text{final}} > 0 \). The solid displacement vector field and the pore pressure field are denoted by \( u \) and \( p \), respectively.

The strong form of the initial–boundary value problem can be stated as follows. Given \( \hat{u}, \hat{t}, \hat{p}, \hat{q}, u_0, \) and \( p_0 \), find \( u \) and \( p \) such that

\[
\nabla \cdot \sigma' - \nabla \cdot p + \rho g = 0 \quad \text{in} \quad \Omega \times \mathbb{T},
\]

\[
\nabla \cdot \hat{u} + \nabla \cdot q = 0 \quad \text{in} \quad \Omega \times \mathbb{T},
\]

subject to boundary conditions

\[
 u(x, t) = \hat{u}(x, t) \quad \text{on} \quad \partial \Omega_u \times \mathbb{T},
\]

\[
 n(x) \cdot \sigma(x, t) = \hat{t}(x, t) \quad \text{on} \quad \partial \Omega_t \times \mathbb{T},
\]

\[
 p(x, t) = \hat{p}(x, t) \quad \text{on} \quad \partial \Omega_p \times \mathbb{T},
\]

\[
 n(x) \cdot q(x, t) = \hat{q}(x, t) \quad \text{on} \quad \partial \Omega_q \times \mathbb{T},
\]

and initial conditions

\[
 u(x, 0) = u_0(x), \quad p(x, 0) = p_0(x),
\]

for all position vectors \( x \in \Omega \) and time \( t \in \mathbb{T} \). Here, Eq. (1) is the linear momentum balance equation governing the solid deformation problem, and Eq. (2) is the mass balance equation governing the fluid flow problem. Also, \( \sigma' \) is the effective stress tensor, \( \rho \) is the density of the porous material, \( g \) is the gravitational vector, and \( q \) is the seepage (Darcy) velocity vector. Note that the above formulation has adopted Terzaghi’s effective stress principle. An overdot denotes material time derivative with respect to the motion of the solid matrix.
The formulation is closed by two constitutive laws: a stress–strain relationship for the effective stress and the generalized Darcy’s law for the seepage velocity. They can be written, respectively, as

\[
\begin{align*}
\sigma' &= \mathbb{C} : \varepsilon, \\
q &= -\kappa \cdot (\nabla p - \rho_f \mathbf{g}),
\end{align*}
\]

(8)

where \(\varepsilon = \nabla^s \mathbf{u}\) is the infinitesimal strain tensor, \(\mathbb{C}\) is the stress–strain tangent tensor, \(\kappa\) is the mobility (the intrinsic permeability over the dynamic viscosity of the pore fluid) tensor and \(\rho_f\) is the density of the pore fluid.

In this work, we consider two types of elastic materials: (1) isotropic linear elasticity, and (2) transversely isotropic elasticity. For transversely isotropic elasticity, the tangent is

\[
\mathbb{C} = \mathbb{C}_{iso} := K (\mathbf{1} \otimes \mathbf{1}) + 2G \left( \mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right),
\]

(10)

where \(K\) and \(G\) are the bulk modulus and the shear modulus of the solid matrix, respectively, and \(\mathbf{1}\) and \(\mathbf{I}\) are the second-order and the fourth-order identity tensors, respectively. For transversely isotropic elasticity, the tangent is given by

\[
\mathbb{C} = \mathbb{C}_{\text{iso}} := \lambda (\mathbf{1} \otimes \mathbf{1}) + 2G_T \mathbf{I} + (a (\mathbf{1} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{1}) + \mathbf{b} \otimes \mathbf{m}) + (G_L - G_T) (\mathbf{1} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{1}).
\]

(11)

Here, \(\lambda, a, b, G_L, G_T\) are elasticity parameters, \(\mathbf{m}\) is a structure tensor constructed by the dyadic product of the unit normal vector to the bedding plane, \((\bullet \otimes \circ)_{ijkl} = (\bullet)_{ij} (\circ)_{jk}\), and \((\bullet \oplus \circ)_{ijkl} = (\bullet)_{ij} (\circ)_{jk}\). The above five elasticity parameters can be converted into \(E_1, E_2, \nu_{12}, \nu_{23},\) and \(G_{12}\), where \(E\) and \(\nu\) denote Young’s modulus and Poisson’s ratio, respectively, and the subscripts \(\perp\) and \(\parallel\) denote the bedding plane normal direction and the bedding plane parallel direction, respectively [64]. The mobility tensor for an isotropic material is simply \(\kappa = \kappa_1 \mathbf{1}\). When the material is transversely isotropic, \(\kappa = \kappa_\perp \mathbf{m} + \kappa_\parallel (\mathbf{1} - \mathbf{m})\) where \(\kappa_\perp\) and \(\kappa_\parallel\) denote mobility values in the bedding plane normal and parallel directions, respectively.

\subsection{2.2. Discretization}

We now discretize the above-described initial–boundary value problem using a locally mass conservative mixed finite element method that applies the CG method to the solid deformation problem (momentum balance equation) and the EG method to the fluid flow problem (mass balance equation). To begin, let the domain \(\mathcal{O}\) be partitioned by shape-regular (satisfying the minimum angle condition [65]) elements \(T\). The set of elements is denoted by \(\mathcal{T}\). The diameter of \(T\) is denoted by \(h_T\), and \(h\) is defined as the largest value of \(h_T\) in \(\mathcal{T}\). Because the objective of this work is to develop stabilized low-order finite elements, we will consider linear \((\mathcal{Q}_1)\) finite element discretization for all variables. Also, without loss of generality, we assume homogeneous Dirichlet boundary conditions for the displacement field, \(i.e., \hat{u} = 0\). Then, for CG discretization of the linear momentum balance equation (1), the finite element space for the displacement vector field and its variation can be defined as

\[
\mathcal{U}_h(\mathcal{T}_h) := \{ \eta \mid \eta \in [C^0(\mathcal{O})]^d, \eta = 0 \text{ on } \partial \mathcal{O}_u, \eta|_T \in [\mathcal{Q}_1(T)]^d, \forall T \in \mathcal{T}_h \}. \tag{12}
\]

Let us denote the CG-discretized displacement vector by \(\mathbf{u}_h \in \mathcal{U}_h(\mathcal{T}_h)\) and the discretized effective stress tensor by \(\sigma'_h\). Through the standard weighted residual procedure, we can develop a discrete variational form of the linear momentum equation as follows: Find \(\mathbf{u}_h \in \mathcal{U}_h(\mathcal{T}_h)\) for all \(\eta \in \mathcal{U}_h(\mathcal{T}_h)\)

\[
- \int_\mathcal{O} \nabla^s \eta : \sigma'_h \, dV + \int_\mathcal{O} p \nabla \cdot \eta \, dV + \int_\mathcal{O} \eta \cdot \rho_f \mathbf{g} \, dV + \int_{\partial \mathcal{O}_h} \eta \cdot \mathbf{n} \, dA = 0. \tag{13}
\]

Here, we have assumed that \(p\) is given for now.

