Using Persistence to Generate Incentives in a Dynamic Moral Hazard Problem*

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Abstract

This paper studies how persistence can be used to create incentives in a continuous-time stochastic game in which a long-run player interacts with a sequence of short-run players. Observation of the long-run player's actions are distorted by a Brownian motion and the actions of both players impact future payoffs through a state variable. For example, a firm or worker provides customers with a product, and the quality of this product depends on both current and past investment choices by the firm. I derive general conditions under which a Markov equilibrium emerges as the *unique* perfect public equilibrium, and characterize the equilibrium payoff and actions in this equilibrium, for any discount rate. I develop an application of persistent product quality to illustrate how persistence creates effective intertemporal incentives in a setting where traditional channels fail, and explore how the structure of persistence impacts equilibrium behavior. This demonstrates the power of the continuous-time setting to deliver sharp insights and a tractable equilibrium characterization for a rich class of dynamic games.

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1 Introduction

Past choices play a central role in determining current and future profitability. For example, a worker's rating on a platform depends on the quality of service she has provided to previous customers. A firm's ability to make high quality products is a function not only of its effort today, but also its past investments in developing technology and training its workforce. A government's success in achieving an objective depends on both past and current policy choices. When actions have a persistent effect on key features of subsequent interactions, and in turn, these features impact earnings, this strengthens incentives to earn and maintain a good rating, develop a quality product or implement a certain policy.

This paper studies how persistence can be used to create incentives in a continuous-time stochastic game in which a long-run player, such as a worker, firm or government, interacts with a sequence of short-run players, such as customers or constituents, and the long-run player's action is imperfectly observed – it is distorted by Brownian motion. Persistence refers to the fact that actions noisily impact an observable state variable, such as a worker's rating, a firm's product quality or a government's policy objective, *and* this state variable influences payoffs. It can capture either an endogenous design choice at the beginning of the game, such as how a rating system aggregates past signals and how a worker is rewarded for a good rating, or an exogenous feature of the environment, such as how past investment influences quality or how past policies map into current outcomes.

I establish general conditions under which a Markov equilibrium emerges as the *unique* perfect public equilibrium (PPE) in this class of stochastic games, and characterize the equilibrium payoffs and actions, for any discount rate. This is a powerful result – in contrast to a folk theorem, it determines what type of equilibria one expects to emerge in such settings and what type of behavior will generate a given payoff. In earlier related work, Faingold and Sannikov (2011) establish a similar result when short-run players have incomplete information about the long-run player's type and the state is the belief that the long-run player is committed to choosing a certain action.

The tractability of the continuous-time setting yields sharp insights into how incentives and payoffs depend on the structure of persistence, such as the depreciation rate of investment, and how the dynamics of behavior depend on observable outcomes, such as the rating of a restaurant or the current level of a policy indicator. The equilibrium characterization can be used to address important design questions, such as designing the optimal reward structure of a rating platform or determining the optimal durability for a production technology, and how these design choices depend on the patience of the long-run player. In Section 2, I provide a specific example of these insights in an application where a firm's quality depends on both current and past investments.

Recent results on repeated games between a long-run and short-run player (Faingold and Sannikov 2011; Fudenberg and Levine 2007) show that the intersection of noise in monitoring and instantaneous adjustment of actions create a genuine challenge in providing intertemporal incentives.[1] Surprisingly, in the analogue of this paper with no persistence, the long-run player cannot earn an equilibrium payoff above the best static Nash payoff. Skrzypacz and Sannikov (2007) show that this issue also arises in games between multiple long-run players in which deviations between individual players are indistinguishable. In contrast, the equilibrium characterization in this paper demonstrates that persistence creates effective intertemporal incentives and enables the long-run player to earn payoffs above the static Nash. This higher payoff stems from both the incentive to invest in building the state and from non-myopic strategic interaction with the short-run players.

The literature on reputation with behavioral types is another important and well-understood mechanism of how to overcome moral hazard in similar settings (Faingold and Sannikov 2011; Fudenberg and Levine 1989, 1992). If consumers believe that there is a chance that the firm is committed to choosing high effort, then the firm will be able to charge a higher price for its product and earn positive profit. A consumer is willing to pay a higher purchase price to a firm with a high reputation (defined as the belief that it is the commitment type), as she believes that there is a very high likelihood that the firm is exogenously committed to providing high quality. Incomplete information creates a form of persistence, as the short-run players' beliefs depend on past action choices. However, fixing a strategic firm's patience, such reputation effects vanish in the ex-ante probability of behavioral types, and so the effectiveness of this persistence via reputation requires a non-trivial fraction of behavioral types.[2]

The connection with the reputational literature motivates several key insights.

[1] Abreu, Milgrom, and Pearce (1991) first examined incentives in repeated games with imperfect monitoring and frequent actions and established that shortening the period between actions has a crucial impact on the ability to structure effective incentives.

[2] Kreps, Milgrom, Roberts, and Wilson (1982); Kreps and Wilson (1982) and Milgrom and Roberts (1982) first demonstrated that reputation, in the form of incomplete information about a player's type, has a dramatic effect on equilibrium behavior. Mailath and Samuelson (2001) show that reputational incentives can also come from a firm's desire to *separate* itself from an incompetent type.

First, when the firm is known to be strategic, I show that auxiliary channels – analogous to beliefs in a reputational model – can also overcome moral hazard. Second, in contrast to the temporary incentives in reputation models (Cripps, Mailath, and Samuelson 2004; Faingold and Sannikov 2011), the incentives in a stochastic game persist in the long-run.[3] Finally, at a theoretical level, I explore the general properties of a stochastic game that has powerful intertemporal incentives. The reputational game can be viewed as a specific type of stochastic game. For instance, if instead of influencing the uncertainty about whether it is a behavioral type, a strategic firm makes a costly initial investment in a new production technology that benefits customers today and in the future, we would see similar intertemporal incentives in the resulting stochastic game.

This final point merits a closer comparison with Faingold and Sannikov (2011), who characterize the unique MPE in the stochastic game that corresponds to a continuous time reputation model. In their paper, payoffs and the evolution of the state take a specific form due to Bayesian updating. My characterization builds on the techniques in their paper to understand more generally what properties of stochastic games are needed for uniqueness of MPE and non-degenerate intertemporal incentives. I analyze a general class of stochastic games that places few restrictions on the process governing the evolution of the state and the structure of payoffs. The key technical advancement is for the case of an unbounded state space and payoff for the long-run player, as it requires significantly different techniques to complete the analysis.

Beyond reputation models with behavioral types, a rich literature analyzes dynamic games with a state variable. Persistence plays a prominent role in these models in that effort is directly linked to future payoffs via the state. The literature can be loosely divided based on the observability of the state variable.

When the state is observable, incentives stem from the direct influence of effort on the transition of the state, as well as from any strategic interaction that is present between players. Ericson and Pakes (1995) were the first to analyze hidden investment and stochastic capital accumulation (the state). They study firm and industry dynamics and establish equilibrium existence when mixed entry/exit strategies are admissable. Their model is similar in spirit to the quality example presented in Section 2. Recent work by Doraszelski and Satterthwaite (2010) modify Ericson and Pakes (1995) to guarantee the existence of a pure strategy MPE in cutoff entry/exit strate-

[3]Long-run reputation effects are also possible in models with behavioral types when consumers cannot observe all past signals (Ekmekci 2011) or the type of the firm is replaced over time.

gies, which is computationally tractable. Neither paper establishes uniqueness, but instead focus on the dynamics associated with a particular MPE. More broadly, MPE is the workhorse solution concept across industrial organization and political economy. Many additional papers study the existence, uniqueness and dynamics of MPE. A comprehensive review of this literature is beyond the scope of this paper.[4]

Several folk theorems exist for discrete time stochastic games with observable states, beginning with a perfect monitoring setting in Dutta (1995), and extending to imperfect monitoring environments in Fudenberg and Yamamoto (2011) and Hörner, Sugaya, Takahashi, and Vieille (2011). My setting differs in that there is a single long-run player and information follows a diffusion process. It is already known that these two changes significantly alter incentives in standard repeated games (compare the folk theorem in Fudenberg, Levine, and Maskin (1994) with the equilibrium degeneracy in Faingold and Sannikov (2011); Fudenberg and Levine (2007)). The intuition is similar for the stochastic games folk theorems and the MPE uniqueness result in this paper.

When the state is unobservable, incentives stem from the long-run player's ability to manipulate the public belief about the state through her effort choice. Cisternas (2016) analyzes a model with hidden states that is similar to the model in this paper. He characterizes necessary conditions for the existence of Markov equilibria, and sufficient conditions in two more restrictive classes of games (linear-quadratic flow payoffs or bounded marginal flow payoffs for the long-run player). Hidden states significantly complicate the model, and it is not possible to establish uniqueness results or a full equilibrium characterization, as it is in this paper with an observable state.

Board and Meyer-ter vehn (2013) study a reputation model without behavioral types by allowing a firm's hidden quality to depend on past effort. In their model, reputation is consumers' belief that the firm has a high quality product. They characterize reputational dynamics and show how incentives depend on the signal process. My paper differs in focus in that the state is observable, there is strategic interaction between the long-run and short-run players (in their paper, the consumers' payoffs do not directly depend on the firm's effort), and the model encompasses different economic settings, including games where payoffs are nonlinear or non-monotonic with respect to the state.

The organization proceeds as follows. Section 2 presents an example. Section 3

[4]The paper also relates to a broad literature on stochastic games and existence of Markov equilibria, beginning with Shapley (1953) and Sobel (1973), who examines equilibrium existence in continuous-time stochastic games with perfect monitoring.

sets up the model and characterizes the structure of PPE. Section 4 presents the three main results: existence of a Markov equilibrium, characterization of the PPE payoff set and uniqueness of a Markov equilibrium in the class of all PPE. Finally, Section 5 presents a set of structural results on the shape of equilibrium payoffs. All proofs are in the Appendix.

2 Example

This section presents two variations of the canonical product choice setting, in which a monopolist firm provides a product to a continuum of short-run consumers and the firm's effort has a persistent effect on the quality of the product. These examples will be used throughout the paper to illustrate the model.

Example 1. At each instant t, the firm chooses an unobservable effort level $a_t \in [0, \overline{a}]$, where $\overline{a} > 0$. Past effort influences quality through an observable state variable

$$X_t = \int_0^t e^{-\theta(t-s)}(a_s ds + dZ_s),$$

where $(Z_t)_{t \geq 0}$ is a standard Brownian motion and $\theta > 0$ determines the decay rate of past effort.[5] A higher state X_t is indicative of higher past effort. The firm's quality depends on both current and past effort, $q(a, X) = (1-\lambda)a + \lambda X$, where $\lambda \in [0, 1]$ captures the relative importance of past effort. Consumers purchase a single unit of the product, and are willing to pay their expected value for the product, $\overline{b} = q(a, X)$.[6] The average discounted profit of the firm is the difference between revenue and the cost of effort,

$$r \int_0^\infty e^{-rt}(\overline{b}_t - a_t^2/2)dt,$$

where $r > 0$ is the discount rate.

In the unique perfect public equilibrium (PPE) with no persistence, $\lambda = 0$, the firm exerts zero effort, $a^* = 0$, and consumers' willingness to pay is zero, $\overline{b}^* = 0$ (application

[5] In a slight abuse of notation, the Lebesgue integral and the stochastic integral are placed under the same integral sign.

[6] In this example, the willingness to pay can become unboundedly large or negative. While this is an undesirable feature from an economic standpoint, the simple formulation allows for a clean illustration of the equilibrium characterization. Example 2 adds non-negativity and budget constraints to this set-up.

of Theorem 3 in Faingold and Sannikov (2011)). Intertemporal incentives break down, despite the fact that the firm would earn more if it could commit to higher effort.

