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A match not made in heaven: on the applicability of mathematics in physics

Arezoo Islami[1]

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Abstract In his seminal 1960 paper, the physicist Eugene Wigner formulated the question of the applicability of mathematics in physics in a way nobody had before. This formulation has been (almost) entirely overlooked due to an exclusive concern with (dis)solving Wigner's problem and explaining the effectiveness of mathematics in the natural sciences, in one way or another. Many have attempted to attribute Wigner's unjustified conclusion—that mathematics is unreasonably effective in the natural sciences—to his (dogmatic) formalist views on mathematics. My goal is to show that this reading misses out on Wigner's highly original formulation of the problem which is presented throughout his body of work in physics as well as in philosophy. This formulation, as I will show, leads us in a new direction in solving the applicability problem.

Keywords Applicability of mathematics · Wigner's Puzzle · Unreasonable effectiveness · Laws of nature · Invariance principles

✉ Arezoo Islami
arezooi@stanford.edu

1 Department of Philosophy, Stanford University, Stanford, USA

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1 Introduction

The relationship between mathematics and physics that started in the wake of modern natural sciences has grown to be much more than a casual affair.[1] It is not only the possibility of this relationship but its enduring strength and ever-increasing significance (especially to physics) that has puzzled many scientists, mathematicians and philosophers for centuries. What seems to be puzzling is the underlying difference between the *relata* of this relationship: physics is the study of inanimate nature, concerned with the discovery of laws of nature, while mathematics is the study of concepts (structures) and operations, which seems to be far removed from the empirical study of the natural world.[2]

It has been argued that given such differences, it is a *miracle* that this relationship has continued to work so well. In the physicist Eugene Wigner's powerful words:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. (Wigner 1960, p. 237)

That this miracle exists, the quotation seems to suggest, is but a wonderful, unexplainable and undeserved gift to us!

There is no shortage of arguments for the claim that the impression of a 'miracle' wears off once we pay closer attention to the structure of modern physics and mathematics and the complex relationship between the two (see Steiner 1998; Sarukkai 2005; Grattan-Guinness 2008; Hamming 1980; Longo 2005; Colyvan 2001; Pincock 2014; Velupillai 2005; Lützen 2011 among others). My goal is to show that, contrary to the common reading, Wigner's paper *also* contains an argument for the *limited* and *reasonable* effectiveness of mathematics in physics. More importantly, a close and contextual reading of Wigner's influential paper, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", on my view, unveils a highly original formu-

[1] Although for Aristotle astronomy, optics, mechanics, and harmonics were subordinate mathematical sciences, they were not considered to be parts of natural science. Moreover, applying mechanical principles to physical objects, which Galileo did, was nothing new. With the rise of modern physics, what is new is that one can give (natural causal) explanation for physical objects in terms of principles that are mathematical in nature. In other words, mathematical principles provide a basis for doing physical science. Along with E.A. Burtt, one might think this shift from understanding physical science as a theory of substantial forms to what involves causal explanations in terms of mathematical principles, involved a pure faith in the deep mathematical structure of the universe. In *The Metaphysical Foundations of Modern Science*, Burtt argues that this trend in works of Galileo, Kepler, Copernicus must be understood based on the 'pure faith' (Burtt 1923).

[2] Physics especially in the last century has become increasingly mathematical to the extent that some parts of physics, string theory for instance, fit better in the description of mathematics. The status of string theory as a theory of physics is controversial. Moreover, here my focus is on characterizing mathematics and physics in the conventional way.

lation of the applicability problem based on the features of the scientific practice that are often overlooked.[3]

I will argue that in this original formulation, (1) the emphasis is solely on modern theoretical physics, unlike how Wigner's paper is often understood. In response to this common understanding, which is prompted partially by the misleading title of the paper, scholars have emphasized the ineffectiveness of mathematics in other natural sciences. Velupillai, for instance, argues for the unreasonable *ineffectiveness* of mathematics in *economics* (Velupillai 2005). Longo and Montévil argue for the *reasonable ineffectiveness* of mathematics in *biology* (Longo and Montévil 2016).[4] Moreover, (2) the question to which Wigner's paper, on my reading, responds is a particular version of the applicability problem: what is it about modern (theoretical) physics that makes mathematics an appropriate language for the formulation of its laws? To answer this question one needs an account of what modern physics is, which, as I will show, is precisely Wigner's lifetime project.[5] Following Wigner, I will argue that (3) the cornerstone of modern physics (the so-called Newtonian paradigm) and its distinguishable feature is the central role of invariance principles in giving structure to the laws of nature. By virtue of these invariance principles, laws of physics are universal and empirically testable. It is precisely these principles that make the mathematical formulation of laws of physics possible.

My reconstruction of Wigner's main thesis provides a deeper and historically-sensitive understanding of the relationship between mathematics and modern theoretical physics which is complex and far from trivial. Understood in this way, in order to solve the applicability problem, and before arriving at metaphysical conclusions about the mathematical structure of our world, we need to reevaluate the existing relationship between mathematics and physics on epistemic grounds.

2 The mysterious relation

In a lecture delivered at New York University in May 1959, the physicist Eugene Wigner invigorated and reintroduced the question of applicability of mathematics,

[3] My paper, first and foremost, aims to contribute to the scholarship on Wigner's problem of the "Unreasonable Effectiveness". By extracting and highlighting what is interesting in Wigner's project, which is almost entirely overlooked by other commentators, I hope to contribute to the scholarship on the question of the applicability of mathematics, in general. To fully answer this question, we need a much more detailed philosophical work about the *relata* of this relationship, which is outside the capacity of this paper.

[4] I am convinced by Longo's argument that Wigner's removal of other natural sciences must be studied carefully. The comparison between biology and physics, in relation to the applicability of mathematics, is enlightening and deepens our understanding of the relationship between mathematics and physics. For an interesting and illuminating discussion see Longo and Montévil (2016) and Longo and Montévil (2013).

[5] This question is often not distinguished from a connected question: what is it about *mathematics* that makes it an appropriate language for modern physics? To provide an answer to the latter question, we need an elaborate account of mathematics. Following the format of Wigner's paper and the trajectory of his work, and given the fact that other commentators in the field have mainly focused on the latter question, I will limit myself to making only some brief remarks about mathematics. The question of what mathematics is, given its applicability in modern physics, is at the center of another paper under construction. Of course to provide a full answer to the applicability problem we need to have an elaborate view of both *relata* of this relationship.

under the striking title *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*–a formulation that has crucially influenced and shaped debates over this problem to this day.[6] Wigner was chosen to deliver the very first lecture of a biannual lecture series, Richard Courant lectures in Mathematical Sciences, at New York University, which were funded as a gift from Richard Courant's friends to celebrate his 70th birthday. These lectures aim to highlight Courant's vision of mathematics and its aesthetic characteristics. There is a remarkable statement from Kurt Friedrichs,

Who presented the gift, highlighting exactly this point:

> One may think that one of the roles mathematics plays in other sciences is that of providing law and order, rational organization and logical consistency, but that would not correspond to Courant's ideas. In fact, within mathematics proper Courant has always fought against overemphasis of the rational, logical, legalistic aspects of this science and emphasized the inventive and constructive, esthetic and even playful on the one hand and on the other hand those pertaining to reality. How mathematics can retain these qualities when it invades other sciences is an interesting and somewhat puzzling question. Here we hope our gift will help.