Next, we discretize the mass balance equation (2) in space using the EG method. We begin this by defining the space of piecewise discontinuous \(k\)th degree polynomial functions as

\[
\mathcal{P}_h^{\text{DG}_k}(\mathcal{T}_h) := \{ \psi \mid \psi \in L^2(\mathcal{O}), \psi|_T \in [\mathcal{Q}_k(T)], \forall T \in \mathcal{T}_h \}. \tag{14}
\]
It is noted that Dirichlet boundary conditions will be weakly imposed for local mass conservation so they are not incorporated in the finite element space. From this definition, the CG finite element space with linear functions can be introduced as

$$\mathcal{P}_h^{CG}(T_h) := \mathcal{P}_h^{DG}(T_h) \cap C^0(\Omega),$$

and the space for piecewise constant function as \( \mathcal{P}_h^{DG_0}(T_h) \). We now define the finite element space for linear EG discretization as \([25]\)

$$\mathcal{P}_h(T_h) := \mathcal{P}_h^{CG}(T_h) + \mathcal{P}_h^{DG_0}(T_h).$$

The above equation means that the EG finite element space is constructed by enriching (adding) piecewise constant functions to the CG finite element space with the same order polynomials. As an example, EG-\( Q_1 \) is the discrete compatibility principle \([66]\). But it will also be shown that stabilization is irrelevant to the choice of a specific interior penalty method. Also, \( h_e \) is the edge length, \( \kappa \) is the harmonic mean of \( \kappa^+ \) and \( \kappa^- \), and \( \alpha \) is a penalty parameter introduced to provide some degree of inter-element continuity for stability. It has been experienced that an \( \alpha \) ranging from 100 to 1000 usually gives nearly the same and satisfactory result, and here we set \( \alpha = 100 \). See Remark 4 of Choo and Lee \([38]\) for more discussion on this parameter. It is also noted that the penalty terms at Dirichlet boundaries are not multiplied by \( \kappa \) to ensure satisfaction of boundary conditions in undrained problems.

Weighted interior penalty methods are used to construct variational formulations for the EG method. Let \( \mathcal{E}_h \) denote the set of all edges at element boundaries, which is decomposed into the set of internal edges \( \mathcal{E}_h^I \) and the set of boundary edges \( \mathcal{E}_h^B \). The set of boundary edges is further decomposed into \( \mathcal{E}_h^\partial = \mathcal{E}_h^{\partial_p} \cup \mathcal{E}_h^{\partial_q} \), where \( \mathcal{E}_h^{\partial_p} \) is the set of edges on the pressure boundary, \( \partial \Omega_p \), and \( \mathcal{E}_h^{\partial_q} \) is the set of edges on the flux boundary, \( \partial \Omega_q \). For an interior edge \( e \in \mathcal{E}_h^I \), let us denote by \( T^+ \) and \( T^- \) the two elements sharing it, and by \( n_e \) the unit normal vector oriented from \( T^+ \) to \( T^- \). At edges, the weighted average operator and the jump operator are defined as

\[
\{ \xi \} := \delta_e \xi |_{T^+} + (1 - \delta_e) \xi |_{T^-},
\]

\[
\llbracket \xi \rrbracket := \xi |_{T^+} - \xi |_{T^-},
\]

respectively. Here, \( \delta_e \in [0, 1] \) is the weight for averaging, which we calculate in this work as

\[
\delta_e = \frac{\kappa^-}{\kappa^+ + \kappa^-}, \quad \kappa^\pm := (n_e)^T \cdot \kappa |_{T^\pm} \cdot n_e. \tag{19}
\]

Then, a discrete variational formulation for the mass balance equation (2) can be constructed as follows. Find \( p_h \in \mathcal{P}_h(T_h) \) such that for all \( \psi \in \mathcal{P}_h(T_h) \)

\[
\sum_{T \in \mathcal{T}_h} \int_T \psi \nabla \cdot \hat{\mathbf{u}}_h \, dV - \sum_{T \in \mathcal{T}_h} \int_T \nabla \psi \cdot \mathbf{q}_h \, dV
+ \sum_{e \in \mathcal{E}_h^I} \int_{e} \| \psi \| p_h \| \mathbf{q}_h \| \, dA - \text{sform} \sum_{e \in \mathcal{E}_h^I} \int_{e} \| p_h \| \| (\kappa \cdot \nabla \psi) \| \, dA
+ \sum_{e \in \mathcal{E}_h^{\partial_p}} \alpha \int_{e} \phi(p_h - \hat{p})(\kappa \cdot \nabla \psi) \cdot n_e \, dA + \sum_{e \in \mathcal{E}_h^{\partial_q}} \alpha \int_{e} \psi(p_h - \hat{p}) \, dA
- \sum_{e \in \mathcal{E}_h^{\partial_q}} \alpha \int_{e} \psi \hat{q} \, dA = 0. \tag{20}
\]
For completeness, we briefly present the conservation properties of Eq. (20) as follows. Let us define the discrete flux vector, \( \bar{q}_h \), as

\[
\bar{q}_h := q_h, \quad \forall T \in \mathcal{T}_h,
\]

(21)

\[
\bar{q}_h \cdot n_e := -\{q_h \cdot n_e\} + \frac{\alpha}{h_e} \kappa \|p_h\|, \quad \forall e \in \mathcal{E}_h^I,
\]