In this paper, I show that persistence incentivizes the firm to choose a positive level of effort and earn positive profits. Theorems 1-3 establish that there is a *unique* perfect public equilibrium (PPE), which is Markov. Assume $\overline{a} > 1/\theta$ to rule out the case where the upper bound on effort is binding. The effort level and profit in this unique equilibrium are characterized as a function of the level of persistence λ, the decay rate of past effort θ and the discount rate r. The firm chooses effort level

$$a(X) = \frac{\lambda}{r+\theta},$$

which is increasing as past effort plays a larger role in determining current quality, effort decays at a slower rate or the firm becomes more patient. Effort is strictly positive for any positive level of persistence, $\lambda > 0$. At stock quality X, the firm's equilibrium continuation value is

$$U(X) = \frac{(1-\lambda)\lambda}{r+\theta} + \frac{\lambda^2}{2(r+\theta)^2} + \left(\frac{r\lambda}{r+\theta}\right) X. \tag{1}$$

This payoff has three components. The first term captures the component of revenue that stems from the impact that *current effort* has on current quality. This arises from the strategic interaction between the firm and consumers; if the consumer expects higher effort today, she is willing to pay a higher price today. The second term captures the firm's future net return on current effort, which arises from the link between current effort and future quality. It is the present value of the higher future prices received for the impact that current effort has on future quality, minus the cost of this effort. In this example, these first two effects are independent of the state, since equilibrium effort is constant for all X. The third term captures the firm's expected return from the current stock quality. It is the expected present value of the revenue the firm would receive from selling a product of quality $q(0, X)$, absent any additional effort and allowing X to decay at rate θ. As the firm becomes more patient, this third term converges to zero, as the current stock quality has a negligible impact on long-run payoffs.

The tractable equilibrium characterization can be used to study the optimal structure of persistence. Intuitively, the firm always wants to reduce the rate of decay of past investment, as $U(X)$ is decreasing in θ. The optimal persistence λ^* depends on the decay rate θ and discount rate r. Suppose the firm starts with no stock, $X_0 = 0$.

If the firm is relatively patient and investment decays slowly, $r + \theta \leq 1$, then it is optimal for quality to be fully determined by past investments, $\lambda^* = 1$. If the firm is less patient or investment decays quickly, $r + \theta > 1$, then it is optimal for quality to partially depend on current investment, $\lambda^* = \frac{r+\theta}{2r+2\theta-1}$. This is calculated directly from maximizing (1).

Example 2. Maintain the same set-up as Example 1, but suppose that prices must be non-negative and consumers have a budget constraint – they cannot pay more than $\overline{B}$. Customers are willing to pay $\bar{b} = \min\{\max\{0, q(a, X)\}, \overline{B}\}$. This example also has a unique PPE, which is Markov. Although a closed form solution for the continuation value and equilibrium effort is not possible, the equilibrium characterization demonstrates that, for any $\lambda > 0$, the firm to chooses a positive level of effort at some states and earns positive profits.

The characterization also yields insight into the shape of the continuation value. Profits are convex in the state at low levels and concave at high levels. When the state is high, consumers are paying close to their maximum, and further quality improvements have little value to the firm. Volatility lowers profits since negative quality shocks reduce revenue more than positive quality shocks increase revenue. On the other hand, when the stock quality is low, the firm faces the potential for substantial gains if quality rises. Revenue is low, so the risk of loss from a negative quality shock is small. For example, if quality is a measure of innovation and the product has value only when X crosses a threshold such that $q(a(X), X)$ is positive, then volatility raises profits.

In both examples, persistence creates long-run incentives for a firm to build its quality. This contrasts with models in which the incentive to produce high quality is derived from consumers' uncertainty over the firm's payoffs, and reputation effects vanish in the long-run (Cripps et al. 2004; Faingold and Sannikov 2011).[7]

[7]Mathematically, persistence creates a long-run effect when there is positive probability that the state variable does not converge to an absorbing state. In this example, there are no absorbing states, and the requirement is satisfied trivially.

3 Model

3.1 Model Set-up

A long-run player and a continuum $I = [0, 1]$ of identical short-run players, indexed by i, play a stochastic game with imperfect monitoring in continuous time. At each instant of time $t \in [0, \infty)$, a publicly observable state variable X_t in nonempty closed interval $\mathcal{X} \subset \mathbb{R}$ determines the information structure, action set and feasible flow payoffs. Denote the initial state by X_0 and the (possibly infinite) upper and lower endpoints of $\mathcal{X}$ by $\overline{X}$ and $\underline{X}$, respectively. At each t, long-run and short-run players simultaneously choose actions a_t from A and b_t^i from $B(X_t)$, respectively, where A is a nonempty compact subset of a Euclidean space and $B(X)$ is a nonempty compact subset of a closed Euclidean space B. Denote the set of feasible short-run player actions and states as $E = \{(b, X) \in B \times \mathcal{X} | b \in B(X)\}$. Assume that the boundary of the feasible set of actions for short-run players grows at most linearly with the state – that is, there exists a $K_b, c_b > 0$ such that for all $(b, X) \in E$, $|b| \leq K_b|X| + c_b$.[8]

Individual actions are privately observed. Players observe the aggregate distribution of short-run players' actions, $\bar{b}_t \in \Delta B(X_t)$, and a noisy public signal of the long-run player's action, $(Y_t)_{t\geq 0}$. Given X_0, the public signal and state evolve stochastically according to a system of stochastic differential equations,

$$\begin{bmatrix} dY_t \\ dX_t \end{bmatrix} = \begin{bmatrix} \mu_y(a_t, \bar{b}_t, X_t) \\ \mu_x(a_t, \bar{b}_t, X_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_y\left(\bar{b}_t, X_t\right) \\ \sigma_x\left(\bar{b}_t, X_t\right) \end{bmatrix} \cdot \begin{bmatrix} dZ_t^y \\ dZ_t^x \end{bmatrix} \tag{2}$$

where $(Z_t^y, Z_t^x)_{t\geq 0}$ is a d-dimensional Brownian motion, $\mu_y : A \times E \to \mathbb{R}^{d-1}$ is the drift of the public signal, $\mu_x : A \times E \to \mathbb{R}$ is the drift of the state variable, $\sigma_y : E \to \mathbb{R}^{(d-1)\times d}$ is the volatility of the public signal and $\sigma_x : E \to \mathbb{R}^d$ is the volatility of the state variable. Assume μ_y, μ_x, σ_y and σ_x are Lipschitz continuous. The drift of the public signal and state provide a signal of the long-run player's action and can also depend on the aggregate action of the short-run players and the state. Volatility is independent of the long-run player's action to maintain the assumption of imperfect monitoring. Each function can be linearly extended to $A \times \Delta E$ or ΔE, respectively, where (in a slight abuse of notation), $\Delta E = \{(\bar{b}, X) \in \Delta B \times \mathcal{X} | \operatorname{supp} \bar{b} \subset B(X)\}$. Let $(F_t)_{t\geq 0}$ represent the filtration generated by public information, $(Y_t, X_t)_{t\geq 0}$. Short-run players receive no

[8] I use $|\cdot|$ to denote the Euclidean norm for vectors.

information about the long-run player's action beyond what is contained in $(F_t)_{t\geq 0}$.

Define a state X as an *absorbing state* if the drift and volatility are both zero, $\mu_x(a,b,X) = \mathbf{0}$ and $\sigma_x(b,X) = \mathbf{0}$ for all $(a,b) \in A \times B(X)$. I assume that the volatility of the state variable is positive at all states in the interior of the state space, which rules out interior absorbing states.[9] This assumption ensures that the future path of the state variable is stochastic, except possibly at the boundary of the state space.[10]

Assumption 1 (Positive Volatility). *For any compact proper subset $I \subset \mathcal{X}$, there exists a $c_I > 0$ with:*

$$\sigma_I = \inf_{\{(b,X)\in B\times I | b\in B(X)\}} |\sigma_x(b,X)|^2 > c_I.$$

Second, I assume that the rows of σ_{yy} are linearly independent, where σ_{yy} are the first $(d-1)$ columns of σ_y, and if σ_x is a linear combination of σ_y, then μ_x is the same linear combination of μ_y. This maintains imperfect monitoring by ensuring that the long-run player's action cannot be inferred from the path of the public signal and state variable.

Assumption 2 (Imperfect Monitoring).

1. *There exists a constant $c_y > 0$ such that $|\sigma_{yy}(b,X)\cdot y| \geq c_y |y|$ for all $y \in \mathbb{R}^{d-1}$ and $(b,X) \in E$.*

2. *If there exists a $(b,X) \in E$ and scalars $(\alpha_1, ..., \alpha_{d-1})$ such that $\sigma_x(b,X) = \sum \alpha_i \sigma_y^i(b,X)$, then there exists an $f : E \to \mathbb{R}$ such that for all $a \in A$,*

$$\mu_x(a,b,X) = \sum \alpha_i \mu_y^i(a,b,X) + f(b,X).$$

Payoffs. The payoff of the long-run player depends on her action, the distribution of short-run players' actions and the state. She seeks to maximize the expected value of her discounted payoff,

$$r\int_0^\infty e^{-rt} g(a_t, \bar{b}_t, X_t)dt$$

where $r > 0$ is the discount rate and $g : A \times E \to \mathbb{R}$ is a Lipschitz continuous function representing the flow payoff, which is linearly extended to $A \times \Delta E$. Short-run players have identical preferences. The payoff of player i in period t depends on her action, the distribution of short-run players' actions, the action of the long-run player and the

[9] It is a straightforward extension to allow for a finite number of interior absorbing states.

[10] If the state space is bounded, then for all $(a,b) \in A \times B(\overline{X})$, $\sigma_x(b,\overline{X}) = 0$ and $\mu_x\left(a,b,\overline{X}\right) \leq 0$, and for all $(a,b) \in A \times B(\underline{X})$, $\sigma_x(b,\underline{X}) = 0$ and $\mu_x\left(a,b,\overline{X}\right) \geq 0$.

state, $h(a_t, b_t^i, \bar{b}_t, X_t)$, where $h : A \times \{(b, \bar{b}, X) \in B^2 \times \mathcal{X} | (b, \bar{b}) \in B(X)^2\} \to \mathbb{R}$ is a continuous function, which is linearly extended to $A \times \{(b, \bar{b}, X) \in B \times \Delta B \times \mathcal{X} | b \in B(X), \operatorname{supp} \bar{b} \subset B(X)\}$. The dependence of payoffs on the state variable creates a form of action persistence, since the state variable depends on prior actions.

To ensure that the expected discounted payoff of the long-run player is well-behaved requires a restriction on either the rate at which the state variable can grow or the flow payoff of the long-run player. Either the drift of the state grows at a linear rate less than the discount rate or the flow payoff is bounded with respect to X.

Assumption 3. *At least one of the following conditions hold.*

1. *The flow payoff g is bounded.*
2. *The drift μ_x has linear growth at a rate less than r in that there exists a $K_\mu \in [0, r)$ and $c_\mu > 0$ such that for all $(a, b, X) \in A \times E$, if $X \geq 0$ then $\mu_x(a, b, X) \leq K_\mu X + c_\mu$ and if $X \leq 0$ then $\mu_x(a, b, X) \geq K_\mu X - c_\mu$.*

No lower bound is necessary on the slope of μ_x for $X > 0$, since a negatively sloped drift pulls the state variable towards zero. Similarly, no upper bound is necessary for $X < 0$. Assumption 3 is trivially satisfied when the state space is bounded.