Following this premise, Wigner's lecture unfolds *his* vision of the "aesthetic" and "playful" aspects of mathematics, and its usefulness in the natural sciences. As we read on, it becomes clear that Wigner's own vision of mathematics and its applicability differ from the attitude that Courant and the audience seem to share and honor. Wigner has to walk a thin line in this lecture to subtly unfurl his vision while he pays homage to Courant and his friends.[7]

As we read Wigner's paper, the lecture that was published a few months later in *Communications in Pure and Applied Mathematics* (1960), we find convincing reasons to think that the conclusion of this lecture (paper) is that the effectiveness of mathematics in natural sciences is unreasonable. That is, the question, call it the *Applicability Problem*–why is it that mathematics is applicable to the natural sciences?—for Wigner has a simple answer: "We don't know. There is no rational explanation for it!"

Right from the start, the lecture's title, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," offers a bold statement of this conclusion. In the body of the paper, Wigner highlights this conclusion in more than one place:

[6] Wigner, of course, wasn't the first person concerned with the applicability of mathematics in natural sciences. In modern times, Kant most prominently raised this issue in *Metaphysical Foundations of Natural Science*: "[I]n any special doctrine of nature there is only as much *proper* science as there is *mathematics* therein. For. proper science, and above all proper natural science requires a pure part lying at the basis of the empirical part, and resting on a priori cognition of natural things." How is this relation possible? Or maybe in Kant's jargon, how is mathematics possible? My paper, however, is focused on the current formulation of the problem, which is reinvigorated and reintroduced by Wigner's lecture.

[7] My argument for this claim is in the following sections. Roughly put, while for Wigner advanced mathematics proceeds based on intra-mathematical criteria such as (formal) beauty and elegance, that is not the case for elementary branches of mathematics such as geometry and algebra. On Wigner's view, the latter is suggested to us by our experience of nature. Moreover, while the puzzle for Courant is the relationship between mathematics and *reality*, for Wigner the problem is the applicability of mathematics to (modern) *physics*, which is our study of inanimate nature. I will discuss these issues further in the course of this paper.

> [T]he enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it. (Wigner 1960, p. 223)

> It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind's capacity to divine them. (Wigner 1960, p. 229)

And finally he ends the lecture by another striking statement of this conclusion:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. (Wigner 1960, p. 237)

The *reasonable* conclusion based on this body of evidence is that Wigner is an advocate of the "unreasonable" effectiveness of mathematics: a reading that is widely accepted among philosophers, mathematicians and physicists in the field including Steiner (1998), Sarukkai (2005), Grattan-Guinness (2008), Hamming (1980), Longo (2005), Colyvan (2001), Pincock (2014) and Velupillai (2005).[8] For example, in his detailed treatment of the problem in *The Applicability of Mathematics as a Philosophical Problem*, Mark Steiner reconstructs Wigner's argument in this way:

1. Mathematical concepts arise from the aesthetic impulse in humans.
2. It is unreasonable to expect that what arises from the aesthetic impulse in humans should be significantly effective in physics.
3. Nevertheless, a *significant* number of these concepts are *significantly* effective in physics.
4. Hence, mathematical concepts are unreasonably effective in physics (Steiner 1998).

Steiner's reading is advocated by many other scholars; this is a reading that has motivated many to offer arguments against Wigner's *unreasonableness*, to suggest solutions to Wigner's mystery and to declare the applicability phenomenon in one way or another *reasonable*.

[8] Although these commentators advocate different views about mathematics and physics, they converge on their reading of Wigner. While for Wigner and Steiner physics is an empirical science, Hamming, for instance, argues that the whole of physics or at least "a surprising amount of it" can be derived deductively by armchair thinking only. It is in this way that physics seems to be almost an *a priori* endeavor, much entangled with and similar to mathematics (Hamming 1980, pp. 88–89). Subtleties of this kind are very important and in my view, they ultimately amount to different formulations of the applicability problem. I have elaborated on these accounts in another work.

3 Unfolding the mysterious

This reading of Wigner's "unreasonable" effectiveness, on my view, is *understandable* but *not* accurate. Despite all this evidence, I argue that Wigner's paper includes a highly original, and historically-sensitive formulation of the applicability problem which emphasizes the limitations of the applicability of mathematics in modern physics and offers a solution to this problem based on invariance principles.[9]

To start, Wigner made understanding the structure of physics the center of his career as a physicist. For him modern physics wouldn't be possible had it not been for the overarching structures that exist among laws of nature- what we call invariance or symmetry principles[10] (this will become clear in what follows). It is in relation to the study of these principles, and their fundamental importance in quantum mechanics, that he was awarded the 1963 Nobel prize "for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles".(The Nobel Foundation, 1963)[11]

On a different historical note, Wigner is lecturing to a particular audience, who have gathered to celebrate Courant's vision: that the mathematics that is invented so playfully and aesthetically by mathematicians ends up in natural sciences to such a remarkable and unexpected degree. It is in this context that we can understand the puzzling format of this lecture. Wigner, following Courant's famous book, *What is Mathematics*, devotes the first, and the most *underdeveloped* part of his lecture to mathematics, and highlights its aesthetic and playful aspects.[12] There, as I will elaborate in the following section, he talks about the construction of concepts, as opposed to the deductive structure of mathematics, as the core of mathematical work. In addi-

[9] Evaluating whether this solution is convincing or not is a separate matter which requires a space of its own. In this paper, I will focus on describing *what it is*.

[10] While Wigner used invariance and symmetry principles interchangeably, the contemporary reader has a more nuanced notion. Symmetry is used to refer to symmetry groups (transformations), such as space-time symmetries, Lorentz and Poincare transformations. The invariance of certain quantities under symmetry transformations is linked to conservation laws (Noether's theorems). For instance the invariance of energy of an isolated physical system under space translations leads to the conservation of linear momentum. Following Wigner, I use both symmetry and invariance principles in the same way.

[11] Invariance principles were known to physicists prior to the twentieth century, but they were considered to be the consequences of and second to laws of nature. Einstein's papers on special and especially general relativity reversed this trend, putting invariance principles first and deriving laws of nature on their basis. This reversal, however, didn't have much of an influence on the work of theoretical physicists outside relativity theories. Wigner's work on invariance principles in quantum mechanics had a profound effect on changing this situation. As Gross writes, "that today principles of symmetry are regarded as the most fundamental part of our description of nature, is in no small part due to the influence of Eugene Wigner." (Gross 1995, p. 46).