(22)

\[
\bar{q}_h \cdot n_e := -q_h \cdot n_e + \frac{\alpha}{h_e} (p_h - \hat{p}), \quad \forall e \in \mathcal{E}_h^\partial,
\]

(23)

\[
\bar{q}_h \cdot n_e := -\hat{q}, \quad \forall e \in \mathcal{E}_h^\partial,
\]

(24)

Then, by letting \( \psi \) be 1 in an element \( T \) and 0 elsewhere, we can see that (20) becomes

\[
\int_T \nabla \cdot \dot{u} \, dV + \sum_{e \in \partial T} \int_e \bar{q}_h \cdot n_e \, dA = 0, \quad \forall T \in \mathcal{T}_h,
\]

(25)

which means that the discrete variational equation is locally (element-wise) conservative. Now, by taking \( \psi = 1 \) throughout the domain, we can also see that

\[
\int_\Omega \nabla \cdot \dot{u} \, dV + \int_{\partial \Omega} \bar{q}_h \cdot n \, dA = 0,
\]

(26)

which indicates that the EG discretization is also globally conservative. For more details on local mass conservation of mixed CG/EG elements and its impact on poromechanical responses, we refer to [38,39].

Lastly, we discretize \( \dot{u}_h \) in Eq. (20) in time. Using the implicit Euler method, we approximate \( \dot{u}_h \) for a given time increment \( \Delta t := t^{n+1} - t^n \) from time \( t^n \) to \( t^{n+1} \) as

\[
\dot{u}_h = \frac{u_h^{n+1} - u_h^n}{\Delta t},
\]

(27)

where the superscripts \( (\cdot)^{n+1} \) and \( (\cdot)^n \) denote the quantities at time \( t^{n+1} \) and \( t^n \), respectively. The time discretization process is completed by substituting Eq. (27) into Eq. (20), and evaluating all other variables in Eqs. (13) and (20) at time \( t^{n+1} \). For notational brevity, we will drop the superscript \( (\cdot)^{n+1} \) for quantities pertaining to time \( t^{n+1} \) in the following.

The fully discrete form of the problem at hand can now be written as follows. Given \( u_h^n \) and \( p_h^n \), find \( (u_h, p_h) \in \mathcal{U}_h(\mathcal{T}_h) \times \mathcal{P}_h(\mathcal{T}_h) \) such that

\[
\mathcal{A}(u_h, p_h; \eta, \psi) = \mathcal{F}(\eta, \psi), \quad \forall (\eta, \psi) \in \mathcal{U}_h(\mathcal{T}_h) \times \mathcal{P}_h(\mathcal{T}_h).
\]

(28)

Here, \( \mathcal{A} \) is the bilinear form, defined as

\[
\mathcal{A}(u_h, p_h; \eta, \psi) := a(u_h, \eta) + b(p_h, \eta) + c(u_h, \psi) + d(p_h, \psi),
\]

with (for IIPG)

\[
a(u_h, \eta) := -\int_\Omega \nabla^h \eta : \sigma_h' \, dV,
\]

(30)
\[ b(p_h, \eta) := \int_{\Omega} p_h \nabla \cdot \eta \, dV , \]  
(31)

\[ c(u_h, \psi) := \sum_{T \in T_h} \int_T \psi \nabla \cdot (u_h - u_h^0) \, dV , \]  
(32)

\[ d(p_h, \psi) := \Delta t \sum_{T \in T_h} \int_T \psi \nabla \cdot q_h \, dV + \Delta t \sum_{e \in E_{\partial p}} h_e \int_e \hat{\psi} \hat{p} \, dA + \Delta t \sum_{e \in E_{\partial q}} \alpha_e h_e \int_e \hat{\psi} \hat{q} \, dA , \]  
(33)

and \( \mathcal{F} \) is the linear functional, defined as

\[ \mathcal{F}(\eta, \psi) := f(\eta) + g(\psi) , \]  
(34)

with

\[ f(\eta) := -\int_{\Omega} \eta \cdot \rho g \, dV - \int_{\partial \Omega} \eta \cdot \hat{t} \, dA , \]  
(35)

\[ g(\psi) := \Delta t \sum_{e \in E_{\partial p}} \alpha_e h_e \int_e \hat{\psi} \hat{p} \, dA + \Delta t \sum_{e \in E_{\partial q}} \int_e \hat{\psi} \hat{q} \, dA = 0 . \]  
(36)

It is noted that \( \Delta t \) has been multiplied to the discrete mass balance equation.

### 3. Stabilization of mixed continuous/enriched Galerkin finite elements

The purpose of this section is to develop stabilized versions of the foregoing mixed CG/EG formulation for which equal-order linear interpolation can be used even in the undrained limit. For this purpose, we first identify the stability condition for mixed CG/EG spaces by phrasing the enriched formulation as a twofold saddle point problem. It will allow us to recognize that the stability condition for mixed CG/EG elements has the same form of the standard inf–sup condition for mixed CG/CG elements, although it takes the EG-approximated pressure field. By appealing to this similarity, we apply two stabilization methods originally proposed for mixed CG/CG methods, namely the PPP and FPL methods, to derive stabilized formulations for the mixed CG/EG elements. It will be shown that the resulting stabilization terms only take the CG part of EG interpolation functions, which is a key finding of this work.

#### 3.1. Stability condition in the undrained limit

First, we briefly explain why a stability problem arises in undrained condition. Consider a poromechanical problem with a small time increment, i.e., \( \Delta t \approx 0 \). In this case, one can see that the bilinear form (33) and the linear functional (36) nearly vanish, meaning that the pore fluid is undrained (no seepage) from the solid matrix. The resulting discrete problem is form-identical to mixed finite element formulations for constrained problems such as Stokes flow. Therefore, in the undrained limit, the pore pressure field \( p_h \) acts as the incompressibility constraint (Lagrange multiplier) of the solid deformation, rather than an independent variable for fluid flow. For such a constrained problem, it is well-known that the combination of finite element spaces is subjected to the inf–sup stability condition [41].