Strategies and equilibrium. A public pure strategy for the long-run player is a stochastic process $(a_t)_{t \geq 0}$ with $a_t \in A$ and progressively measurable with respect to $(F_t)_{t \geq 0}$. Likewise, a public pure strategy for a short-run player is an action $b_t^i \in B(X_t)$ progressively measurable with respect to $(F_t)_{t \geq 0}$. Given that small players have identical preferences, it is without loss of generality to work with aggregate strategies $(\bar{b}_t)_{t \geq 0}$. The long-run player's expected discounted payoff at time t under strategy $S = (a_t, \bar{b}_t)_{t \geq 0}$ is given by

$$V_t(S) \equiv E_t \left[r \int_0^\infty e^{-rs} g(a_s, \bar{b}_s, X_s) ds \right]$$

I restrict attention to pure strategy perfect public equilibria (PPE), as defined in Sannikov (2007).

Definition 1. *A public strategy profile $S = (a_t, \bar{b}_t)_{t \geq 0}$ is a perfect public equilibrium if, after all public histories,*

$$V_t(S) \geq V_t(S') \text{ a.s.}$$

for all public strategies $S' = (a'_t, \bar{b}'_t)_{t\geq 0}$ *with* $(\bar{b}'_t)_{t\geq 0} = (\bar{b}_t)_{t\geq 0}$ *almost everywhere, and*

$$b \in \arg\max_{b' \in B(X_t)} h(a_t, b', \bar{b}_t, X_t) \qquad \forall b \in \operatorname{supp} \bar{b}_t.$$

Timing. At each instant t, players observe the current state X_t, choose actions, and then nature stochastically determines payoffs, the public signal and next state given the current state and action profile.

3.2 Equilibrium Structure

This section extends a recursive characterization of PPE to the current setting. Given strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$, define the long-run player's continuation value as the expected value of the future discounted payoff at time t,

$$W_t(S) \equiv E_t \left[r \int_t^\infty e^{-r(s-t)} g(a_s, \bar{b}_s, X_s) ds \right]. \tag{3}$$

The expected average discounted payoff at time t can be represented as

$$V_t(S) = r \int_0^t e^{-rs} g(a_s, \bar{b}_s, X_s) ds + e^{-rt} W_t(S). \tag{4}$$

The following lemma establishes that $(V_t(S))_{t\geq 0}$ is a martingale (therefore, $E|V_t(S)| < \infty$ for all $t \geq 0$), and $(W_t(S))_{t\geq 0}$ is bounded with respect to $(X_t)_{t\geq 0}$.

Lemma 1. *Assume Assumption 3. For any public strategy profile* $S = (a_t, \bar{b}_t)_{t\geq 0}$, *initial state* X_0 *and path of the state variable* $(X_t)_{t\geq 0}$ *that evolves according to (2) given* S,

1. $V_t(S)$ *is a martingale.*
2. *There exists a* $K_W > 0$ *such that* $|W_t(S)| \leq K_W(1 + |X_t|)$ *for all* $t \geq 0$.

The result follows from Assumption 3, which ensures that either the state grows at a slow enough rate relative to the discount rate or large values of the state don't allow for unboundedly large flow payoffs. It is similar in spirit to Lemma 1 in Strulovici and Szydlowski (2015), which establishes that the value function of an optimal control problem is finite and satisfies a linear growth condition with respect to the state.

Establishing $E|V_t(S)| < \infty$ and characterizing the growth rate of $(W_t(S))_{t\geq 0}$ is not required for models that have a uniformly bounded continuation value.

The next lemma characterizes the evolution of the continuation value and the long-run player's incentive constraint in a PPE. It is the analogue of Theorem 2 in Faingold and Sannikov (2011), allowing for an unbounded state space and flow payoff.

Lemma 2. *Assume Assumptions 1, 2 and 3. A public strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$ is a PPE with continuation values $(W_t)_{t\geq 0}$ if and only if for some (F_t)-measurable process $(\beta_t)_{t\geq 0}$ in $\mathcal{L}$,*

1. *Given $(\beta_t)_{t\geq 0} = (\beta_{yt}, \beta_{xt})_{t\geq 0}$, $(W_t)_{t\geq 0}$ satisfies*

$$\begin{aligned} dW_t &= r\left(W_t - g(a_t, \bar{b}_t, X_t)\right)dt + r\beta_{yt}(dY_t - \mu_y(a_t, \bar{b}_t, X_t)dt) \\ &\quad + r\beta_{xt}(dX_t - \mu_x(a_t, \bar{b}_t, X_t)dt). \end{aligned} \tag{5}$$

2. *There exists a $K, M \geq 0$ such that $|W_t| \leq M + K|X_t|$ for all $t \geq 0$, with $K = 0$ if g is bounded.*

3. *Strategies $(a_t, \bar{b}_t)_{t\geq 0}$ are sequentially rational for almost all $t \geq 0$,*

$$a_t \in \arg\max_{a' \in A} \; g(a', \bar{b}_t, X_t) + \beta_{yt}\mu_y(a', \bar{b}_t, X_t) + \beta_{xt}\mu_x(a', \bar{b}_t, X_t) \;\; a.s. \tag{6}$$

$$b \in \arg\max_{b' \in B(X_t)} \; h(a_t, b', \bar{b}_t, X_t) \quad \text{for all } \; b \in \operatorname{supp} \bar{b}_t \;\; a.s. \tag{7}$$

The continuation value of the long-run player is a stochastic process that is measurable with respect to public information, $(F_t)_{t\geq 0}$. The drift $(W - g)$ captures the difference between the continuation value and the flow payoff; this is the expected change in the continuation value. The volatility β determines the sensitivity of the continuation value to the public signal and state; future payoffs are more sensitive when the respective volatility is larger. Sequential rationality for the long-run player depends on the trade-off between an action's impact on flow payoffs today and its expected impact on future payoffs through the drift of the public signal and state variable, weighted by β. This condition is analogous to the one-shot deviation principle in discrete time. From the Martingale Representation Theorem, the continuation value and incentive constraint are linear with respect to $(\beta_t)_{t\geq 0}$, which lends significant tractability to the model.

Definition 2 uses the condition for sequential rationality to specify an auxiliary static game parameterized by the state variable and the volatility of the continuation value.

Definition 2. *Let $S^* : \mathcal{X} \times \mathbb{R}^d \rightrightarrows A \times \Delta B$ denote the correspondence of static Nash equilibrium action profiles in the auxiliary game parameterized by (X, z_y, z_x),*

$$S^*(X, z_y, z_x) = \left\{ (a, \bar{b}) : \begin{array}{c} a \in \arg\max_{a' \in A} g(a', \bar{b}, X) + z_y \mu_y(a', \bar{b}, X) \\ + z_x \mu_x(a', \bar{b}, X) \\ b \in \arg\max_{b' \in B(X)} h(a, b', \bar{b}, X) \qquad \forall b \in \operatorname{supp} \bar{b} \end{array} \right\}. \tag{8}$$

In any PPE strategy profile $(a_t, \bar{b}_t)_{t \geq 0}$, given stochastic process $(X_t, \beta_t)_{t>0}$, the action profile at time t must be a Nash equilibrium of the auxiliary static game, $(a_t, \bar{b}_t) \in S^*(X_t, \beta_{yt}, \beta_{xt})$.

The final assumption requires that this auxiliary game has a unique static Nash equilibrium with an atomic distribution over small players' actions.

Assumption 4. *For all $(X, z_y, z_x) \in \mathcal{X} \times \mathbb{R}^d$, S^* is non-empty, single-valued and returns $\bar{b} = \delta_b$ for some $b \in B(X)$, where δ_b is the Dirac measure on action b. S^* is Lipschitz continuous on every bounded subset of $\mathcal{X} \times \mathbb{R}^d$.*

Under Assumption 4, the stage game must have a unique static Nash equilibrium. This rules out coordination games and some games with strategic complementarities. Similarly for the dynamic game, the assumption requires that for any weight z on future payoffs, there is a unique optimal action profile. This rules out some games in which the dynamic setting creates a strategic complementarity. Assumption 4 still allows for a broad class of games, including games in which actions are strategic substitutes or strategic complements with a unique fixed point. It is easy to verify, as it is stated in terms of the primitives of the model. When it fails and S^* is not single-valued, the correspondence may not be lower hemicontinuous and different techniques are necessary to characterize Markov equilibrium payoffs (Faingold and Sannikov 2011).

Assumption 4 does not trivially guarantee the equilibrium uniqueness result in the dynamic game. A PPE is characterized by a path of incentive weights $(z_{y,t}, z_{x,t})_{t \geq 0}$ that satisfy the conditions from Lemma 2. Assuming that there is a unique optimal action profile for every (X, z_y, z_x) does not preclude the existence of multiple paths $(z_{y,t}, z_{x,t})_{t \geq 0}$ that satisfy these conditions, and hence, multiple PPE. In fact, in discrete time, many games satisfy an analogous assumption and have multiple non-trivial equilibria.[11] The uniqueness result in this paper stems from showing that, in fact, there is a unique path $(z_{y,t}, z_{x,t})_{t\geq 0}$ that characterizes a PPE, and this path is Markov in the state.

We illustrate that the examples introduced in Section 2 satisfy the assumptions outlined above.

Example 1, cont. In this example, $\mu(a, \bar{b}, X) = a - \theta X$, $g(a, \bar{b}, X) = \bar{b} - a^2/2$, $B(X) = [-X, \bar{a} + X]$, $\sigma_x(\bar{b}, X) = 1$ and there is no public signal ($d = 1$). The boundaries of the feasible action set for short-run players are linear in X, and μ and g are Lipschitz continuous. The volatility is positive and constant, so Assumption 1 is satisfied and there are no absorbing states. Assumption 2 is irrelevant, since there are no public signals. The state variable has negative drift when X is high and positive drift when X is low. Therefore, part (2) of Assumption 3 is satisfied. In the auxiliary game parameterized by (X, z_x) (suppressing z_y since there is no public signal), the firm maximizes $\bar{b} - a^2/2 + z_x(a - \theta X)$, which yields $a(X, z_x) = z_x$. Therefore, consumers are willing to pay $q(a(X, z_x), X) = (1 - \lambda)z_x + \lambda X$. Both the firm and consumers' actions are unique and Lipschitz continuous in (X, z_x), satisfying Assumption 4.

Example 2, cont. The only difference from Example 1 is the set of actions for short-run players is independent of X, $B = [0, \overline{B}]$. Assumptions 1-4 are still satisfied. When the consumers' wtp is bounded, the firm's flow payoff is also bounded. Therefore, part (1) of Assumption 3 is also satisfied. Now, $q(a(X, z_x), X) = \max\{0, \min\{(1 - \lambda)z_x + \lambda X, \overline{B}\}\}$.

3.3 Discussion of Model

It is well known that the public signal can be used to punish or reward the long-run player by allowing future equilibrium play to depend on the realization of the public signal. A stochastic game adds a second channel for intertemporal incentives – the long-run player's action impacts the evolution of the state, which in turn effects the future

[11] The analogous assumption in discrete time is more complex, as the incentive weights are functions rather than scalars. The continuation value can change according to any function $z : (\mathcal{Y}, \mathcal{X}) \to W$, where $\mathcal{Y}$ is the signal space and W is the set of feasible payoffs. The simple scalar representation is possible in continuous time because the continuation value changes linearly with respect to Brownian information, a property that does not hold in discrete time.

flow payoff and information structure. The process $(\beta_t)_{t\geq 0}$ characterized in Lemma 2 captures all channels for intertemporal incentives.