[12] With the exception of a few comments, mostly in this paper, one cannot find much evidence on Wigner's views about mathematics. This is very telling, on my view, about the side of the applicability problem that Wigner found interesting. Unlike what most scholars in the field emphasize, the illuminating part of this paper on my view, is how Wigner understands the conditions for the possibility of mathematization of physics. That is, the aspects of modern theoretical physics that make for its mathematizability. Given the context of his intellectual environment, as Ferrerirós has recently argued, one can see the influence of the Hilbert school formalism on his account of mathematics. For an argument see Ferreirós (forthcoming).

tion, in the course of the lecture, he restates and reemphasizes Courant's puzzlement about the relationship between mathematics and nature, to the extent that he calls it a "miracle" (a word that Courant might have used).[13] Given the occasion, we can imagine that the audience knew much about mathematics and shared, to some degree this puzzlement about the relationship between mathematics and reality. In course of the lecture, Wigner delicately turns Courant's question (about the relation of mathematics to reality) to the relationship between mathematics and *physics*.[14] And finally to celebrate this sense of bewilderment, he ends the lecture with what he calls *a more cheerful note*, quoted before, where he calls the applicability phenomenon a " a wonderful gift which we neither understand nor deserve".[15]

Historical notes aside, a close and contextual reading of this paper makes a stronger case for rejecting the common reading of the "unreasonable effectiveness". Wigner devotes the big bulk of his paper to modern physics, its particular hierarchical structure, and how mathematics is used in formulating laws of nature. In other words, for Wigner, there is a *Why* question and a *How* question. The Why question is "Why is mathematics useful in physics?", which is just the applicability problem (in physics). The How question is "How is mathematics useful in physics?". To answer the Why question Wigner first tries to address the How question. What I try to emphasize in this paper is that by focusing on his response to the How question, we realize that Wigner's question of applicability is the question, what is it about *physics* that makes mathematics an appropriate language for the formulation of its laws? Besides a few rhetorical comments that Wigner makes in this lecture, most prominently in the beginning and at the end, the whole body of the work, closely (and contextually) studied, offers an argument for the *reasonable* and *limited* effectiveness of mathematics in modern physics.

Moreover, while the common reading emphasizes his striking remarks on the "unreasonable" effectiveness, it cannot make sense of the lengthy passages in which Wigner elaborates his views on invariance principles. Read more carefully, we see a tension that exists between Wigner's remarks on the limitations of the applicability of mathematics and the importance of invariance principles, on one hand and his comments about the applicability of mathematics as *miraculous*, *unreasonable* and *undeserved*, on the other. My goal is to show that we can make sense of this tension, if we pay attention to his emphasis on modern theoretical physics. The rest of the paper aims to make this clear.

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[13] Of course, Courant was not the only one. Wigner's close friend and brother in law, Dirac had stated this puzzlement in more than one place. Mathematical beauty was also a concern for Walter Dubislav and Michael Polanyi, both of whom are mentioned in course of Wigner's lecture.

[14] Wigner, moreover, makes an effort to close his lecture by saying that lots of work remains to be done to solve this mystery "a gift we neither understand, nor deserve". As it were, he is saying, there is much more work left for the other Courant lecturers to do in this relation. See http://cims.nyu.edu/webapps/content/lectures/home for more information on these lectures.

[15] It is quite remarkable that he calls this bafflement a *cheerful* note, whereas it should have been a sad conclusion, an unsolved problem.

4 Motivation and limitations

Before proceeding further, let's make a few remarks about the most common and immediate skeptical response to the applicability problem. The skeptic might stop here and raise doubts about whether the applicability problem is a genuine philosophical problem deserving an answer or a mere pseudo-problem. One objection to this effect points out that a significant part of mathematics that is applicable in the natural sciences was devised exactly for this purpose, by scientists themselves in response to specific practical needs. Take the example of calculus which was invented by Newton and Leibniz for the purpose of its use in the study of motion. It shouldn't surprise anyone that calculus has the utility that it does in mechanics. Moreover, the objection continues, not all mathematical concepts are used in the natural sciences. Most of what is called *pure* mathematics is divorced from any practical use whatsoever.[16]

Wigner raises and responds to both of these questions but these responses are often overlooked by other scholars in the field:[17]

> It is true, of course, that physics chooses certain mathematical concepts for the formulation of the laws of nature, and surely only a fraction of all mathematical concepts is used in physics. It is true also that the concepts which were chosen were not selected arbitrarily from a listing of mathematical terms but were developed, in many if not most cases, independently by the physicist and recognized then as having been conceived before by the mathematician. (Wigner 1960, p. 229)

In response to the first objection, it is true that a piece of mathematics like calculus was originally invented for the purpose of its use, but this doesn't explain why the more advanced concepts in mathematical analysis, which were developed independently of the primary motivation, ended up playing important roles in physics. Note that in expanding mathematical concepts, mathematicians have rigor, generality, simplicity, etc. in mind and for the most part, they aren't concerned with applicability. So although the usefulness of the former, original group of concepts in calculus was expected, the usefulness of the latter and more advanced concepts is *unexpected* and *deserves an explanation*.

There are also many mathematical concepts (and theorems) that were not devised with a practical purpose in mind and only later, sometimes centuries later, were found to have an application. Complex numbers provide a striking example of the foregoing.

[16] This is a controversial position about pure mathematics. It is true that a big part of pure mathematics *today* is not useful. Yet the situation might change in the future and in fact many mathematicians who develop highly abstract concepts hope that one day their mathematical work is useful in the natural sciences, technology and so on. Here I am not providing argument for the skeptical account on pure mathematics and its lack of applicability. I am only recounting it.

[17] Hamming, for instance, in his *partial explanations* for Wigner's puzzle suggests that: (1)"we see what we look for", and (2) "We select the kind of mathematics to use. That is when the mathematical tools that we have chosen are not adequate in a particular case, we choose a different kind." (Hamming 1980, pp. 87–89). These points haven't gone unnoticed by Wigner as the following quote suggests.

They were invented in the sixteenth century as solutions to cubic equations and in the twentieth century to everyone's bafflement and surprise, ended up in the most fundamental equations of quantum mechanics such as Heisenberg's commutation relations and Schrödinger's wave equation.

There are also numerous examples of mathematical concepts that were devised to solve a specific scientific problem and only later they appeared in a completely unexpected and unrelated field. Take the example of matrices. Matrices were developed first in the eighteenth century in attempts to solve eigenvalue problems,[18] which originated in physical experiments- in particular in relation to differentials (of dynamical) equations $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$. Around the nineteenth century they were extricated from their physical context only to reappear again in Heisenberg's mechanics to represent quantum quantities and their non-commutative multiplication.[19]

It has been argued that in all these cases mathematics is bound to play such a role because mathematical concepts provide the simplest possible formalism. This is not the case, of course, when we think of the mathematical concepts that are used in our physical theories today. "Let us not forget", Wigner notes, "that the Hilbert space of quantum mechanics is the complex Hilbert space, with a Hermitian scalar product. Surely to the unpreoccupied mind, complex numbers are far from natural or simple and they cannot be suggested by physical observations. Furthermore, the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics." (Wigner 1960, p. 229)

As a response to the second objection, it is true that a only a fraction of mathematics is applicable in physics, and it is true that we should not only look at the successful cases and ignore the rest. It is moreover true that many mathematically formulated theories of physics are considered false in the light of our current standards.[20] However, it is not true that these considerations eliminate or dissolve the applicability problem. They limit it to its proper domain. The limited applicability of mathematics in formulating laws of physics still requires an explanation.[21]

[18] Suppose that A is a square matrix. An eigenvector of this matrix is a vector x with the same direction as Ax. That is $Ax = \lambda x$, where λ is a constant. Eigenvectors are these special vectors that when multiplied by a matrix, they are only stretched, or shrunk or reversed (or left unchanged). Now this constant λ is the eigenvalue of the matrix A. For instance if A is the identity matrix then all vectors are eigenvectors of it, since $Ax = x$ with eigenvalue $\lambda = 1$. (In defining eigenvalues here, I am using the modern definition.)