To identify the inf–sup stability condition for the mixed CG/EG elements, let us recall that the EG finite element space has been constructed by adding the space of CG finite elements and the space of piecewise constant functions. Thus, \( p_h \) and \( \psi \) are decomposed into the CG part and the piecewise constant part as

\[ p_h = p_{CG}^1 + p_{DG}^0 , \]
(37)

\[ \psi = \psi_{CG}^1 + \psi_{DG}^0 , \]  
(38)
where \( p_h^{CG}, \psi_h^{CG} \in \mathcal{T}_h^{CG}(T_h) \) and \( p_h^{DG}, \psi_h^{DG} \in \mathcal{T}_h^{DG}(T_h) \). Inserting Eqs. (37) and (38) into the bilinear forms (31) and (32), respectively, we get
\[
\begin{align*}
    b(p_h, \eta) &= b(p_h^{CG}, \eta) + b(p_h^{DG}, \eta), \\
    c(u_h, \psi) &= c(u_h, \psi_h^{CG}) + c(u_h, \psi_h^{DG}).
\end{align*}
\]  
(39)
(40)

The above equations show that the bilinear forms of the enriched discretization are mathematically analogous to those of a problem with two constraint fields. Therefore, we can rephrase the mixed CG/EG formulation in the undrained limit as the following twofold saddle point problem:
\[
\begin{align*}
    a(u_h, \eta) + b(p_h^{CG}, \eta) + b(p_h^{DG}, \eta) &= f(\eta), \\
    c(u_h, \psi_h^{CG}) &= 0, \\
    c(u_h, \psi_h^{DG}) &= 0.
\end{align*}
\]  
(41)

The inf–sup condition for a twofold saddle point problem has been derived by Howell and Walkington [67]. For this problem, the inf–sup condition can be written as
\[
\sup_{\eta \in \mathcal{T}_h, \eta \neq 0} \frac{b(p_h^{CG}, \eta) + b(p_h^{DG}, \eta)}{\|\eta\|_1} \geq C(p_h^{CG} + p_h^{DG}), \quad \forall (p_h^{CG}, p_h^{DG}) \in \mathcal{T}_h^{CG} \times \mathcal{T}_h^{DG},
\]  
(42)

with \( C > 0 \) independent of \( h \). Now, as done by Choo and Borja [52] for double-porosity poromechanics, we combine the bilinear forms and use the triangle inequality to arrive at
\[
\sup_{\eta \in \mathcal{T}_h, \eta \neq 0} \frac{b(p_h^{CG} + p_h^{DG}, \eta)}{\|\eta\|_1} \geq C\|p_h^{CG} + p_h^{DG}\|_0, \quad \forall (p_h^{CG}, p_h^{DG}) \in \mathcal{T}_h^{CG} \times \mathcal{T}_h^{DG}.
\]  
(43)

Recalling that \( p_h = p_h^{CG} + p_h^{DG} \), we can rewrite the above equation as
\[
\sup_{\eta \in \mathcal{T}_h, \eta \neq 0} \frac{b(p_h, \eta)}{\|\eta\|_1} \geq C\|p_h\|_0, \quad \forall p_h \in \mathcal{T}_h.
\]  
(44)

Importantly, the form of Eq. (44) is the same as that of the standard inf–sup condition for mixed CG/CG elements. The difference is that now the enriched pore pressure field, \( p_h = p_h^{CG} + p_h^{DG} \), instead of the standard CG pressure field, enters the stability condition. Thus it is hypothesized that a stabilization method originally developed for the standard inf–sup condition may be applied for stabilizing enriched mixed finite elements, provided that the enriched field is treated properly. Drawing on this hypothesis, we apply two types of stabilization methods popular in the mixed CG/CG literature to develop stabilized formulations for mixed CG/EG elements.

3.2. PPP-based stabilized formulation

We first develop a stabilized formulation based on the PPP method proposed by Bochev et al. [57]. The PPP method is motivated by a weaker inf–sup condition that equal-order linear finite elements satisfy, which is given by
\[
\sup_{\eta \in \mathcal{T}_h, \eta \neq 0} \frac{b(p_h, \eta)}{\|\eta\|_1} \geq C_1\|p_h\|_0 - C_2\|p_h - \Pi p_h\|_0, \quad \forall p_h \in \mathcal{T}_h,
\]  
(45)

with \( C_1 > 0 \) and \( C_2 > 0 \) independent of \( h \), and \( \Pi : L^2(\Omega) \mapsto R_0 \) a projection operator from the \( L^2 \) space to the piecewise constant space \( R_0 \). Comparing Eqs. (44) and (45), one can identify that the last term on the right hand side of Eq. (45) indicates the deficiency in the inf–sup stability of equal-order linear elements.

To counteract this stability deficiency, the PPP method augments the following term to the mass balance equation [50,52] (after time discretization)
\[
S_{PPP}(p_h, \psi) := \sum_{T \in \mathcal{T}_h} \tau_{PPP} \int_T (\psi - \Pi \psi)(p_h - p_h^n - \Pi(p_h - p_h^n)) \, dV,
\]  
(46)
such that the discrete problem is modified to be
\[
\mathcal{A}(\mathbf{u}_h, p_h; \eta, \psi) + S_{\text{PPP}}(p_h, \psi) = \mathcal{F}(\eta, \psi). \tag{47}
\]
Here, \(\tau_{\text{PPP}} > 0\) is the stabilization parameter of the PPP method, and the piecewise projection operator \(\Pi\) is specifically defined as
\[
\Pi \psi = \frac{1}{V_T} \int_T \psi \, dV, \quad \Pi p_h = \frac{1}{V_T} \int_T p_h \, dV, \quad \forall T \in \mathcal{T}_h \tag{48}
\]
with \(V_T\) the volume of element \(T\). In other words, \(\Pi\) calculates the average of a field variable within an element.