The linear structure of the continuation value with respect to β_t precludes effective intertemporal incentives in a repeated game with a short-run player (Faingold and Sannikov 2011). On the boundary of the payoff set, it is not possible to tangentially transfer continuation payoffs between players due to the short-run player, and non-tangential transfers must be linear, which results in the continuation value exceeding its boundary for any $\beta_t > 0$. Thus, transfers must be zero, $\beta_t = 0$ for all t. By analogous reasoning, in a stochastic game, $\beta_t = 0$ at the state that yields the maximum payoff across all states (if such a state exists). However, in a stochastic game, β_t can depend on the state and it is possible to use non-zero linear transfers at other states without the continuation value escaping its boundary. This can lead to non-trivial intertemporal incentives that are not possible in a repeated game.[12]

The assumption that the volatility of the state is positive ensures that all states can be reached from the current state in an arbitrarily small period of time. Therefore, deviations alter the measure of the path of the state but not the support. This plays a crucial role in the structure of incentives. Consider a deviation from a_t to $\tilde{a}_t$. Given that the support of $(X_s)_{s>t}$ is the same under both strategies, the difference in the continuation value depends on the difference in measure that each strategy induces over $(X_s)_{s>t}$, which in turn depends on the difference in the drift, $\mu(\tilde{a}_t, \cdot) - \mu(a_t, \cdot)$. Combined with linearity, this yields the tractable incentive constraint for the long-run player characterized in (6).

4 Equilibrium Analysis

This section presents the main results of the paper. I establish the existence of Markov equilibria, characterize the correspondence of PPE payoffs of the long-run player, and derive conditions under which there is a *unique* PPE, which is Markov.

[12] Non-linear incentive structures, such as value-burning, are ineffective in both repeated and stochastic games, because the expected losses from false punishment exceed the expected gains from cooperating (Fudenberg and Levine 2007; Skrzypacz and Sannikov 2007).

4.1 Existence of Markov Equilibria

First I construct the set of Markov equilibria, which establishes existence and characterizes equilibrium behavior and payoffs. Given $(X, z) \in \mathcal{X} \times \mathbb{R}$, let $(a(X, z), b(X, z)) \equiv S^*(X, \mathbf{0}, z/r)$ be the optimal action profile in the equilibrium of the auxiliary static game with incentive weights $z_y = \mathbf{0}$ on the public signal and $z_x = z/r$ on the state, and let $g^*(X, z) \equiv g(S^*(X, \mathbf{0}, z/r), X)$, $\mu^*(X, z) \equiv \mu_x(S^*(X, \mathbf{0}, z/r), X)$ and $\sigma^*(X, z) \equiv \sigma_x(b(X, z), X)$ be the value of the flow payoff, drift and volatility, respectively, in this equilibrium. By Assumption 4, g^*, μ^* and σ^* are Lipschitz continuous in (X, z). Note that $g^*(X, 0)$ corresponds to the static Nash equilibrium payoff of the original game at state X.

In a Markov equilibrium, the continuation value and equilibrium actions depend solely on the state variable and are independent of the public signal. Theorem 1 constructs the set of Markov equilibria, with continuation values specified as the solution(s) $U : \mathcal{X} \to \mathbb{R}$ to an ordinary differential equation and actions as the equilibrium actions of the auxiliary static game with $z_x = U'/r$ and $z_y = \mathbf{0}$.

Theorem 1. *Suppose Assumptions 1, 2, 3 and 4 hold. Given any initial state X_0, iff $U : \mathcal{X} \to \mathbb{R}$ is a solution to the optimality equation,*

$$rU(X) = rg^*(X, U'(X)) + U'(X)\mu^*(X, U'(X)) + \frac{1}{2}U''(X) \left|\sigma^* \left(X, U'(X)\right)\right|^2, \qquad (9)$$

with linear growth (and bounded if g is bounded), then U characterizes a Markov equilibrium with:

1. *Equilibrium payoffs $U(X_0)$;*
2. *Continuation values $(W_t)_{t\geq 0} = (U(X_t))_{t\geq 0}$;*
3. *Equilibrium actions uniquely specified by $(a_t, \bar{b}_t) = S^*(X_t, 0, U'(X_t)/r)$.*

The optimality equation has at least one continuous solution that has linear growth (is bounded) and lies in the range of feasible payoffs for the long-run player. Thus, there exists at least one Markov equilibrium.

The continuation value is equal to the sum of the equilibrium flow payoff rg^* and the expected change in the continuation value $U'\mu^* + \frac{1}{2}U'' \left|\sigma^*\right|^2$. This expected change has two components: (i) the interaction between the drift of the state and the slope of the continuation value and (ii) the interaction between the volatility of the state and the concavity of the continuation value. If the continuation value is increasing in the state, then higher drift increases the expected change in the continuation value. If the continuation value is concave ($U'' < 0$), it is more sensitive to negative shocks than positive shocks. Positive and negative shocks are equally likely. Therefore, volatility decreases the expected change in the continuation value.

The equilibrium incentive weight is proportional to the slope of the continuation value, $(\beta_{yt}, \beta_{xt}) = (0, U'(X_t)/r)$. The impact of the current action on future payoffs depends on how quickly the continuation value changes with respect to the state and how the current action affects the drift of the state. When the state is at the value(s) that yields the maximum equilibrium payoff across all states, $U' = 0$ and therefore the continuation value is independent of the Brownian information, $\beta_x = 0$. This ensures that the continuation value does not escape the payoff set and in these periods, the long-run player acts myopically. However, at other states, the continuation value depends on Brownian information ($\beta_x \neq 0$), which creates non-degenerate incentives.

Example 1, cont. From the auxiliary game, $(a(X,z), b(X,z)) = (z/r, (1-\lambda)z/r + \lambda X)$. Therefore, $g^*(X,z) = (1-\lambda)z/r + \lambda X - z^2/2r^2$, $\mu^*(X,z) = z/r - \theta X$ and trivially, $\sigma^*(X,z) = 1$. Given initial state X_0, any solution to

$$rU(X) = (1-\lambda)U'(X) + \lambda r X + U'(X)^2/2r - U'(X)\theta X + \frac{1}{2}U''(X), \qquad (10)$$

with linear growth characterizes a Markov equilibrium. It is easy to verify that the continuation value (1) presented in Section 2 is a solution to (10), with slope $U'(X) = r\lambda/(r+\theta)$. Therefore, (1) characterizes a Markov equilibrium with equilibrium actions $a(X, U') = \lambda/(r+\theta)$ and $b(X, U') = \lambda(1-\lambda)/(r+\theta) + \lambda X$.

Example 2, cont. Substituting the constrained wtp $b(X,z) = \max\{0, \min\{(1-\lambda)z/r + \lambda X, \overline{B}\}\}$ into (9) yields a similar expression to (10). However, a closed for solution is no longer possible.

Outline of Proof. The first step is to show that if a Markov equilibrium $(a_t^*, \overline{b}_t^*)$ exists, then continuation values must be characterized by the solution to the optimality equation (9). In a Markov equilibrium, continuation values take the form of $W_t = U(X_t)$ for some function U. Using Ito's formula to differentiate U with respect to X_t

yields an expression for the law of motion of the continuation value in any Markov equilibrium,

$$dU(X_t) = U'(X_t)\mu_x(a_t^*, \overline{b}_t^*, X_t)dt + \frac{1}{2}U''(X_t)\left|\sigma_x(\overline{b}_t^*, X_t)\right|^2 dt + U'(X_t)\sigma_x(\overline{b}_t^*, X_t)dZ_t$$

The continuation value must also follow the law of motion (5) from Lemma 2. Matching the drifts of these two laws of motion yields the optimality equation, while matching the volatilities yields the incentive weights $(\beta_{yt}, \beta_{xt}) = (\mathbf{0}, U'(X_t)/r)$. The second step is to show that the optimality equation has at least one solution that lies in the range of feasible payoffs for the long-run player and has linear growth (is bounded when g is bounded).

The innovative part of the proof lies in establishing existence of a solution to the optimality equation when the state space is unbounded, particularly when g is also unbounded. I show by construction that there exist lower and upper solutions to the optimality equation, $\alpha : \mathcal{X} \to \mathbb{R}$ and $\beta : \mathcal{X} \to \mathbb{R}$, that have linear growth. This is only possible when the maximum drift of the state has linear growth at rate less than r (Assumption 3). The lower and upper solutions characterize bounds on the solution to the optimality equation, $\alpha \leq U \leq \beta$. Next I show that the bound on the optimality equation grows linearly with respect to U', and therefore the optimality equation does not grow too quickly (technically speaking, it satisfies a growth condition on any compact subset of the state space). These conditions establish that the optimality equation has a twice continuously differentiable solution with linear growth. When g is bounded, the lower and upper solutions are constant, which establishes existence of a bounded solution.

The final step is to show that the continuation value and actions characterized above constitute a Markov equilibrium. Given a solution U and an action profile uniquely specified at state X_t by $(a_t^*, \overline{b}_t^*) = S^*(X_t, 0, U'(X_t)/r)$ (where uniqueness follows from Assumption 4), the state variable evolves uniquely according to (2), the continuation value $(U(X_t))_{t\geq 0}$ satisfies the law of motion (5) and the action profile satisfies the conditions for sequential rationality, (6) and (7). Therefore, $(a_t^*, \overline{b}_t^*, U(X_t))$ constitute a PPE. Since the state evolves uniquely and actions are uniquely specified as a function of the state, *each* solution to the optimality equation characterizes a unique Markov equilibrium. If there are multiple solutions, then there will be multiple Markov equilibria. There are no Markov equilibria other than those characterized by these solutions.

4.2 The PPE Payoff Set

Next, I show that in any PPE, the long-run player cannot achieve a payoff above the highest or below the lowest Markov equilibrium payoff. Let $\xi : \mathcal{X} \rightrightarrows \mathbb{R}$ denote the correspondence that maps a state onto the corresponding set of PPE payoffs of the long-run player, and let $\Upsilon : \mathcal{X} \rightrightarrows \mathbb{R}$ denote the analogous correspondence for Markov equilibrium payoffs.

Theorem 2. *Assume Assumptions 1, 2, 3 and 4. Then for any state $X \in \mathcal{X}$, the set of PPE payoffs of the long-run player at state X is equal to the convex hull of the set of Markov equilibrium payoffs at state X, $\xi(X) = \mathrm{co}(\Upsilon(X))$.*

The impossibility of the long-run player achieving a PPE payoff above the highest Markov payoff yields insight into the role that persistent effect of actions play in generating intertemporal incentives. In a Markov equilibrium, the public signal is ignored. When this Markov equilibrium yields the highest equilibrium payoff, it precludes the existence of equilibria that use the public signal to build stronger incentives. Thus, the ability to generate effective intertemporal incentives in stochastic games solely stems from the link the state variable creates between the current action and future feasible payoffs.

Outline of Proof. The proof uses an escape argument similar to other papers in the literature, including Faingold and Sannikov (2011). Suppose a PPE $(W_t)_{t \geq 0}$ yields a payoff higher than the maximum Markov equilibrium payoff U at state X_0 and let $D_t = W_t - U(X_t)$ be the difference between these two payoffs. The innovative parts of the proof are to establish that indeed the volatility of D_t is bounded away from 0 on an unbounded state space and to show that the escape argument can be applied to unbounded flow payoffs when the state does not grow too quickly. When the volatility of D_t is positive, D_t will grow arbitrarily large with positive probability, independent of X_t. By Lemma 1, $|W_t(S)|$ is bounded with respect to X_t. Thus, D_t can only grow arbitrarily large when X_t grows arbitrarily large, leading to a contradiction. Faingold and Sannikov (2011) rely on the compactness of the state space to show that the volatility of D_t achieves a minimum that is bounded away from 0 and rely on the boundedness of the flow payoff to reach a contradiction when D_t grows arbitrarily large, and thus their proofs do not immediately extend to an unbounded state space or flow payoff.