[19] For an elaborate view of this history see Thomas Hawkins, "The Theory of Matrices in nineteenth Century" (Hawkins 1974).

[20] Wigner gives a few examples of these theories such as Bohr's early theory of atom, Ptolemy's epicycles, and the free electron theory. In all of these cases there were amazing numerical agreements between the theory and experiment and were formulated in the language of mathematics (Wigner 1960, p. 236).

[21] The relationship between mathematics and physics, as illustrated in these cases (where the constructions in mathematics, independent of their applicability, were successfully applied in physics) is far from trivial. We understand this by looking at the intricate history of the development of mathematics and physics. The skeptic's skepticism as well as the mystic's mysticism have their root in precisely ignoring this history, and hence cutting the philosophical investigation too short.

5 Wigner's question

Wigner defines what he calls the *Empirical Law of Epistemology* (hereafter ELE) as "the appropriateness and accuracy of the mathematical formulation of the laws of nature in terms of concepts chosen for their manipulability". Why Wigner calls this an *empirical* law and how it relates to *epistemology* will become clear in the following. For now, let's focus on the content of the law. Some instances of ELE: the second law of motion in Newtonian Mechanics is formulated in terms of the second derivative of position with respect to time; the axioms of quantum mechanics are formulated using a complex Hilbert space, where the states are defined as vectors in Hilbert space and the observables are eigenvalues of the self-adjoint operators and the list goes on indefinitely. These are just a few instances of a more general phenomena of the successful use of mathematical concepts in physical theories. What seems to be striking is that virtually *every* law of physics is formulated mathematically.

Our situation in this case is like "a man who was provided with a bunch of keys and who, having to open several doors in succession, always hit on the right key on the first or second trial" (Wigner 1960, p. 223). This would hardly surprise anyone, had there been a unique coordination between the keys and the doors- each door having its own distinctly marked key. The situation of the *lucky* man only surprises and puzzles us since the keys (advanced mathematical concepts) aren't made for these locks (regularities in nature). Mathematicians develop these advanced concepts simply for their mathematical qualities: manipulability, formal beauty, generality, etc., with no attention to their applicability in the study of nature. Mathematics for Wigner is a science governed just by such standards, which are internal to mathematics.[22]

Wigner defines mathematics as "a science of skillful operations with concepts and rules invented just for this purpose" (Wigner 1960, p. 224). This is a description that holds for most of mathematics, Wigner claims, and in particular for the kind of mathematics that plays an important role in physics (especially in quantum mechanics). The rest of the mathematical concepts belonging to elementary mathematics and especially elementary geometry, "were formulated to describe entities which are directly suggested to us by the actual world."(Ibid) The latter group doesn't pose a particular problem (for Wigner), or a puzzle, it is rather the group of more advanced concepts whose use "borders on mystery".[23]

[22] Formal beauty for Wigner is closely tied to rigor, simplicity and generality. As an example, in expanding the concept of 'number' to include negative numbers, mathematicians chose a definition that preserves many rules that hold between positive numbers such as associativity, commutativity, existence of a null member 0 (with addition), and associativity, commutativity, existence of a null member 1 (with multiplication).This expansion made definition of the set of integers $\mathbb{Z}$ possible which is a superset of the set of natural numbers ($\mathbb{N}$) forming an algebraic group with the operation of $+$ (unlike $\mathbb{N}$).The reason is that in $\mathbb{Z}$ every number m has its inverse with respect to $+$, $-$m in $\mathbb{Z}$ (such that m+($-$m)=0). Through such expansion, mathematicians were able to prove many theorems about integers, with the set of natural numbers as a spacial and limited case. This is what Wigner means by more rigor and generality. In addition, they made the proof of many theorems simpler and provided the solution to many unsolvable equations such as $(x + 1 = 0)$.

[23] There is already a refinement that Wigner adds to Courant's ideas: not *all* mathematical concepts are developed playfully and aesthetically. There are some that were in a way "forced" on us by nature, by the actual world.

What Wigner emphasizes is that in inventing new concepts and operations, the mathematician turns her attention completely away from nature- unconcerned with usefulness and applicability of her creation in the study of nature. G. H. Hardy in his book, *The Mathematician's Apology* insists that pure mathematics has no utility (as utility seems to compromise beauty). And that beauty is the essence of mathematics, as well as the arts:

> The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy 1940, p. 85).

Wigner emphasizes that, (1) these advanced concepts are *invented* (in contrast to discovered, or suggested to us by nature) and, (2) they are invented *only* for the purpose of mathematics (with no attention to their applicability).[24] It is *not* therefore, the deductive structure of mathematics that "borders on mystery", rather, it is the peculiar expansion of concepts based on intra-mathematical criteria that makes their later use (in natural sciences) look unexpected and unexplainable.

Take the example of complex numbers: numbers that include the imaginary i, the square root of -1. Mathematically defined, a complex number is a number of the form $z = x + iy$, where x and y are real numbers ($x, y \in \mathbb{R}$) and $i = \sqrt{-1}$. x is called the real part and y the imaginary part of z.[25] Wigner points out: "Certainly nothing in our experience suggests the introduction of these quantities" (Wigner 1960, p. 225). These numbers were first defined in the sixteenth century to provide solutions for cubic equations. At that time, it was so strange to think of the square roots of negative numbers that mathematicians couldn't possibly accept them. They were called "sophistic", "inexplainable", "nonsensical", "absurd" or "impossible" (Buzaglo 2002, p. 10). Later in an attempt to use these numbers, mathematicians called them "imaginary", or as Leibniz put it, "existing only in mind", dismissing the prior descriptions- "nonsensical", "absurd" and "impossible".[26]

These strange creatures gradually found their way to different parts of mathematics, and most surprisingly, to the theories of physics to an extent that they are considered

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[24] Wigner's definition of (advanced) mathematics is distinctly formalist, which is not so hard to understand given his ties to the Hilbert school. In this paper, he makes reference to Hilbert, Michael Polanyi and Walter Dubislav. In his recent paper, Josè Ferreiròs focuses on Walter Dubslav and his influence on Wigner's views, which he characterizes as 'strict formalism' Ferreirós (forthcoming). On Wigner's view (a) mathematical concepts -invented, not deduced- are governed by certain game-like rules, (b) the criteria for developing such concepts, isn't their link to nature, or our experience of nature, but their formal characteristics which are internal to mathematics. Wigner states, "The great mathematician fully, almost ruthlessly, exploits the domain of permissible reasoning and skirts the impermissible. That his recklessness does not lead him to a morass of contradictions is a miracle itself" (Wigner 1960, p. 224).

[25] All real numbers are thus complex numbers with $y = 0$.