To obtain the correct stabilization term for the EG method, Eq. (46) has to be expanded further. Because \(p_h\) and \(\psi\) are additively decomposed, the projections of \(p_h\) and \(\psi\) are also decomposed as
\[
\Pi p_h = \Pi p_h^{CG} + \Pi p_h^{DG_0}, \quad \Pi \psi = \Pi \psi^{CG_1} + \Pi \psi^{DG_0}. \tag{49}
\]
Here, because \(p_h^{DG_0}\) and \(\psi^{DG_0}\) are piecewise constant functions
\[
\Pi p_h^{DG_0} = p_h^{DG_0}, \quad \Pi \psi^{DG_0} = \psi^{DG_0}. \tag{50}
\]
Therefore,
\[
p_h - \Pi p_h = p_h^{CG_1} + p_h^{DG_0} - \Pi p_h^{CG_1} - \Pi p_h^{DG_0} = p_h^{CG_1} - \Pi p_h^{CG_1},
\]
\[
\psi - \Pi \psi = \psi^{CG_1} + \psi^{DG_0} - \Pi \psi^{CG_1} - \Pi \psi^{DG_0} = \psi^{CG_1} - \Pi \psi^{CG_1}. \tag{51}
\]
Inserting the above two equations into Eq. (46), we get
\[
S_{\text{PPP}}(p_h, \psi) := \sum_{T \in \mathcal{T}_h} \tau_{\text{PPP}} \int_T (\psi^{CG_1} - \Pi \psi^{CG_1})[p_h^{CG_1} - (p_h^{CG_1})^n - \Pi \{p_h^{CG_1} - (p_h^{CG_1})^n\}] \, dV. \tag{53}
\]
This expression shows that the stabilization term should only be augmented to the CG part of the EG-discretized variables, not to the entire part. We have confirmed this aspect through numerical examples presented in the next section.

As the stabilization term is applied only to the CG part of the pressure space, we adopt \(\tau_{\text{PPP}}\) from the stabilized mixed CG/CG elements for coupled poromechanics proposed by White and Borja [50], given by
\[
\tau_{\text{PPP}} = \frac{1}{2G}, \tag{54}
\]
where \(G\) is the shear modulus of the solid matrix, introduced earlier in the previous section. It is noted that this parameter is adapted from 1/2 in the original PPP method formulated for a non-dimensionalized Stokes problem [57], analogizing the viscosity of fluid to the shear modulus of the solid matrix. For transversely isotropic elasticity, we replace \(G\) with the lower value of \(G_L\) and \(G_T\) to ensure that the stabilization parameter is sufficiently large.

### 3.3. FPL-based stabilized formulation

The second type of stabilized formulation we develop in this work is based on the FPL method. This method is based on a stabilization method for the Stokes problem suggested by Brezzi and Pitkäranta [68] and has been tailored to mixed CG/CG elements for coupled poromechanics by Truty and Zimmerman [49]. In this method, the discrete problem is modified as
\[
\mathcal{A}(\mathbf{u}_h, p_h; \eta, \psi) + S_{\text{FPL}}(p_h, \psi) = \mathcal{F}(\eta, \psi), \tag{55}
\]
where the FPL stabilization term is given by (after time discretization)
\[
S_{\text{FPL}}(p_h, \psi) := \sum_{T \in \mathcal{T}_h} \tau_{\text{FPL}} \int_T \nabla \psi \cdot \nabla (p_h - p_h^n) \, dV. \tag{56}
\]
Here, \(\tau_{\text{FPL}}\) is the stabilization parameter of the FPL method. Specific expressions for this parameter have been proposed by previously based on 1D analysis of mixed CG/CG elements [9,49].
Again, to identify the correct stabilization term, we expand Eq. \((56)\) recalling that \(p_h := p_{h}^{CG_1} + p_{h}^{DG_0}\) and \(\psi := \psi^{CG_1} + \psi^{DG_0}\). Note that
\[
\nabla p_h := \nabla p_{h}^{CG_1} + \nabla p_{h}^{DG_0} = \nabla p_{h}^{CG_1},
\]
\[
\nabla \psi := \nabla \psi^{CG_1} + \nabla \psi^{DG_0} = \nabla \psi^{CG_1},
\]
because of the gradient of a constant function is zero. Therefore, Eq. \((56)\) becomes
\[
S_{FPL}(p_h, \psi) := \sum_{T \in T_h} c_{FPL} \int_T \nabla \psi^{CG_1} : \nabla \left[ p_{h}^{CG_1} - (p_{h}^{CG_1})^n \right] \, dV.
\]
This equation indicates that the FPL stabilization term is also applied to the CG part of the EG-discretized variables only.

Because Eq. \((59)\) only involves the CG finite element space, the stabilization parameter \(c_{FPL}\) derived for mixed CG/CG elements may be used. Here, we adopt the parameter proposed by Preisig and Prévost [9], given by
\[
c_{FPL} = \max \left( \frac{h_T^2}{4M} - \kappa_T \Delta t, 0 \right),
\]
where \(\kappa_T := \kappa|_T\) and \(M\) is the P-wave modulus of the solid matrix. For transversely anisotropic materials, we substitute the lower value of \(\kappa_{\perp}\) and \(\kappa_{\parallel}\) into \(\kappa\) and calculate \(M\) using \(\lambda\) and the lower value of \(G_L\) and \(G_T\). Note that the max operator allows us to prevent \(c_{FPL}\) becoming negative as \(\Delta t\) becomes larger. Therefore, the FPL stabilization term naturally disappears when \(\Delta t\) is sufficiently larger and the inf–sup condition becomes irrelevant.

3.4. Remarks on PPP- and FPL-based stabilized formulations

Before closing this section, let us briefly discuss and compare the two types of stabilized formulations. First, the PPP stabilization term \((53)\) contains the time derivative of the pore pressure field minus its element-wise average. Therefore, this term can be interpreted as artificial fluid compressibility added to counteract the inf–sup condition in the incompressibility limit. Because fluid compressibility is a local property, the PPP stabilization term is calculated in an element-wise manner without a length parameter, and it does not change the numerical flux in the conservation property. By contrast, the FPL stabilization term \((59)\) contains the gradient of the pore pressure field, which implies that it augments numerical fluid diffusion to avoid a strictly undrained condition. As such, this term is non-local in nature, taking a stabilization parameter dependent on the grid size and the fluid mobility. The stabilization parameter \((60)\) has been derived analytically for a 1D poromechanical problem. While this parameter may be suboptimal when the flow condition is far from a 1D condition, it seems to work reasonably well for complicated problems as will be shown in the next section. It is also noted that the stabilization term is consistent in the sense that it vanishes as \(\Delta t \to 0\) and \(h_T \to 0\).