Equilibrium Degeneracy without Persistent Actions. If the state evolves independently of the long-run player's action, then there is no link between the current action and future feasible payoffs and it is not possible to generate effective intertemporal incentives. The unique PPE is one in which the long-run player acts myopically and plays the Nash action profile of the static game at state X. This is the analogue of the equilibrium degeneracy result for repeated games in Fudenberg and Levine (2007) and Faingold and Sannikov (2011).

Corollary 1. *Assume Assumptions 1, 2, 3 and 4 and suppose μ_x is independent of action a for all X. Then in the unique PPE, $(a_t, \overline{b}_t) = S^*(X_t, 0, 0)$ for all $t \geq 0$ and the continuation value $U(X_t) = E_t\left[r\int_t^\infty e^{-rs} g^*(X_s, 0)dt\right]$ is characterized by the unique solution to the optimality equation.*

4.3 Equilibrium Uniqueness

This section establishes when there is a unique PPE, which is Markov. The main step is to determine when the optimality equation has a unique feasible solution. When this is the case, by Theorem 1, there is a unique Markov equilibrium and by Theorem 2, PPE payoffs are uniquely specified as the payoffs in this unique Markov equilibrium. Uniqueness of equilibrium actions follows almost immediately.

The limit behavior of U and U' as the state approaches its boundary, play a key role in determining when the optimality equation has a unique feasible solution. Any two feasible solutions that satisfy the same boundary conditions cannot differ on the interior of the state space, due to the structure of the optimality equation, and are therefore equivalent solutions. Thus, establishing that all feasible solutions satisfy the same boundary conditions is necessary and sufficient to establish that the optimality equation has a unique feasible solution.

Assumptions 5, $5'$ and $5''$ outline three sets of sufficient conditions on g^*, μ^* and σ^* to guarantee unique boundary conditions. The two key components of these assumptions are to rule out oscillation and establish that there are unique incentive weights at the boundary.

4.3.1 Unbounded Flow Payoff.

Assumption 5 establishes a set of sufficient conditions for a unique Markov equilibrium when g is unbounded.

Assumption 5. *Suppose g is unbounded.*

1. *Additive separability: there exist Lipschitz continuous functions $g_1, \mu_1 : \mathcal{X} \to \mathbb{R}$ and $g_2, \mu_2 : \mathbb{R} \to \mathbb{R}$ such that $g^*(X, z) = g_1(X) + g_2(z)$ and $\mu^*(X, z) = \mu_1(X) + \mu_2(z)$.*

2. *Monotonicity: there exists a $\delta > 0$ such that for $|X| > \delta$, $g_1'(\cdot) + z\mu_1'(\cdot)/r$ is monotone for all $z \in \mathbb{R}$.*

3. *Volatility: $|\sigma^*|^2$ is Lipschitz continuous.*

Monotonicity ensures that the slope of the continuation value U' has a well-defined limit, while additive separability ensures that the limit of U' is uniquely determined. The Lipschitz continuity of $|\sigma^*|^2$ guarantees that two different solutions cannot have the same limit slope.

Theorem 3 establishes uniqueness and characterizes the boundary conditions. Note that (13) and (14) characterize the boundary slope and continuation value in terms of the primitives of the model, and are simple to derive in applications.

Theorem 3. *Assume Assumptions 1, 2, 3, 4 and 5. Then for each $X_0 \in \mathcal{X}$, there exists a unique PPE, which is Markov. The unique solution U of (9) with linear growth satisfies*

$$\lim_{X \to p} U(X) - y(X) = g_2(z_p) + z_p \mu_2(z_p)/r \tag{11}$$

$$\lim_{X \to p} U'(X) = z_p \tag{12}$$

for $p \in \{-\infty, \infty\}$, where

$$z_p = \lim_{X \to p} \frac{rg_1(X)}{rX - \mu_1(X)} \tag{13}$$

is the the asymptotic incentive weight and y denotes the continuation value that the long-run player would earn from repeated play of the static Nash equilibrium profile,

$$y(X) \equiv -\phi(X) \int \frac{rg_1(X)}{\phi(X)\mu_1(X)} dX \tag{14}$$

and $\phi(X) \equiv \exp(\int r/\mu_1(X) dX)$.

Boundary condition (11) establishes that the continuation value converges to the payoff that the long-run player would earn by playing the static Nash equilibrium profile plus a constant, $g_2(z_p) + z_p\mu_2(z_p)/r$. If this constant is positive, then as the state becomes large (or small), the long-run player's payoff is strictly higher than the payoff from playing the static Nash profile. This constant has an important interpretation. The first term, $g_2(z_p)$, is the *equilibrium strategic interaction* between the long-run and short-run players at the boundary of the state space. It is the portion of the equilibrium flow payoff that arises from strategic interaction and captures the effect of the long-run player's action a on the short-run players' aggregate action, net of the cost of a. The second term, $z_p\mu_2(z_p)/r$ is the *investment effect*, or the impact of the long-run player's action on the state. This is measured by the portion of the drift that arises from the long-run player's action, which captures how the state changes with respect to this action, multiplied by the slope of the continuation value, which determines how the continuation value changes with respect to the state.

Boundary condition (12) establishes that the slope U' converges to a unique limit slope (13) that depends on the ratio of the growth rate of the flow payoff to the growth rate of the drift. When this slope is positive, it is possible to sustain non-trivial intertemporal incentives across the entire state space. This ability is an important and novel insight of the paper. If it is possible to sustain non-trivial incentives at the boundary of the state space, then incentives are permanent in the sense that they don't dissipate with time, regardless of the asymptotic behavior of $\{X_t\}$.

Example 1, cont. This example falls under Assumption 5, as the long-run player's flow payoff is unbounded. As shown above, $g^*(X, z) = (1-\lambda)z/r + \lambda X - z^2/2r^2$ and $\mu^*(X, z) = z/r - \theta X$, which are linearly separable in (X, z). Define $g_1(X) = \lambda X$, $g_2(z) = (1-\lambda)z/r - z^2/2r^2$, $\mu_1(X) = -\theta X$ and $\mu_2(z) = z/r$. Note that $g_1' + z\mu_1'/r = \lambda - z\theta/r$ is monotone in z. Therefore, Assumption 5 is satisfied and there is a unique PPE. Since I already verified that the continuation value (1) and actions presented in Section 2 are a Markov equilibrium, this must constitute the unique PPE.

Next, use Theorem 3 to derive the boundary conditions as $X \to \infty$. When the continuation value is linear in X, as in this example, the boundary conditions as $X \to \infty$ are sufficient to characterize the continuation value for all X. From (13), the limit slope as X grows large is

$$z_\infty = \lim_{X\to\infty} \frac{r\lambda X}{rX + \theta X} = \frac{r\lambda}{r+\theta}.$$

Note this is the slope of the continuation value (1) presented in the example; since the solution is linear in X, the slope is constant across X. Plugging in $\mu_1 = -\theta X$, $\phi(X) = \exp(-\int (r/\theta X)dX) = X^{-r/\theta}$, and from (14),

$$y(X) = -X^{-r/\theta} \int \frac{r\lambda X}{-X^{-r/\theta}\theta X} dX = \frac{r\lambda}{r+\theta} X \tag{15}$$

Plugging $z_\infty = \frac{r\lambda}{r+\theta}$ into the right hand side of (11) yields

$$g_2(z_\infty) + z_\infty \mu_2(z_\infty)/r = \frac{(1-\lambda)\lambda}{r+\theta} + \frac{\lambda^2}{2(r+\theta)^2}. \tag{16}$$

Plugging (15) and (16) into boundary condition (11) yields the limit continuation value,

$$\lim_{X\to\infty} U(X) - \frac{r\lambda}{r+\theta} X = \frac{(1-\lambda)\lambda}{r+\theta} + \frac{\lambda^2}{2(r+\theta)^2}.$$

In fact, since the solution is linear, (11) is satisfied for all X and the continuation value is equal to $y(X) + g_2(z_\infty) + z_\infty \mu_2(z_\infty)/r$ (it is straightforward to verify that this is equal to (1)).

Outline of Proof. The innovative part of this proof is to establish the boundary conditions for an unbounded flow payoff and state space. Let

$$\psi(X, z) \equiv g^*(X, z) + \frac{z}{r} \mu^*(X, z)$$

be the value of the long-run player's incentive constraint in the equilibrium of the auxiliary static game with incentive weights $z_y = \mathbf{0}$ on the public signal and $z_x = z/r$ on the state. Given the monotonicity assumption and the Lipschitz continuity of μ^* and g^*, the limits of $\psi(X, z)/X$ and $\psi'(X, z)$ exist and are equal as $X \to p \in \{-\infty, \infty\}$. Denote these limits by $\psi_\infty(z)$. Let U be a linear growth solution to (9). For any z such that $U'(X) = z$ infinitely often for all $\delta > 0$, $|X| > \delta$ and U not affine, the shape of (9) will alternate between convex and concave at slope z. Therefore, $\psi(X, z)$ will lie above $U(X)$ when (9) is concave at slope z, and below $U(X)$ when (9) is convex at slope z. This guarantees that $\lim_{X\to\infty} U(X)/X = \lim_{X\to\infty} U'(X)$ exist and are equal. Denote these limits by $z_\infty \in \mathbb{R}$.

The linear growth of U and the Lipschitz continuity of $|\sigma^*|^2$ guarantee that the impact of the second derivative converges to zero, $\lim_{X\to\infty} |\sigma^*(X, U'(X))|^2 U''(X) = 0$. Therefore, from (9), $\lim_{X\to\infty}(U(X) - \psi(X, U'(X)))/X = 0$, and from the above derivation, $\lim_{X\to\infty}(U(X) - \psi(X, U'(X)))/X = z_\infty - \psi_\infty(z_\infty)$. Combining these results yields $z_\infty = \psi_\infty(z_\infty)$, and the limit slope z_∞ must be a fixed point of ψ_∞. Assumption 5 requires that g^* and μ^* are separable in (X, z), which is sufficient to ensure that ψ_∞ has a unique fixed point. Thus, any linear growth solution U to (9) has the same limit slope z_∞.

From (9) and the uniqueness of the limit slope, there exists a $c \in \mathbb{R}$ such that any linear growth solution U to (9) satisfies $\lim_{X\to\infty} U(X) - U'(X)\mu_1(X)/r - g_1(X) = c$. Consider the linear first order differential equation

$$y(x) - y'(x)\mu_1(x)/r - g_1(x) - c = 0. \tag{17}$$

When the growth rate of μ_1 is in $[0, r)$, then there is a unique linear growth solution y of (17), and when the growth rate of μ_1 is less than zero, then any two linear growth solutions y_1 and y_2 satisfy $\lim_{x\to\infty} y_1(x) - y_2(x) = 0$. Therefore, any linear growth solution to (9) satisfies $\lim_{X\to\infty} U(X) - y(X) = 0$ for all linear growth solutions y to (17).

Let U and V be two linear growth solutions to (9). Then by the above reasoning, both solutions satisfy the same boundary conditions. Therefore, for $p \in \{-\infty, \infty\}$, $\lim_{X\to p} U(X) - V(X) = 0$ and $\lim_{X\to p} U'(X) = \lim_{X\to p} V'(X) = z_p$, where z_p is the unique fixed point of ψ_p. Similar to Faingold and Sannikov (2011), if there exists an X such that $U(X) - V(X) > 0$, the structure of (9) prevents these solutions from satisfying the same boundary conditions for at least one boundary.