[26] The word "imaginary" was first coined by Descartes in 1630 to reflect his idea that For every equation of degree n, we can imagine n roots which do not correspond to any real quantity. See Magic of Complex numbers in Mandic and Lee Goh (2009) for more details.

indispensable.[27] Penrose captures what seems to be "mysterious" and "baffling" to most scientists in this situation:

> While at first, it may seem that the introduction of such square roots of negative numbers was just a device- a mathematical invention designed to achieve a specific purpose- it later becomes clear that these objects are achieving far more than that for which they were originally designed. ..When Cardano introduced his complex numbers [in 1539], he could have no inkling of the many magical properties which were to follow- properties which go under various names, such as Cauchy integral formula, the Riemann mapping theorem, the Lewy extension property. (Penrose 1989, 94–95)

As a result, the adverse situation surrounding these numbers changed. It became clear that, as Jacques Hadamard has once said, "The shortest path between two truths in the real domain passes through the complex domain."[28] And not only did they permeate different areas of mathematics but they played an important role in physics.

As the Penrose quote suggests, complex numbers that were primarily invented for their manipulability (given how complex conjugates work), and usefulness in finding real solutions to the cubic equations, later showed many surprising properties. Moreover, they connected (formerly disconnected) areas of mathematics such as trigonometry with exponential functions, for instance, in Euler's formula $e^{ix} = \cos x + i \sin x$ and its remarkable consequence: Euler's identity $e^{i\pi} = -1$. That these mathematical properties of complex numbers, their striking manipulability in mathematical operations, led to their use in theories of physics is an instance of ELE.[29]

Before proceeding further, let's emphasize that Wigner's characterization of mathematics is distinctly formalist, formalism being a popular view of mathematics in the mid twentieth century.[30] Along with Wigner, many mathematicians and scientists of his generation such as John von Neumann and Walter Dubislav advocated this view of mathematics.[31] The formalist characterization of mathematics makes the relationship between mathematics and physics appear more mysterious than it need be. As Ferrierós rather dramatically puts it, it creates "miracles out of thin air" (Ferreirós forthcoming). Or as Unger and Smolin argue, it turns the applicability of mathematics

[27] The case of complex numbers in quantum mechanics is the focus of another paper, in which I argue that this way of characterizing their development is not historically accurate. Wigner, Penrose and many other commentators make a similar mistake in overlooking historical details regarding the development of these numbers, for instance, with regard to their geometrical representation. In this way, they turn the applicability of complex numbers in physics into a more puzzling and mysterious phenomenon.

[28] Quoted in Jean-Pierre Kahane's 1991 article *Jacques Hadamard* (Kahane 1991, p. 26).

[29] Buzaglo writes, "Eventually it became clear that without this "nonsense" there would be no mathematics- or at least no modern mathematics. Moreover, without expansions it is hard to see how we could progress in physics" (Buzaglo 2002, p. 11).

[30] I am convinced by Ferrerirós' argument in his recent paper, where he traces Wigner's formalism to his connections to the Hilbert school. Not only does Wigner think that advanced mathematical concepts are invented, but also he emphasizes that they are invented *just* for the purpose of their manipulability. It is this emphasis on 'just' that reveals his formalist tendencies about mathematics. See Ferreirós (forthcoming) and Lützen (2011) for a clear characterization of Wigner's formalism.

[31] See Ferrerirós' recent paper on the historical context of Wigner's 'strict formalism' and his relation to Walter Dubislav (Ferreirós forthcoming).

into a happy accident (Unger and Smolin 2014). [32] Getting things right about mathematics, as we will see is not Wigner's concern either in this paper or elsewhere. He devoted a big bulk of the "Unreasonable" paper and most of his career in physics to characterizing the hierarchical structure of modern physics.

5.1 Laws of nature

To further capture the mystery of ELE, Wigner describes his account of laws of physics, where these mathematical concepts appear. The physicist, unlike the mathematician, is interested in inanimate nature and in *discovering* "regularities" that exist among the natural phenomena. The objective of physics, defined in this way, is not to explain (inanimate) nature but to explain "regularities in the behavior of the object" (Wigner 1964a, p. 39). The scope of physicist's objective is, therefore, restricted to the domains in which these regularities can be found. And "the great success of physics is due", Wigner notes in his Nobel lecture, "to [this] restriction in its objective."(Ibid)[33]

The world we live in is very complex, and its future seems to be uncertain and unpredictable. Amidst this complexity, it is quite surprising and very fortunate for us that we are able to discover *any* regularities: "it is …a miracle that certain regularities in the events could be discovered". Wigner calls this a *miracle* in part because these regularities among events *exist*, and also because we *are* able to discover them. We would fail to discover regularities among events had it been the case that events were uncorrelated or human cognition was unable to identify these patterns. Of course, it is true that if the events around us didn't have any structure and if we couldn't understand and influence them, our life would have been completely different. Stronger still, "there would be no such thing as we call life". However that doesn't mean the regularities among events had to have the precision that they have.[34] (Wigner 1963, p. 29)

It is a contingent fact that these regularities exist and a contingent fact that we can discover them. What partially explains our success in discovering regularities among

[32] While it might seem that either formalists or Platonists have more difficulty explaining the applicability of mathematics, Colyvan argues that the question of applicability of mathematics has its full force for different schools of philosophy of mathematics from nominalists to Platonists. For an interesting discussion see Colyvan (2001).

[33] P. Roman in *Why Symmetry* also notes: "Nature, be it even objective reality, just *is*. It cannot be 'explained', at least not as far as science is concerned. Existence is a primary category, including, by the way, ourselves, too. (Which would imply that the explainer himself must be explained.) And certainly science is much more than 'the description of Nature' …We have gone far beyond such a casual phenomenology and even empiricism. We want to 'understand', and we have in part succeeded. …And the twentieth century German writer Hermann Hesse tells us: 'Every science is …a kind of ordering, simplifying; an attempt to make digestible for the spirit that which is indigestible.' Indeed, we are safe to say: *Science is the attempt to correlate individual phenomena and events into a coherent framework (or systems of such frameworks)*" (Roman 2004, p. 2).

[34] Wigner writes, "If we look a little deeper into the situation we realize that we would not live in the same sense that we do if the events around us had no structure. .. There would be no way our volition could manifest itself and there would be no such thing as that which we call life, This does not mean, of course, that life depends on the existence of the unbelievable precision and accuracy of the correlation between events which our laws of nature express and indeed the precision of these laws has all elements of a miracle that we can think of."

events is an abstraction, a division between initial conditions and regularities that physicists have used since Newton's age:

> Man, has, therefore devised an artifice that permits the complicated nature of the world to be blamed on something which is called accidental and thus permits him to abstract a domain in which simple laws can be found. The complications are called initial conditions, the domain of regularities, laws of nature. Unnatural as this division of the world structure may appear from a detached point of view, and probable though that the possibility of such a division has its own limits, the underlying abstraction is probably one of the most fruitful ones that human mind has made (Wigner 1949, p. 3).