4. Numerical examples

This section verifies and investigates the performance of the proposed stabilized formulations through four numerical examples of various complexity. In each example, we compare numerical solutions obtained by the following three types of mixed finite elements:
1. CG-Q_1/EG-Q_1: Linear mixed CG/EG elements, Eq. \((28)\).
2. CG-Q_1/EG-Q_1-PPP: Stabilized linear mixed CG/EG elements using the PPP method, Eq. \((47)\).
3. CG-Q_1/EG-Q_1-FPL: Stabilized linear mixed CG/EG elements using the FPL method, Eq. \((55)\).
Fig. 2. Geometry and boundary conditions of the 1D consolidation problem.

It is noted that CG-$Q_1$/EG-$Q_1$ would be unstable for an undrained problem, whereas CG-$Q_1$/EG-$Q_1$-PPP and CG-$Q_1$/EG-$Q_1$-FPL are deliberately designed to be stable for the same problem. Because the focus of this work is on undrained behavior, we will present and discuss numerical results at the very first time step of each example, unless specified otherwise.

The three types of mixed CG/EG elements have been implemented in a modified version of Geocentric, a massively parallel mixed CG/CG finite element code for poromechanics utilizing a block-preconditioned Newton–Krylov solver [69]. The code relies heavily on open source scientific computing libraries including the deal.II finite element library [70], p4est mesh handling library [71], and the Trilinos project [72].

4.1. 1D consolidation

The purpose of our first example is to verify whether the stabilized CG/EG formulations perform well as their CG/CG counterparts. For this purpose, we select Terzaghi’s 1D consolidation problem, a simple poromechanical problem that can be solved analytically and has been used by previous works on stabilization of CG/CG elements (e.g., [9,50,56]). Fig. 2 shows the geometry and boundary conditions of the problem. The soil domain is homogeneous and isotropic, with material parameters of $K = 500$ kPa, $\nu = 0.2$, and $\kappa = 10^{-11}$ m$^2$/kPa s. The analytical solution of Terzaghi’s problem is given by

$$P(Z,T) = \sum_{m=0}^{\infty} \frac{4}{\pi(2m+1)} \sin \left(\frac{\pi(2m+1)}{2}Z\right) e^{-\left(\pi(2m+1)/2\right)^2 T},$$

where $P$, $Z$, and $T$ are dimensionless pressure, depth, and time, defined as

$$P := p/W, \quad Z := z/H, \quad T := (c_v/H^2)t,$$

respectively. Here, $c_v$ is the coefficient of consolidation, which is calculated to be $1.0 \times 10^{-8}$ m$^2$/s from the above material parameters. Note that this $c_v$ value is realistic for a soft clay.

We solve this problem using three types of mixed CG/EG elements: CG-$Q_1$/EG-$Q_1$, CG-$Q_1$/EG-$Q_1$-PPP, and CG-$Q_1$/EG-$Q_1$-FPL. Also, to check whether the stabilization methods are mesh sensitive, we use two meshes of different discretization levels, namely $H/h = 20$ and $H/h = 40$. The two meshes are structured with quadrilateral elements.

Fig. 3 presents numerical results at the first time step when $t = 30$ min ($T = 1.8 \times 10^{-5}$). The analytical solution at this time is also shown for comparison. The results show that the CG-$Q_1$/EG-$Q_1$ formulation is inherently unstable, as can be seen from the spurious oscillations in the pore pressure field—which are typical manifestation of the inf–sup instability. The pressure oscillations are more severe in the coarser mesh. By contrast, the stabilized finite elements do not show this type of pressure oscillation, meaning that they do not suffer from the inf–sup stability problem. It is noted that the minor oscillation in the PPP solution is because of the pressure boundary condition in the undrained limit, instead of lack of inf–sup stability. We have confirmed this by checking to see that the same oscillation persists even when a high-order displacement interpolation is used. Because the PPP
while doubling the number of elements, as described in Choo and Lee [38]. Table 1 shows the L2-norm of spatial discretization. To isolate spatial errors, we quadruple the number of time steps (starting from 25 steps) for mixed CG/CG elements. To confirm the FPL method is exactly derived for this kind of 1D problem, so the method works optimally for this problem. See also Preisig and Prévost [9] and Yoon and Kim [56] in which the FPL method shows the same performance for mixed CG/CG solution for Terzaghi’s problem. Therefore, it can be concluded that both stabilization methods perform very similarly to their original versions for mixed CG/CG elements.

An important requirement for a stabilization method for poromechanics is that it should preserve numerical accuracy under drained condition wherein the formulation is not subject to the inf–sup condition. For the FPL method, this is achieved by not adding the stabilization term \( (59) \) when \( \Delta t < h_1^2/(4k_T M) \), see Eq. (60). In other words, when the time increment is greater than a certain limit, the stabilized formulation is identical to CG-Q1/EG-Q1. For the PPP method, while the stabilization term \( (53) \) is always added to the variational equation, this term has been shown to have very little effect on drained problems for mixed CG/CG elements. To confirm that both stabilized formulations do not compromise the numerical accuracy under drained condition, we carry out error convergence tests with respect to the analytical solution (61). We particularly choose the analytical solution at \( T = 0.1 \) when the problem is certainly in the drained regime. The convergence tests are performed using three levels of spatial discretization. To isolate spatial errors, we quadruple the number of time steps (starting from 25 steps) while doubling the number of elements, as described in Choo and Lee [38]. Table 1 shows the \( L^2(\Omega) \)-norm and \( H^1(\Omega) \)-seminorm errors of the two stabilized formulations upon mesh refinement. One can see that both the CG-Q1/EG-Q1-PPP and CG-Q1/EG-Q1-FPL show the expected optimal order of convergences. The error convergence test results confirm that both of the stabilized formulations work well throughout the entire range of drainage conditions.