4.3.2 Bounded Flow Payoff.

Assumptions $5'$ and $5''$ establish two sets of sufficient conditions for unique boundary conditions when g is bounded, for an unbounded and bounded state space, respectively.

Assumption $5'$. *Suppose g is bounded and $\mathcal{X} = \mathbb{R}$. Then there exists a $\delta > 0$ such that for $|X| > \delta$, $g^*(\cdot, 0)$ is monotone.*

Monotonicity ensures that the continuation value has a well-defined limit; this is not necessary when $\mathcal{X}$ is bounded, since U has bounded variation.

Assumption 5″. *Suppose g is bounded and $\mathcal{X}$ is compact. Then $\{\underline{X}, \overline{X}\}$ are absorbing states.*

Theorem 4 establishes uniqueness and characterizes the boundary conditions.

Theorem 4. *Assume Assumptions 1, 2, 3 and 4, and either Assumption 5′ or 5″. Then for each $X_0 \in \mathcal{X}$, there exists a unique PPE, which is Markov. The unique bounded solution U of (9) satisfies*

$$\lim_{X \to p} U(X) - g^*(X, 0) = 0 \tag{18}$$

$$\lim_{X \to p} \mu^*(X, U'(X))U'(X) = 0 \tag{19}$$

for $p \in \{\underline{X}, \overline{X}\}$.

Assumptions 5′ and 5″ ensure that intertemporal incentives dissipate at the boundary states. Monotonicity ensures that U' converges to 0 when $\mathcal{X}$ is unbounded, rather than oscillating, while absorbing boundary states ensure that the equilibrium drift converges to 0 when $\mathcal{X}$ is compact. If either the drift or the slope of the continuation value approaches 0 at the boundary, then incentives dissipate, as either the impact of the action on the state or the sensitivity of the continuation value to changes in the state approach 0. Thus, $\mu^*(X, U')U'$ converges to zero in both cases, and the long run player's incentive constraint collapses to the static incentive constraint. When incentives collapse at the boundary, the continuation value approaches the static Nash payoff, as shown in (18).

Even if incentives dissipate at the boundary, the state may not converge to its boundary and therefore, the boundary conditions in Theorem 4 do not preclude the existence of permanent incentives. If the state space is unbounded and Assumption 5′ is violated, then the continuation value can perpetually oscillate and non-trivial incentives are possible at the state grows large or small. If 5″ is violated and the continuation value is pulled away from the boundary quickly enough, then non-trivial incentives are possible at the boundary. When these assumptions fail, uniqueness will still obtain in some settings (a rigorous characterization of these settings is beyond the scope of this paper).

Example 2, cont. This example falls under Assumption 5′, as the long-run player's flow payoff is bounded but the state space is unbounded. With bounded wtp, $g^*(X,z) = \max\{0, \min\{(1-\lambda)z/r + \lambda X, \overline{B}\}\} - z^2/2r^2$ and $\mu^*(X,z) = z/r - \theta X$. Therefore, $g^*(X,0) = \max\{0, \min\{\lambda X, \overline{B}\}\}$ is monotone and Assumption 5′ is satisfied. By Theorem 4, there is a unique PPE. From (19), the limit slope as X grows large is 0. Plugging $g^*(X,0)$ into (18) yields the boundary conditions for the continuation value, $\lim_{X\to\infty} U(X) = \overline{B}$ and $\lim_{X\to-\infty} U(X) = 0$. Although it is not possible to characterize a closed-form expression for equilibrium effort, the boundary conditions and the continuity of U establish that equilibrium effort will be non-zero at some states.

5 Properties of Equilibrium Payoffs

The optimality equation yields rich insights into how the correspondence of PPE payoffs of the long-run player is tied to the underlying structure of the game. This section characterizes properties of the shape of equilibrium payoffs when there is a unique Markov equilibrium.

5.1 The Shape of Equilibrium Payoffs

The static Nash equilibrium payoff, $g^*(\cdot, 0)$, is a key determinant of the shape of U. Note $g^*(\cdot, 0)$ is straightforward to derive from the primitives of the game (specifically, g and h). Proposition 1 relates the monotonicity or single-peakedness of U to the monotonicity or single-peakedness of $g^*(\cdot, 0)$.

Proposition 1. *Assume Assumptions 1, 2, 3 and 4 and* (9) *has a unique solution.*

1. *If $g^*(\cdot, 0)$ is (strictly) monotonically increasing (decreasing) on $\mathcal{X}$, then U is (strictly) monotonically increasing (decreasing) on $\mathcal{X}$.*

2. *If $g^*(\cdot, 0)$ is single-peaked with a maximum (minimum) and $g^*(\underline{X}, 0) = g^*\left(\overline{X}, 0\right)$, then U is single-peaked with a maximum (minimum).*

3. *$g^*(\cdot, 0)$ constant on $\mathcal{X} \Leftrightarrow U$ constant on $\mathcal{X}$.*

Example 1, cont. As shown above, $g^*(X,z) = (1-\lambda)z/r + \lambda X - z^2/2r^2$. Therefore, the static Nash payoff is $g^*(X,0) = \lambda X$, which is strictly increasing in X. By

Proposition 1, U is strictly increasing in X. This is easily verified in the closed form for the continuation value (1).

Example 2, cont. Similarly, $g^*(X,0) = \max\{0, \min\{\lambda X, \overline{B}\}\}$ is increasing in X, and therefore, U is increasing in X.

If $g^*(\cdot,0)$ is not monotonic or single-peaked, it is still possible to partially characterize the shape of U. Proposition 2 gives a rich analytical description of the shape of U, including showing that the number of extrema of $g^*(\cdot,0)$, denoted n_g, bounds the number of PPE payoff extrema for the long-run player, n_U, and the shape of $g^*(\cdot,0)$ on subsets of the state space partially determines the number and type of PPE payoff extrema on this same subset.

Proposition 2. *Assume Assumptions 1, 2, 3 and 4 and* (9) *has a unique solution. Let $I \subset \mathcal{X}$ and $n_U(I)$ denote the number of strict interior extrema of U on $I \subset \mathcal{X}$.*

1. $n_U \leq n_g < \infty$.
2. *If $g^*(\cdot,0)$ is constant on I, then $n_U(I) \leq 1$.*
3. *If $g^*(\cdot,0)$ is strictly monotonic on I, then $n_U(I) \leq 2$ and U is not constant on I. If $g^*(\cdot,0)$ is strictly increasing (decreasing) on I and $n_U(I) = 2$, with X_1 a minimum and X_2 a maximum, then $X_1 < X_2$ ($X_1 > X_2$).*
4. *Suppose I contains a boundary state, $\underline{X} \in I$ or $\overline{X} \in I$. If $g^*(\cdot,0)$ is monotonic on I, then $n_U(I) \leq 1$ and if $g^*(\cdot,0)$ is constant on I, then $n_U(I) = 0$. If $g^*(\cdot,0)$ is strictly increasing (decreasing) on I and $n_U(I) = 1$, then if $\underline{X} \in I$, U has an interior maximum (minimum) on I; and if $\overline{X} \in I$, then U has an interior minimum (maximum) on I.*

The intuition for the proof stems from the behavior of U at interior extrema. If there is an extremum at state X, then $U'(X) = 0$ and the optimality equation simplifies to

$$rU(X) = rg^*(X,0) + U''(X) \left|\sigma^*(X,0)\right|^2 /2,$$

which depends on the static Nash payoff $g^*(X,0)$ and whether the extremum is a maximum or minimum, determined by the sign of $U''(X)$. Applying Proposition 2 to specific settings will yield structural empirical predictions about how equilibrium behavior and actions change with the state.

5.2 Equilibrium Payoff Bounds

If the flow payoff of the long-run player is bounded, then the correspondence of PPE payoffs of the long-run player is also bounded. Let $\overline{W} \equiv \sup_{X\in\mathcal{X}} U(X)$ and $\underline{W} \equiv \inf_{X\in\mathcal{X}} U(X)$ be the highest and lowest PPE payoffs of the long-run player across all states, and let X_H and X_L denote the (possibly infinite) states that yield these payoffs. Let $g^*(X_H, 0)$ and $g^*(X_L, 0)$ be the static Nash equilibrium payoffs at X_H and X_L, where, with a slight abuse of notation, if $X_H \in \{-\infty, \infty\}$ then $g^*(X_H, 0) = \limsup_{X\to X_H} g^*(X, 0)$ and analogously for $X_L \in \{-\infty, \infty\}$. Proposition 3 establishes that the correspondence of PPE payoffs of the long-run player, ξ, is bounded by $g^*(X_H, 0)$ and $g^*(X_L, 0)$.

Proposition 3. *Assume Assumptions 1, 2, 3 and 4 and g is bounded. Then the highest (lowest) PPE payoff of the long-run player across all states is bounded above (below) by the static Nash payoff of the long-run player at the corresponding state,*

$$g^*(X_L, 0) \leq \underline{W} \leq \overline{W} \leq g^*(X_H, 0).$$

This bound follows directly from the optimality equation. Suppose there is an interior state X_H such that $\overline{W} = U(X_H)$. When $X_t = X_H$, W_t must have a weakly negative drift and zero volatility so as not to exceed $\overline{W}$. From Lemma 2, this implies $g(a_t, \bar{b}_t, X_H) \geq \overline{W}$. From Theorem 1, $U'(X_H) = 0$, and therefore $g(a_t, \bar{b}_t, X_H) = g^*(X_H, 0)$. Combining these conditions yields $\overline{W} \leq g^*(X_H, 0)$. If the continuation value is sufficiently flat around X_H (i.e. $U''(X_H) = 0$) or X_H is an absorbing state, then $\overline{W} = g^*(X_H, 0)$. Otherwise, $\overline{W} < g^*(X_H, 0)$, as either the continuation value or the state changes too quickly at X_H to maintain $g^*(X_H, 0)$. Note that when $W_t \in (\underline{W}, \overline{W})$, W_t may lie above or below the static Nash payoff at the corresponding state, $g^*(X_t, 0)$.

Although in general it is not possible to characterize X_H from the primitives of the game (X_H does not necessarily correspond to the state that maximizes $g^*(\cdot, 0)$), and therefore the payoff bounds in Proposition 3, a weaker bound that can be characterized from the primitives of the model immediately follows. The correspondence of PPE payoffs of the long-run player, ξ, is bounded above (below) by the highest (lowest) static Nash payoff across all states.

Corollary 2. *Assume Assumptions 1, 2, 3 and 4 and g is bounded. Then*

$$\inf_{X\in\mathcal{X}} g^*(X, 0) \leq \underline{W} \leq \overline{W} \leq \sup_{X\in\mathcal{X}} g^*(X, 0).$$

6 Conclusion

This paper shows that persistence provides an important channel for intertemporal incentives and develops a tractable method to characterize Markov equilibrium behavior and payoffs. The tools developed in this paper will yield insights into equilibrium behavior in a broad range of settings, from industrial organization to political economy to macroeconomics. Once functional forms are specified for payoffs and the evolution of the state, it is straightforward to use Theorem 1 to construct a Markov equilibrium. This in turn can be used to derive empirically testable comparative statics and predictions about the dynamics of equilibrium behavior based on observable features of the environment. Future research can use this framework to address design questions in specific applications, such as determining the optimal structure of persistence.