Regularities determine the time evolution of a system (if a physical system is in the state of s_1 at time t_1, it will be in the state s_2 at time t_2) whereas initial conditions are about the state of the system at the outset of its evolution. The laws of nature are statements of these regularities.[35] What makes the formulation of the laws of nature possible is abstracting away from initial conditions, the domains in which regularities cannot be found. While it is difficult to find the exact "inventor" of such artifice, it has been around since the time of Newton. Kepler still tried to find the magnitudes of the orbits of the planets while he was trying to discover laws of planetary motion. Newton realized that without such a division it is impossible to formulate laws of motion.[36]

Arguably, the division between laws of nature and initial conditions is a condition for the possibility of modern physics and it is as prominent, in quantum mechanics as it was in the classical physics. Steven Weinberg writes,

> In modern quantum mechanics as well as in Newtonian mechanics there is a clear separation between the conditions that tell us the initial state of a system (whether the system is the whole universe, or just a part of it), and the laws that govern its subsequent evolution. (Weinberg 1992, p. 34)

Similarly, Frank Wilczek writes:

> Indeed, classical physics teaches us that the size of planetary orbits is not the sort of thing we should aspire to predict. It makes a sharp distinction between the basic laws, which govern the evolution of systems in time, and are expected to be simple, and the initial conditions, which must be given from outside. The equations of classical physics can be applied to any number of different types of solar system, having different sizes and shapes. There is nothing in Newton's

[35] By virtue of characterizing laws in this way, Wigner takes a route different from a more metaphysically inclined way of understanding laws, as statements of a necessary relation between events. He rather offers an anti-realist account of laws as statements of correlations between events, as opposed to a more strict causal conception, where causality is tightly linked to metaphysical necessity. Using an anti-realist undertone, he defines physics not as concerned with nature but with regularities that we *experience* among events.

[36] Wigner writes, "The surprising discovery of Newton's age is just the clear separation of laws of nature on the one hand and initial conditions on the other. The former are precise beyond anything reasonable; we know virtually nothing about the latter" (Wigner 1964a, p. 40).

> laws of gravity and mechanics, nor for that matter in the other pillar of classical physics, Maxwell's electrodynamics, that could serve to fix a definite size. (Wilczek 1999, p. 303)

Take the case of freely falling bodies. Galileo observed that two rocks, dropped from the same height at the same time, reach the ground at the same time. This was surprising because it didn't matter whether they were dropped from a building or a leaning tower, whether they were dropped by a man or a woman, whether they were dropped on a sunny day or a rainy day, etc. Not only was the result of the experiment independent of so many conditions that could have affected it, such as the ones just mentioned, but Galileo also realized that the length of the time that it takes two objects to reach the ground is independent of their shape, size and material. This surprising feature contains the statement of the law which can be expressed mathematically. The length of the time t taken for a freely falling body to travel distance d is only dependent on the distance:

$$t = \sqrt{\frac{2d}{g}}$$

or equivalently for the travelled distance:

$$d = \frac{1}{2}gt^2$$

where g is the gravity of earth approximately 9.81 m/s^2 close to the surface of the earth. In the context of the second law of motion in Newtonian mechanics, it means that the gravitational force that acts on a falling body (F) is independent of its size, material and shape and is only proportional to its mass (m): $F = mg$.

The second law of motion in this case only states the relationship between the mass of the falling body and the gravitational force that acts on it. It doesn't say anything about the initial conditions for this law: why the object has the mass that it does, why the earth on which the experiment is done has a particular mass, why the earth has a specific gravity g, etc- which are the initial conditions for this law.[37]

What is striking, moreover, is the fact that the result of the experiment is dependent on a very small set of conditions. In this case, the time that takes a falling body to reach the ground is only proportional to the distance. This, of course, didn't have to be the case. We can imagine a world in which the result of every experiment is dependent on an indefinite number of conditions. Without these surprising features, the argument goes, it would be impossible to formulate laws of nature.

[37] As another example take the case of Halley's comet. Bigelow and Pargetter write: "If we want to explain why, say, Halley's comet returns about every seventy-six years, we will use the laws of mechanics, but we will also have recourse to matters of particular fact. For instance, we will have to feed in details on the mass of the comet, its position and velocity at some given moment, the paths of the planets, and so forth. These facts are clearly not laws of nature - they are particular, not general, and they not only could have been otherwise, but have been otherwise, and will be otherwise again. They are what we call initial conditions (Bigelow and Pargetter 1991, pp. 300–301)."

The laws of nature, however, are only a small part of our knowledge of the inanimate world. The rest, what we call *initial conditions*, only appear in laws as the antecedents. The laws are otherwise silent on them. Understood in this way, laws of nature are conditional statements with these initial conditions as their antecedents[38]:

$$(A, B, C, \ldots \to X_1, X_2, \ldots) \tag{1}$$

$A, B, C, \ldots$ are these initial conditions and $X_1, X_2, \ldots$ are events that follow from them.[39].

5.2 Invariance principles

What makes it possible for us to formulate the division between initial conditions and regularities is the fact that the correlations between events hold under different circumstances, or as mathematicians put it, under different transformations. Had it been the case, for instance, that a regularity would change from one point to another, and from a particular time to another, we wouldn't have been able to formulate laws. A surprising feature of Galileo's case is that the regularity that he discovered is true not only in Italy and in Galileo's time but virtually at any place on earth and at any time. That is, this regularity is *invariant* with respect to changes in time and place. The invariance with respect to changes (transformations) in space and time, or equivalently, the assumption about the homogeneity of space and the uniformity of time, is an example of what we call an *invariance* or *symmetry principle*.

Without this invariance, we couldn't assume that the result of the experiments done in the lab or carried out on a small sample of observations hold *universally*. More generally, we couldn't make the assumption that with the same initial conditions, the result will be the same. We couldn't assume that the laws of nature will hold under different circumstances.[40] The underlying idea is that, for instance, there is nothing particular about the time in which we make the observations nor about the location in which the experiments are being done. This is what gives laws their universal character. Moreover, had it been the case that regularities change from one place to another, and from one time to the next, it would be impossible to test the accuracy of laws. Without this principle, as Wigner writes, "physical theories could have been given no foundation of fact".

Invariance principles, therefore, enable us to predict relations between new events based on the established relations between other events. If we know that events $A, B, C, \ldots$ entail events $X_1, X_2, \ldots$ then invariance principles enable us to infer that the occurrence of $T(A), T(B), T(C), \ldots$ also entails $T(X_1), T(X_2), \ldots$, where T is a transformation between events mapping one event to another with respect to a relevant coordinate (time, position, velocity, etc). That is:

[38] The commas in these formula can be replaced by the logical conjunction $'\wedge'$.

[39] This formulation doesn't make the universal form of these laws explicit. A better form could be $\forall x(A(x), B(x), \ldots \to X_1(x), X_2(x), \ldots)$.

[40] This is a necessary condition for any causal analysis of nature that enables one to make predictions.

$$(A, B, C, \ldots \to X_1, X_2, \ldots) \to (T(A), T(B), T(C), \ldots \to T(X_1), T(X_2), \ldots) \quad (2)$$

In general, there are three categories of such principles of invariance, which Wigner calls *older principles of invariance*.[41] The first two are transformations with respect to movement in space (Euclidean transformation) and movement in time (Time displacement). The third, the less intuitive transformation- also called Galilean principle of relativity- is an invariance with respect to transformation between the coordinates of two reference frames moving in constant velocity with respect to each other.[42] What these principles say is that laws of nature are invariant under three geometrical transformations: spatial, temporal and uniform velocity. They are geometrical in the sense that they don't change events but change only their location in space and time and their motion.[43]

The invariance principles give structure to the laws of nature: they are *laws of laws of nature*. In the same way that laws of nature are statements of regularities among events, invariance principles are statements of regularities among laws. Our knowledge of inanimate nature, modern physics, thus contains a hierarchy.