### 4.2. 3D foundation

To extend our investigation to a 3D problem, we now simulate a flexible foundation applied on the top center of a 3D ground. Taking advantage of symmetry, we model a quarter of the domain as depicted in Fig. 4. On the top boundary, the rectangular foundation part is subjected to a compressive traction and impermeable, whereas the rest part is free of traction and subjected to zero pressure to allow drainage. All other boundaries are supported by rollers and impermeable. The material parameters are assigned to be the same as the previous example. Similar to the previous example, we solve this problem with two meshes, one having 4,096 hexahedral elements and the other having 32,768 hexahedral elements. The element sizes are uniform in both meshes.

Fig. 5 shows pore pressure solutions after 30 min of loading obtained by the three mixed CG/EG formulations. Note that the pore pressure fields shown here are excess pore pressures normalized by the magnitude of the foundation load. One can see that the CG-Q1/EG-Q1 produces severe pressure oscillations in both meshes, confirming its lack of inf–sup stability. These oscillations are absent in the results of CG-Q1/EG-Q1-PPP and CG-Q1/EG-Q1-FPL, so it can be concluded that both PPP and FPL methods allow us to obtain stable solutions for a 3D problem. It can also be seen that the PPP method is now free of any pressure oscillation because the loading boundary is impermeable in this problem. However, the PPP and FPL solutions still show slight differences, particularly right below the footing. Fig. 6 compares the two stabilized pressure solutions along the centerline of the footing. It can be seen that the FPL method consistently gives lower excess pore pressures at the top while the difference becomes smaller with mesh refinement. This implies that the 1D nature of the FPL stabilization parameter may lead to slightly more diffusive result in a 3D setting. Except this difference, however, the PPP and FPL results are almost the same.

<table>
<thead>
<tr>
<th>( H/h )</th>
<th>CG-Q1/EG-Q1-PPP</th>
<th>CG-Q1/EG-Q1-FPL</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) error</td>
<td>Rate</td>
<td>( H^1 ) error</td>
</tr>
<tr>
<td>20</td>
<td>9.463e−04</td>
<td>9.763e−03</td>
</tr>
<tr>
<td>40</td>
<td>2.321e−04</td>
<td>2.03</td>
</tr>
<tr>
<td>80</td>
<td>5.805e−05</td>
<td>2.00</td>
</tr>
</tbody>
</table>
Fig. 3. Pore pressure solutions (normalized by the applied load) of the three mixed CG/EG formulations to the 1D consolidation problem at $t = 30$ min ($T = 1.8 \times 10^{-5}$).
4.3. Strip loading on transversely isotropic porous media

In the third example, we shift our attention to transversely isotropic materials whose bedding planes are not aligned with grids. The main purpose of this example is to examine whether the stabilized formulations retain the ability of mixed CG/EG discretization to accommodate arbitrary anisotropy. It also allows us to see whether the stabilization parameter of the FPL method would be appropriate for anisotropic problem despite the fact that it has been derived from a 1D configuration.

Fig. 7 illustrates the setup of this problem in which strip loading is applied on the top center of transversely isotropic materials with different orientations of bedding planes. The problem is similar to the previous one but now it is under plane strain condition and involves a transversely isotropic material. The five elasticity parameters for transverse isotropy are assigned as follows: $E_1 = 1$ MPa, $E_2 = 2$ MPa, $v_{12} = 0.25$, $v_{23} = 0.25$, and $G_{12} = 0.4$ MPa. The mobility values in the bedding plane normal and parallel directions are set as $\kappa_\perp = 10^{-11}$ m$^2$/kPa s and $\kappa_\parallel = 10^{-9}$ m$^2$/kPa s, respectively. We note that it is the solid anisotropy that governs the pore pressure field in undrained condition because excess pore pressure will develop to render solid deformation incompressible. We consider three orientations of bedding planes, namely, $\theta = 10^\circ$, $\theta = 45^\circ$, and $\theta = 80^\circ$, and discretize the domain by 2560 quadrilateral elements in a structured manner. Because the grid structure is aligned with none of the bedding plane orientations, this setup allows us to indirectly investigate the performance of stabilization in an unstructured mesh.

Figs. 8–10 show the pore pressure solutions after 30 min of loading for the three bedding plane orientations. The excess pore pressure solutions are again normalized by the applied load. It can be seen that the pore pressure responses are significantly affected by the orientation of the bedding plane. Also, interestingly, when the bedding plane direction is $80^\circ$, the maximum excess pore pressure below the loading is appreciably lower than the applied load. Regardless of these orientation-dependent results, however, both the PPP- and FPL-based stabilization terms successfully remove the oscillations in the CG-$Q_1$/EG-$Q_1$ results. Furthermore, in this example, the numerical solutions of the CG-$Q_1$/EG-$Q_1$-PPP and CG-$Q_1$/EG-$Q_1$-FPL are virtually identical. This finding suggests that both
Fig. 5. Pore pressure solutions (normalized by the applied load) of the three mixed CG/EG elements to the 3D foundation problem at $t = 30$ min. Displacement fields are magnified by a factor of 500.
Fig. 6. Pore pressure solutions (normalized by the applied load) of the two stabilized mixed CG/EG elements along the centerline of the 3D foundation.

Fig. 7. Geometry and boundary conditions of the strip loading problem. The material is transversely isotropic with respect to the bedding plane depicted in the figure.

of the stabilized formulations remain robust for arbitrarily anisotropic materials, although they were originally developed for problems with isotropic material parameters.