Furthermore, the equilibrium characterization can be used for structural estimation. Markov equilibria have an intuitive appeal in empirical work, due to their simplicity and dependence on payoff relevant variables to structure incentives. Players do not need to condition on past behavior in a complex way, as actions and payoffs are fully determined by the current value of the state. Establishing that a Markov equilibrium exists and is unique provides a strong justification for focusing on this equilibrium concept.

A Appendix

A.1 Proofs from Section 3

Proof of Lemma 1. Suppose g is unbounded. By Assumption 3, there exists a $k \in [0, r)$ and $c > 0$ such that for all $(a, b, X) \in A \times E$, if $X \geq 0$ then $\mu_x(a, b, X) \leq kX + c$ and if $X \leq 0$ then $\mu_x(a, b, X) \geq kX - c$. Lipschitz continuous functions have linear growth. Therefore, by Lipschitz continuity of g and σ_x, the compactness of A and the assumption that $|b| \leq K_b|X| + c_b$ for all $(b, X) \in E$, there exists a $K_g, K_\sigma, c > 0$ such that for all $(a, b, X) \in A \times E$, $|g(a, b, X)| \leq K_g(\frac{c}{k} + |X|)$ and $|\sigma_x(b, X)| \leq K_\sigma(1 + |X|)$.

Step 1: Derive a bound on $E_\tau \left|g(a_t, \bar{b}_t, X_t)\right|$, the expected flow payoff at time t conditional on available information at time $\tau \leq t$. This bound will be independent of the strategy profile. Define $f : \mathcal{X} \to \mathbb{R}$ as

$$f(X) \equiv \begin{cases} K_g(\frac{c}{k} - X) & \text{if } X \leq -1 \\ -\frac{1}{8}K_g X^4 + \frac{3}{4}K_g X^2 + \frac{3}{8}K_g + K_g\frac{c}{k} & \text{if } X \in (-1, 1) \\ K_g(\frac{c}{k} + X) & \text{if } X \geq 1 \end{cases}$$

Note $f \in \mathcal{C}^2$, $f \geq 0$, $|f'| \leq K_g$ and

$$f''(X) = \begin{cases} 0 & \text{if } |X| \geq 1 \\ \frac{3}{2}K_g(1 - X^2) & \text{if } |X| < 1 \end{cases}$$

Ito's Lemma holds for any $\mathcal{C}^2$ function. Given a strategy profile $S = \left(a_t, \bar{b}_t\right)_{t\geq 0}$, initial state $X_\tau < \infty$ and path of the state variable $(X_t)_{t\geq\tau}$ that evolves according to (2),

$$\begin{aligned} f(X_t) &= f(X_\tau) + \int_\tau^t \left(f'(X_s)\mu_x(a_s, \bar{b}_s, X_s) + \frac{1}{2}f''(X_s)\left|\sigma_x\left(\bar{b}_s, X_s\right)\right|^2\right) ds + \int_\tau^t f'(X_s)\sigma_x\left(\bar{b}_s, X_s\right) \cdot dZ_s \\ &\leq f(X_\tau) + \int_\tau^t \left(K_g\left(k\left|X_s\right| + c\right) + 3K_g K_\sigma^2\right) ds + K_g K_\sigma \int_\tau^t \left(1 + \left|X_s\right|\right) dZ_s \\ &\leq f(X_\tau) + k\int_\tau^t f(X_s)ds + 3K_g K_\sigma^2 (t - \tau) + K_g K_\sigma \int_\tau^t \left(1 + \left|X_s\right|\right) dZ_s \end{aligned}$$

for all $t \geq \tau$, where the first inequality follows from $f'(X)\mu_x(a, b, X) \leq K_g(k|X| + c)$, $\frac{1}{2}f''(X)|\sigma_x(b, X)|^2 \leq 3K_g K_\sigma^2$ and $f'(X)\sigma_x(b, X) \cdot z \leq K_g K_\sigma(1 + |X|)z$ for all $z \in \mathbb{R}^d$ and for all $(a, b, X) \in A \times E$, and the second inequality follows from the definition of f. The addition of the absolute value sign in $f'(X)\mu_x(a, b, X) \leq K_g(k|X| + c)$ follows from the sign of f' and the bound on $\frac{1}{2}f''(X)|\sigma_x(b,X)|^2$ follows from $f''(X)|\sigma_x(b,X)|^2 = 0$ if $|X| \geq 1$ and if $|X| < 1$,

$$\begin{aligned} f''(X)\left|\sigma_x(b,X)\right|^2 &= \frac{3}{2}K_g\left(1-X^2\right)\left|\sigma_x(b,X)\right|^2 \\ &\leq \frac{3}{2}K_g\left(1-X^2\right)K_\sigma^2\left(1+|X|\right)^2 \\ &\leq 6K_gK_\sigma^2. \end{aligned}$$

Taking expectations and noting that $(1+|X_s|)$ is square-integrable on $[\tau, t]$, so the expectation of the stochastic integral is zero,

$$\begin{aligned} E_\tau[f(X_t)] &\leq f(X_\tau) + 3K_gK_\sigma^2(t-\tau) + k\int_\tau^t E_\tau\left[f(X_s)\right]ds \\ &\leq \left(f(X_\tau) + 3K_gK_\sigma^2(t-\tau)\right)e^{k(t-\tau)} \end{aligned}$$

where the last line follows from Gronwall's inequality. Note that $|g(a,b,X)| \leq f(X)$ for all $(a,b,X) \in A \times E$. Therefore,

$$e^{-r(t-\tau)}E_\tau\left|g(a_t,\bar{b}_t,X_t)\right| \leq e^{-r(t-\tau)}E_\tau[f(X_t)] \leq \left(f(X_\tau) + 3K_gK_\sigma^2(t-\tau)\right)e^{-(r-k)(t-\tau)}$$

Step 2: Show that if $X_t < \infty$, then $W_t(S) < \infty$.

$$\begin{aligned} |W_t(S)| &= \left|E_t\left[r\int_t^\infty e^{-r(s-t)}g(a_s,\bar{b}_s,X_s)ds\right]\right| \\ &\leq rE_t\left[\int_t^\infty e^{-r(s-t)}\left|g(a_s,\bar{b}_s,X_s)\right|ds\right] \\ &\leq r\int_t^\infty e^{-r(s-t)}E_t\left|g(a_s,\bar{b}_s,X_s)\right|ds \\ &\leq r\int_t^\infty \left(f(X_t) + 3K_gK_\sigma^2(s-t)\right)e^{-(r-k)(s-t)}ds \\ &= \left(\frac{r}{r-k}\right)f(X_t) + \frac{3rK_gK_\sigma^2}{(r-k)^2} \end{aligned}$$

which is finite for any $X_t < \infty$ and $k < r$. Also, given that f has linear growth, there exists a $K > 0$ such that

$$|W_t(S)| \leq K(1+|X_t|).$$

Step 3: Show $E|V_t(S)| < \infty$ for any $X_0 < \infty$. By similar reasoning to Step 2,

$$E\left|V_t(S)\right| \;=\; E\left|E_t\left[r\int_0^\infty e^{-rs}g(a_s,\bar{b}_s,X_s)ds\right]\right| \le E\left[r\int_0^\infty e^{-rs}\left|g(a_s,\bar{b}_s,X_s)\right|ds\right]$$

which is finite for any $X_0 < \infty$ and $k < r$.

Step 4: Show $E_t[V_{t+k}(S)] = V_t(S)$.

$$\begin{aligned}
E_t[V_{t+k}(S)] &= E_t\left[r\int_0^{t+k} e^{-rs}g(a_s,\bar{b}_s,X_s)ds + e^{-r(t+k)}W_{t+k}(S)\right] \\
&= r\int_0^t e^{-rs}g(a_s,\bar{b}_s,X_s)ds \\
&\quad +E_t\left[r\int_t^{t+k} e^{-rs}g(a_s,\bar{b}_s,X_s)ds + e^{-r(t+k)}E_{t+k}\left[r\int_{t+k}^\infty e^{-r(s-(t+k))}g(a_s,\bar{b}_s,X_s)ds\right]\right] \\
&= r\int_0^t e^{-rs}g(a_s,\bar{b}_s,X_s)ds + e^{-rt}W_t(S) = V_t(S).
\end{aligned}$$

By steps 3 and 4, $V_t(S)$ is a martingale.

If g is bounded with respect to X, then, trivially, $V_t(S)$ and $W_t(S)$ are bounded for all $t \ge 0$ and $X_0 \in \mathcal{X}$, and showing $V_t(S)$ is a martingale follows from step 4. Also note that if $\mathcal{X}$ is bounded, then g is bounded. □

Proof of Lemma 2.

Evolution of the continuation value. From Lemma 1, $V_t(S)$ is a martingale. Take the derivative of $V_t(S)$ wrt t:

$$dV_t(S) = re^{-rt}g(a_t,\bar{b}_t,X_t)dt - re^{-rt}W_t(S)dt + e^{-rt}dW_t(S)$$

By the martingale representation theorem, there exists a progressively measurable process $(\beta_t)_{t\ge 0}$ such that V_t can be represented as $dV_t(S) = re^{-rt}\beta_t^\top\sigma\left(\bar{b}_t,X_t\right)dZ_t$, where $\sigma = [\sigma_y^T,\ \sigma_x^T]^T$. Combining these two expressions for $dV_t(S)$ yields the law of motion for the continuation value:

$$\begin{aligned}
dW_t(S) &= r\left(W_t(S) - g(a_t,\bar{b}_t,X_t)\right)dt + r\beta_t^\top\sigma\left(\bar{b}_t,X_t\right)dZ_t \\
&= r\left(W_t(S) - g(a_t,\bar{b}_t,X_t)\right)dt + r\beta_{yt}\left[dY_t - \mu_y(a_t,\bar{b}_t,X_t)dt\right] \\
&\quad +r\beta_{xt}\left[dX_t - \mu_x(a_t,\bar{b}_t,X_t)dt\right]
\end{aligned}$$

where $\beta_t = (\beta_{yt}, \beta_{xt})$ is a vector of length d. The component β_{yt} captures the sensitivity of the continuation value to the public signal, while the component β_{xt} captures the sensitivity of the continuation value to the state variable. Lemma 1 establishes that any continuation value has linear growth with respect to X_t and is bounded when g is bounded.