> There is a strange hierarchy in our knowledge of the world around us. Every moment brings surprises and unforeseeable events…truly the future is uncertain. There is, nevertheless, a structure in the events around us, that is, correlations between the events of which we take cognizance. It is this structure, these correlations, which science wishes to discover, or at least the precise and sharply defined correlations. …We know many laws of nature and we hope and expect to discover more. Nobody can foresee the next such law that will be discovered. Nevertheless, there is a structure in the laws of nature which we call the laws of invariance. This structure is so far-reaching in some cases that laws of nature were guessed on the basis of the postulate that they fit into the invariance structure. …This then, the progression from events to laws of nature, and from laws of nature to symmetry or invariance principles, is what I meant by the hierarchy of our knowledge of the world around us. (Wigner 1963, pp. 28–29)

At the bottom, the first tier of this hierarchy is comprised of events that we observe in this world. The regularities we discern are the second tier of the hierarchy. And invariance principles make up the third tier of the hierarchy of our knowledge.

[41] As opposed to this group of invariance principles, with which we are on *terra cognita*, newer principles of invariance aren't necessarily continuous, global or geometrical. With them we are in *terra incognita*. Symmetries of general theory of relativity, symmetries of quantum mechanics such as permutation symmetry are among the latter group. These principles, unlike the first group, have no roots in the history of science, hence with them we enter *terra incognita.*

[42] In Newton's *Principia* the third invariance principle is recognized as a corollary to the laws of motion, that is, as a consequence of the laws and not one of their underlying assumptions. Huygens, unlike Newton, considered this principle as a basic postulate from which to derive laws of motion. But as it stands, and presented initially in Newton's manuscripts, it is an independent assumption.

[43] These three symmetry principles, for Wigner, are symmetries of both Newtonian mechanics and special relativity (Wigner 1964a, p. 45).

In the same way that the complexity of events around us (first tier) has made it almost impossible for physicists to study them, and thus has motivated them to study structures (regularities) that exist among them, the complexity of laws of nature (second tier) has motivated physicists to study structures that exist among laws (the regularities displayed by laws). This third tier is comprised of invariance principles. While invariance principles help physicists to better understand the underlying structure of their theories, they also serve as guides to test and predict new laws of nature, in this way they are the fundamental part of our description of nature. Invariance principles must be seen as properties of laws of nature which put constraints on what we take to be “acceptable” forms that laws of nature *can* take.

Take the example of symmetries in quantum mechanics- Wigner’s contribution to theoretical physics. The state of a particle, in quantum mechanics is described by a vector in a complex Hilbert space, $|\Psi\rangle$. Wigner defined a symmetry transformation as a map

$$\mathcal{T} : \Psi \mapsto T(\Psi)$$

where T is an operator on the Hilbert space that preserves $|\langle\Psi, \Phi\rangle|$ for all Φ and Ψ. (The reason for this condition is that physical predictions are given by transition probabilities $|\langle\Psi, \Phi\rangle|^2$.) Wigner proved the groundbreaking result that: T is either linear and unitary: $T\langle\Psi, \Phi\rangle = \langle T\Psi, T\Phi\rangle$ or anti-linear and anti-unitary: $T\langle\Psi, \Phi\rangle = \langle\Phi, \Psi\rangle^*$. (The latter are time-reversal transformations) This theorem which is called Wigner’s theorem is a cornerstone of quantum mechanics.[44]

Now due to linearity of the symmetry transformation, if $|\Psi\rangle$ is a state of the system $T|\Psi\rangle$ is also an state of the system where T is an operator in the Hilbert space corresponding to the symmetry transformation $\mathcal{T}$. Moreover, the superposition of any two states is also an state of the system. That is $|\Psi\rangle + T|\Psi\rangle$ is a another allowed state of the system. The former has a counterpart in classical mechanics . That is, if $x(t)$ is a solution to an equation of motion which is invariant with respect to the translation S, then $S(x(t))$ which is the translated $x(t)$ is also a solution to the equation. What is really striking is that the latter property, due to superposition, has no correspondence in the classical mechanics. That’s why Wigner writes: “In quantum theory invariance principles permit even further reaching conclusions than in classical mechanics.”[45]

It would be too hasty, on Wigner’s view, to conclude that it had to be the case that this structure exists among laws of nature. Our world is *contingently* such that these structures exist. Moreover, the existence of this hierarchy is not a necessary condition for our experience of the world in a transcendental, Kantian way. It is only in our (incomplete) state of knowledge about the events that we need laws and in our incomplete understanding of laws that we need invariance principles. In other words, once we discover all laws of nature, or the ultimate law of nature (if there is

[44] A linear operator is an operator that preserves addition and multiplication. An operator U on a Hilbert space H, $U : H \rightarrow H$ is called unitary if $UU^* = U^*U = I$ where U^* is the adjoint of U and I is the identity transformation. A unitary operator acting on a vector changes its direction but preserves the length.

[45] For a more detailed discussion see Gross (1995).

such a thing), invariance principles would lose their role and importance. Similarly if we discover all events in the world, we don't need laws, let alone the mathematical formulation of them. Wigner notes,

> If we already knew the position of the planet at all times, the mathematical relations between these positions which the planetary laws furnish would not be useful but might still be interesting. They might give us a certain pleasure and perhaps amazement to contemplate [Similarly], if we knew all the laws of nature, or the ultimate law of nature, the invariance properties of these laws would not furnish us new information. They might give us a certain pleasure and perhaps amazement to contemplate, even though they would not furnish any new information. (Wigner 1964b, p. 17)

Therefore, invariance principles are not conditions for the possibility of our cognition of the world, rather, they are conditions for the possibility of *our modern physics*, which would lose its effect had we known all events in the world, past, present and future.

6 Wigner's solution

Now it starts to look like we are coming close to a solution to Wigner's applicability problem—what is it about physics that makes mathematics an appropriate language for the formulation of its laws? Wigner's answer is based on the hierarchical structure of modern physics, and more precisely on the role of invariance principles in this structure.

The invariance principles give laws their universality, a feature that makes their mathematical formulation possible. Mathematics is a science that is concerned with the study of formal structures and the most general relations that exist among objects, abstracted from their temporality and particularity.[46] And symmetry or invariance principles help us to catalogue and most importantly identify these structures. As Weyl has argued, mathematics deals with structures in two basic ways: a) by topology, b) by algebra. Topological structures use mostly analysis while algebraic structures use mostly construction and composition. The central question of topology is how an element is related (located) with respect to the other elements. The question of algebra is how two elements are combined to get a third one. Given the diversity of algebraic systems, symmetry principles are required for identifying and analyzing structures. And that's why they are so important to the study of physics. See Weyl's papers on "Topology and Abstract Algebra as Two Roads of Mathematical Comprehension" Weyl (1995).

[46] More precisely, mathematics and physics are the result of a similar abstraction where relations between objects are pushed to their limit, abstracted from their particularity. Of course the subject of mathematics is more idealized and hence farther from the physical world. A closer study of this abstraction and idealization is subject of another paper under construction.