4.4. Fluid injection into a damaged zone

The fourth and last example investigates the performance of stabilization methods in the presence of localized heterogeneity in material properties, which is common in geomechanics due to plasticity, damage, and fracture (e.g., [73–79]). To this end, we simulate injection of fluid into a localized damaged zone in a low-permeability material, which has relevance to hydraulic fracturing and other subsurface engineering activities. The damaged zone in this example is introduced through a phase-field/gradient-damage formulation which is now widely used in the computational mechanics community (e.g., [80,81]). In essence, we represent the damaged zone by a phase-field variable, \( d \), which denotes a fully-damaged region by \( d = 1 \), an intact matrix region by \( d = 0 \), and a partially damaged zone by \( 0 < d < 1 \). Here we use a standard phase-field formulation in the literature [80] and incorporate fluid pressure in the damaged zone by modifying the momentum balance equation as described in
Fig. 8. Pore pressure solutions (normalized by the applied load) of the three mixed CG/EG elements to the strip loading problem at $t = 30$ min, when the bedding plane direction $\theta = 10^\circ$. Displacement fields are magnified by a factor of 500.
Fig. 9. Pore pressure solutions (normalized by the applied load) of the three mixed CG/EG elements to the strip loading problem at $t = 30$ min, when the bedding plane direction $\theta = 45^\circ$. Displacement fields are magnified by a factor of 500.
Fig. 10. Pore pressure solutions (normalized by the applied load) of the three mixed CG/EG elements to the strip loading problem at $t = 30$ min, when the bedding plane direction $\theta = 80^\circ$. Displacement fields are magnified by a factor of 500.
Santillán et al. [82]. For brevity, we omit a detailed explanation of the phase-field formulation. We also note that the phase-field formulation is used here to simulate undrained behavior in a localized zone, rather than evolution of damage or fracture.

The setup of this problem is depicted in Fig. 11. The domain is a 10-m wide square possessing a vertically oriented damaged zone. The intact matrix region \((d = 0)\) is isotropic and homogeneous with material parameters of \(E = 20 \text{ GPa}, \nu = 0.2,\) and \(\kappa = 10^{-13} \text{ m}^2/\text{kPa s}\). The fully-damaged region \((d = 1)\) has anisotropic permeability of which the vertical component is 10\(^6\) times greater than other components. The other permeability components are set to be equal to the matrix permeability. The fully-damaged region has zero stiffness under tensile strain. These permeability and stiffness values of the damaged and intact zones are interpolated by a standard degradation function in phase-field modeling. As for boundary conditions, the left and bottom boundaries are constrained by zero boundary conditions, whereas the right and top boundaries are subjected to constant pressure of 10 MPa. All boundaries are impermeable.

To solve this problem numerically, we discretize the domain by quadrilateral elements of a uniform size that meets the requirement for resolving the phase-field regularization [80]. Then, we model fluid injection by adding a source term of 0.5 mm/s/m\(^3\) to a small circular region at the center of the domain. The radius of the fluid injection region is equal to the length parameter for phase-field regularization. Also, now we investigate whether the stabilization terms work well for all of the three types of interior penalty methods, namely the IIPG, SIPG, and NIPG methods, even when a high permeability jump exists. It is noted that, although it was not shown previously for brevity, all the three methods led to nearly identical results for the previous examples.

Figs. 12 to 14 present the pressure solutions of the three mixed CG/EG elements at \(t = 1 \text{ s}\), obtained by the IIPG, SIPG, and NIPG methods, respectively. The figures show that the CG-\(Q_1/EG-Q_1\) is unstable again, exhibiting different oscillation patterns according to the interior penalty method used. Regardless of these different instability patterns, however, both the PPP and FPL stabilization terms effectively suppress the oscillations, as can be seen from the solutions obtained by the CG-\(Q_1/EG-Q_1\)-PPP and CG-\(Q_1/EG-Q_1\)-FPL. This confirms that the proposed stabilization terms are valid irrespective of which interior penalty method is used because the terms should be added only to the CG part of the enriched pressure space. For a more quantitative comparison, in Fig. 15 we plot pressure profiles along the \(x\) and \(y\) axes that cross the center of the domain. The plots in this figure clearly show that both the CG-\(Q_1/EG-Q_1\)-PPP and CG-\(Q_1/EG-Q_1\)-FPL can capture sharp changes in the pressure field in the localized damaged zone, showing little difference in all cases. Therefore, it can be concluded that both types of stabilized CG/EG formulations may interchangeably be used also for heterogeneous problems.
Fig. 12. Pressure solutions of the three mixed CG/EG elements to the fluid injection problem at $t = 1$ s, when the IIPG method is used. Color bar in kPa.
Fig. 13. Pressure solutions of the three mixed CG/EG elements to the fluid injection problem at $t = 1$ s, when the SIPG method is used. Color bar in kPa.
Fig. 14. Pressure solutions of the three mixed CG/EG elements to the fluid injection problem at $t = 1\ s$, when the NIPG method is used. Color bar in kPa.
Fig. 15. Pressure profiles along the $x$ and $y$ axes crossing the center of the domain.
5. Closure

This paper has proposed stabilized mixed CG/EG formulations that enable the use of equal-order linear interpolation for coupled poromechanical problems even in undrained condition. The major challenge tackled in this work is a new stability condition resulted from the pressure space enrichment for local mass conservation. We have identified that the new stability condition can be derived from a twofold inf–sup condition and has the same form of the standard inf–sup condition for mixed CG/CG elements. Drawing on this finding, we have adopted two stabilization methods originally devised for mixed CG/CG elements, namely the PPP and FPL methods. We have shown that, when applied to the mixed CG/EG discretization, they both lead to stabilization terms that should be augmented only to the CG part of the EG finite element space, not to the enrichment part. Thus the stabilization terms require little additional effort for implementation, preserve the original matrix structure, and are independent of the choice of a specific interior penalty method. Through numerical examples, it has been demonstrated that both of the two stabilized mixed CG/EG formulations are effective and robust under various conditions of drainage, space dimension, anisotropy, and heterogeneity.

The stabilized mixed CG/EG formulations can serve as an efficient means for obtaining locally mass conservative solutions to poromechanical problems that involve low-permeability materials and/or small time increments. It is noted that previous work [38,39] has already shown that local mass conservation in fluid flow can also be critical to accurate simulation of flow-induced deformations. The stabilized formulations will also help accelerate poromechanical simulations involving transport phenomena in low-permeability media (e.g., [40]). Beyond poromechanics, the proposed stabilization methods may also be applied to allow for equal-order interpolation for mixed CG/EG discretization of other form-identical constrained problems. Furthermore, the methodology presented in this work may also help stabilize other types of mixed finite elements in which the constraint field is enriched by additional functions.

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References