Based on the existence of the principles of invariance, mathematics has provided an appropriate language for the study of laws in modern physics:[47] that is ELE has the status of a *law*. Had it been the case that the regularities in nature with which physics is concerned lack this overarching structure, mathematics would lose its usefulness in their formulation (as laws of nature). Moreover, it is very important to bear in mind that the regularities and invariances of nature that allow the success of mathematics are only approximate. There is no conservation of energy, globally speaking, in our universe, because there is no time translation symmetry. The universe, or space, doesnt look the same now vs. billions of years ago. However, on small enough distance and time scales it has these approximate properties.

With the discovery of the new laws of nature, moreover, there is no guarantee that mathematics will continue to play such an important role in physics. The disquieting feature, that ELE and all empirical laws have, is that we don't know their limitations. Moreover, we might in the future come up with a different set of laws, or a unified theory of physics and biology, for instance, which doesn't share any of the qualities that our current laws of nature have. This "unnerving possibility" is the reason why ELE is catalogued as the *empirical* law of epistemology.

Note that it didn't have to be the case that the regularities in nature have such an overarching structure. Similarly it didn't have to be the case that the regularities in nature have a precision that allows a mathematical formulation. Both of them are dependent on the existence of invariance principles which are contingent aspects of the world we live in. The emphasis is that, unlike what most scientists might believe today, these principles are based on experience rather than *a priori* truths. What seems to have prompted the misconception about the *a priori* status of these principles, is at least in part, Einstein's work on special relativity and the *reversal of the trend*. As Wigner puts it, "Einstein's work on special relativity established the older principles of invariance so firmly that we have to be reminded that they are based only on experience" (Wigner 1949, p. 5).

To be more precise, such invariances are features of our observations of the world we live in. We can check their validity (verify or falsify them) by setting up situations that are symmetric with respect to invariant properties under investigation and see whether the symmetry remains preserved.[48] For instance we can see whether it is *really* true that if we drop an object somewhere at one point, it will fall in the same way as it

47 Wigner devotes the bulk of his paper to three examples: Planetary Motion, Heisenberg's Matrix Mechanics, and Quantum Electrodynamics, in order to show the appropriateness of the language of mathematics for theories of physics.

48 As an example take the case of the principle of parity which was rejected due to Wu's experiment. Wigner formalized the principle of conservation of parity (mirror-symmetry) in 1927 as a principle according to which our world and its mirror image behave in the same way, with only difference that left and right are reversed. Wu's experiment, carrie din 1956, was an experiment for checking the truth of parity conservation for beta decay. To all physicists surprise, the result of the experiment was the parity is violated (in fact its violation was maximal) . It was surprising because the physicists believed the conservation of parity holds everywhere. Wigner writes, "Only very elementary theory was necessary to see that Wu's experiment was in conflict with parity principle" (Wigner 1963, p. 31) which led ultimately to the refutation of parity conservation in this case.

falls somewhere else. As this implies, such invariance principles (older invariance principles) are empirical.[49]

As it stands, ELE doesn't add another tier to Wigner's hierarchy: in this sense it is different from laws of nature and invariance principles. However, like invariance principles, it states a property of the laws of nature. Every law of nature, according to ELE, involves mathematical concepts and relations. That is, the correlations between events at the first tier are expressed mathematically (in form of laws at the second tier). What made it possible for the laws to be formulated based on events, is what precisely makes it possible for them to be formulated mathematically.

Now the pessimist might say that Wigner's ELE is at best "an article of faith of the theoretical physicist", especially when it concerns the *future* use of mathematics. Well, that might be the case, Wigner responds, but "what [is] called our article of faith, can be well supported by actual examples". The point is that the history of physics is full of examples that give (empirical) grounds to this "faith": which Wigner calls *Empirical Law of Epistemology*.

7 Conclusion

Wigner's paper closely and contextually studied, I argued, provides an original formulation and solution to the applicability problem in the context of the practice and history of modern theoretical physics. First, he turns a metaphysical question about the relation of mathematics to *reality*, Dirac and Courant's question, into an epistemological question about the relationship between mathematics and *physics* and then solves the latter by appealing to invariance principles. The emphasis here is on aspects of modern physics that are often overlooked, in particular, the division between events, laws of nature and invariance principles.

As a consequence of understanding the applicability problem in this way, we cannot immediately infer that we live in a mathematical universe, *a la* Max Tegmark or that our universe is a mathematical object, *a la* Ted Sider. ELE is a law of epistemology, not a law of nature, in the sense that it guides *our* current study of nature in modern physics. Moreover, it is not a necessity that every study of nature must be mathematical: neither our current biology, for the most part, nor the Aristotelean physics are mathematical.

The mathematical formulation of a physical theory, moreover, must not be taken as a guide to its truth. The danger is that the physicist might take his arrival at a beautiful formulation a sign that he has arrived at truth.[50] Such physicist forgets what Wigner calls the "nightmare of the theorist": the existence of numerous examples of theories, with elegant mathematical formulation, and "alarmingly accurate" description of a group of phenomena, which are nonetheless considered to be false (by our standards). Wigner lists a couple of these examples: Bohr's early atomic theory and free-electron

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49 Note that the situation is not quite the same for testing the validity of every older invariance principle. Given the gravitational field of the earth, on which we do our experiments, for instance, it might be extremely hard (if not impossible) to compare experiments on earth to check the rotational invariance. For an interesting discussion on possible differences see (Wigner 1995, 390–392).

50 Steiner for instance gives examples of this attitude among physicists in *Applicability of Mathematics as a Philosophical Problem*.

theory. The latter theory in solid-state physics states that a metal has a gas of electrons free to move within it (which then explains -among other things- the high electrical conductivity of metals).[51] This theory, as Wigner writes, "gives a marvelously accurate picture of many, if not most, properties of metal, semiconductors and insulators." (Wigner 1960, p. 236)[52]

Wigner's formulation, on my view, is quiet on what it is about mathematics that makes it an appropriate language for its use in modern physics. To answer this question, we need an elaborate account of mathematics (in general), based on its history and practice.[53] Moreover, to explain makes a *specific* mathematical concept useful in a specific connection (in physics) needs a case by case study, which requires the space of its own.

In a subtle and rather complicated attempt at deconstructing a mystery among an audience gathered to celebrate this mystery, Wigner emphasizes the limitations of ELE, "the nightmare of the theorist", and its uncertain future. Then to end on a more *cheerful* note, he *changes* his tone,

> The miracle of the appropriateness of the language of mathematics for the formulation of laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

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[51] I am not sure why Wigner said that the free electron theory is considered false.The free electron theory in condensed matter is not considered false. Its a pretty good working model that is still taught in condensed matter course. Bohrs early atomic theory, on the other hand, is not correct because it fails to work for any atom except hydrogen. The latter suffices to make Wigner's point about the relationship between mathematical formulation and truth.

[52] Beauty as a guide to truth, a slogan of theoretical physics and a regulative principle in its practice, gives the theorist emotional encouragement that she needs, echoing Hardy's famous statement: "Beauty is the first test: there is no permanent place in this world for ugly mathematics." But this emotional encouragement, taken as a true statement about the fundamental structure of the universe, can be very misleading.

[53] Lützen 2011 argues that the origin of physically useful mathematical concepts is physical (Lützen 2011).

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