Sequential rationality. Consider strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ played from period τ onwards and alternative strategy $(\widetilde{a}_t, \bar{b}_t)_{t\geq 0}$ played up to time τ. Recall that all values of X_t are possible under both strategies, but that each strategy induces a different measure over sample paths $(X_t)_{t\geq 0}$. At time τ, the state variable is equal to X_τ. Action a_τ will induce

$$\begin{bmatrix} dY_\tau \\ dX_\tau \end{bmatrix} = \begin{bmatrix} \mu_y(a_\tau, \bar{b}_\tau, X_\tau) \\ \mu_x(a_\tau, \bar{b}_\tau, X_\tau) \end{bmatrix} dt + \sigma(\bar{b}_\tau, X_\tau) \cdot \begin{bmatrix} dZ^y_\tau \\ dZ^x_\tau \end{bmatrix}$$

whereas action $\widetilde{a}_\tau$ will induce

$$\begin{bmatrix} dY_\tau \\ dX_\tau \end{bmatrix} = \begin{bmatrix} \mu_y(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) \\ \mu_x(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) \end{bmatrix} dt + \sigma(\bar{b}_\tau, X_\tau) \cdot \begin{bmatrix} dZ^y_\tau \\ dZ^x_\tau \end{bmatrix}$$

Let $\widetilde{V}_\tau$ be the expected average payoff conditional on information at time τ when the long-run player follows $\widetilde{a}$ up to τ and a afterwards, and let W_τ be the continuation value when the long-run player follows strategy $(a_t)_{t\geq 0}$ starting at time τ.

$$\widetilde{V}_\tau = r\int_0^\tau e^{-rs} g(\widetilde{a}_s, \bar{b}_s, X_s)ds + e^{-r\tau}W_\tau$$

Consider changing τ so that long-run player plays strategy $(\widetilde{a}_t, \bar{b}_t)$ for another instant: $d\widetilde{V}_\tau$ is the change in average expected payoffs when the long-run player switches to $(a_t)_{t\geq 0}$ at $\tau + d\tau$ instead of τ. When long-run player switches strategies at time τ,

$$\begin{aligned}
d\widetilde{V}_\tau &= re^{-r\tau}\left[g(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) - W_\tau\right]d\tau + e^{-r\tau}dW_\tau \\
&= re^{-r\tau}\left[g(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) - g(a_\tau, \bar{b}_\tau, X_\tau)\right]d\tau + re^{-r\tau}\beta_{y\tau}\left[dY_\tau - \mu_y(a_\tau, \bar{b}_\tau, X_\tau)d\tau\right] \\
&\quad + re^{-r\tau}\beta_{xt}\left[dX_t - \mu_x(a_t, \bar{b}_t, X_t)dt\right] \\
&= re^{-r\tau}\left[\left[g(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) - g(a_\tau, \bar{b}_\tau, X_\tau)\right]d\tau + \beta_{y\tau}\left[\mu_y(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) - \mu_y(a_\tau, \bar{b}_\tau, X_\tau)\right]d\tau\right] \\
&\quad + re^{-r\tau}\left[\beta_{x\tau}\left[\mu_x(\widetilde{a}_\tau, \bar{b}_\tau, X_\tau) - \mu_x(a_\tau, \bar{b}_\tau, X_\tau)\right]d\tau + \beta_\tau^\top\sigma\left(\bar{b}_t, X_t\right)dZ_t\right]
\end{aligned}$$

There are two components to this strategy change: how it affects the immediate flow payoff and how it affects future public signal Y_t and future state X_t, which impact the continuation value. The profile $(\widetilde{a}_t, \overline{b}_t)_{t\geq 0}$ yields the long-run player a payoff of:

$$\begin{aligned}\widetilde{W}_0 &= E_0\left[\widetilde{V}_\infty\right] = E_0\left[\widetilde{V}_0 + \int_0^\infty d\widetilde{V}_t\right] \\ &= W_0 + E_0\left[r\int_0^\infty e^{-rt}\left\{\begin{array}{c} g(\widetilde{a}_t, \overline{b}_t, X_t) + \beta_{y\tau}\mu_y(\widetilde{a}_\tau, \overline{b}_\tau, X_\tau) + \beta_{x\tau}\mu_x(\widetilde{a}_\tau, \overline{b}_\tau, X_\tau) \\ -g(a_t, \overline{b}_t, X_t) - \beta_{y\tau}\mu_y(a_\tau, \overline{b}_\tau, X_\tau) - \beta_{x\tau}\mu_x(a_\tau, \overline{b}_\tau, X_\tau)\end{array}\right\} dt\right]\end{aligned}$$

If

$$g(a_t, \overline{b}_t, X_t)+\beta_{yt}\mu_y(a_t, \overline{b}_t, X_t)+\beta_{xt}\mu_x(a_t, \overline{b}_t, X_t) \geq g(\widetilde{a}_t, \overline{b}_t, X_t)+\beta_{yt}\mu_y(\widetilde{a}_t, \overline{b}_t, X_t)+\beta_{xt}\mu_x(\widetilde{a}_t, \overline{b}_t, X_t)$$

holds for all $t \geq 0$, then $W_0 \geq \widetilde{W}_0$ and deviating to $S = (\widetilde{a}_t, \overline{b}_t)$ is not a profitable deviation. A strategy $(a_t)_{t\geq 0}$ is sequentially rational for the long-run player if, given $(\beta_t)_{t\geq 0}$, for all t:

$$a_t \in \arg\max\ g(a', \overline{b}_t, X_t) + \beta_{yt}\mu_y(a', \overline{b}_t, X_t) + \beta_{xt}\mu_x(a', \overline{b}_t, X_t).$$

A.2 Proof of Theorem 1

Let $\psi(X, z) \equiv g^*(X, z) + \frac{z}{r}\mu^*(X, z)$ be the value of the long-run player's incentive constraint in the equilibrium of the auxiliary static game with incentive weights $z_y = \mathbf{0}$ on the public signal and $z_x = z/r$ on the state.

Form of Optimality Equation. In a Markov equilibrium, the continuation value and equilibrium actions are characterized as a function of the state variable as $W_t = U(X_t)$, $a_t^* = a(X_t)$ and $\overline{b}_t^* = \overline{b}(X_t)$. By Ito's formula, if a Markov equilibrium exists, the continuation value will evolve according to:

$$\begin{aligned}dU(X_t) &= U'(X_t)dX_t + \frac{1}{2}U''(X_t)\left|\sigma_x\left(\overline{b}_t^*, X_t\right)\right|^2 dt \\ &= U'(X_t)\mu_x(a_t^*, \overline{b}_t^*, X_t)dt + \frac{1}{2}U''(X_t)\left|\sigma_x\left(\overline{b}_t^*, X_t\right)\right|^2 dt + U'(X_t)\sigma_x\left(\overline{b}_t^*, X_t\right) dZ_t\end{aligned}$$

Matching the drift of this expression with the drift of the continuation value characterized in Lemma 2 yields the optimality equation for strategy profile $(a^*, \overline{b}^*)$,

$$U''(X) = \frac{2r\left(U(X) - g(a^*, \overline{b}^*, X)\right)}{\left|\sigma_x\left(\overline{b}_t^*, X_t\right)\right|^2} - \frac{2\mu_x(a^*, \overline{b}^*, X)}{\left|\sigma_x\left(\overline{b}_t^*, X_t\right)\right|^2}U'(X) \tag{20}$$

which is a second order non-homogenous differential equation.

Matching the volatility characterizes the process governing incentives, $\beta_y = 0$ and $\beta_x = U'(X)/r$. Intuitively, the continuation value and equilibrium actions are independent of the public signal in a Markov equilibrium; this is born out mathematically by the condition $\beta_y = 0$. Plugging these into the condition for sequential rationality,

$$S^*(X, 0, U'(X)/r) = \left\{ (a^*, \overline{b}^*) : \begin{array}{l} a^* = \arg\max_{a \in A} g(a, \overline{b}^*, X) + U'(X)\mu_x(a, \overline{b}^*, X)/r \\ \overline{b}^* = \arg\max_{b \in B(X)} h(a^*, b, \overline{b}^*, X) \end{array} \right\}.$$

which is unique by Assumption 4.

Existence of solution to optimality equation. Define $f : \mathcal{X} \times \mathbb{R}^2 \to \mathbb{R}$ as:

$$f(X, U, U') \equiv \frac{2r}{|\sigma^*(X, U')|^2} (U - \psi(X, U'))$$

which is continuous on $\text{int}(\mathcal{X})$. I establish that the second order differential equation $U'' = f(X, U, U')$ has at least one solution $U \in \mathcal{C}^2$ that takes on values in the interval of feasible payoffs for the long-run player, and therefore, is a solution to (20).

Case 1: Unbounded State Space. Theorem 5.6 from Coster and Habets (2006), which is based on Schmitt (1969), gives sufficient conditions for the existence of a solution to a second order differential equation defined on $\mathbb{R}^3$. The Theorem is reproduced below.

Theorem 5 (Coster Habets (2006))**.** *Let $\alpha, \beta \in C^2(\mathbb{R})$ be functions such that $\alpha \leq \beta$, $D = \{(t, u, v) \in \mathbb{R}^3 | \alpha(t) \leq u \leq \beta(t)\}$ and $f : D \to \mathbb{R}$ be a continuous function. Assume that α and β are such that for all $t \in \mathbb{R}$,*

$$f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t) \text{ and } \beta''(t) \leq f(t, \beta(t), \beta'(t)).$$

Assume that for any bounded interval I, there exists a positive continuous function $H_I : \mathbb{R}^+ \to \mathbb{R}$ that satisfies the Nagumo condition,[13]

$$\int_0^\infty \frac{s ds}{H_I(s)} = \infty, \tag{21}$$

and for all $t \in I$, $(u, v) \in \mathbb{R}^2$ with $\alpha(t) \leq u \leq \beta(t)$, $|f(t, u, v)| \leq H_I(|v|)$. Then the equation $u'' = f(t, u, u')$ has at least one solution $u \in \mathcal{C}^2(\mathbb{R})$ such that for all $t \in \mathbb{R}$, $\alpha(t) \leq u(t) \leq \beta(t)$.

[13] The Nagumo condition is a growth condition on the second order differential equation $f(X, U, U')$ and plays an important role in demonstrating the existence of solutions of the boundary value problem.

Lemma 3. *If $\mathcal{X} = \mathbb{R}$, then (9) has at least one solution $U \in C^2(\mathbb{R})$ that lies in the range of feasible payoffs for the long-run player.*

Proof. Suppose $\mathcal{X} = \mathbb{R}$. Then (9) is continuous on $\mathbb{R}^3$. Define $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ as

$$\alpha(X) := \begin{cases} \alpha_1 X - c_a & \text{if } X \leq -1 \\ \frac{1}{8}\alpha_1 X^4 - \frac{3}{4}\alpha_1 X^2 - \frac{3}{8}\alpha_1 - c_a & \text{if } X \in (-1, 1) \\ -\alpha_1 X - c_a & \text{if } X \geq 1 \end{cases}$$

$$\beta(X) := \begin{cases} -\beta_1 X + c_b & \text{if } X \leq -1 \\ -\frac{1}{8}\beta_1 X^4 + \frac{3}{4}\beta_1 X^2 + \frac{3}{8}\beta_1 + c_b & \text{if } X \in (-1, 1) \\ \beta_1 X + c_b & \text{if } X \geq 1 \end{cases}$$

for some $\alpha_1, \beta_1, c_a, c_b \geq 0$. Note that $\alpha, \beta \in C^2(\mathbb{R})$ and $\alpha \leq \beta$. Then α and β are lower and upper solutions to (9) if there exist $\alpha_1, \beta_1, c_a, c_b \geq 0$ such that for all $X \in \mathbb{R}$,

$$\frac{2r}{|\sigma(X, \alpha'(X))|^2} \left(\alpha(X) - \psi(X, \alpha'(X))\right) \leq \alpha''(X)$$

and

$$\beta''(X) \leq \frac{2r}{|\sigma(X, \beta'(X))|^2} \left(\beta(X) - \psi(X, \beta'(X))\right).$$

Step 1: By Assumption 3, $\exists k \in [0, r)$ and $c \geq 0$ such that $\mu^*(X, z) \leq kX + c$ for all $X \geq 0$ and $\mu^*(X, z) \geq kX - c$ for all $X \leq 0$. Show that there exist $\alpha_1, \beta_1, c_a, c_b \geq 0$ such that α, β are lower and upper solutions to (9). Note this step does not require g to be bounded.

Step 1a: Find a bound on ψ. By Lipschitz continuity and the fact that g^* and μ^* are bounded in z, $\exists k_g, k_m \geq 0$ such that

$$\begin{aligned} |g^*(X, z) - g^*(0, z)| &\leq k_g |X| \\ |\mu^*(X, z) - \mu^*(0, z)| &\leq k_m |X| \end{aligned}$$

for all (X, z). Therefore, $\exists \underline{g}_1, \underline{g}_2, \overline{g}_1, \overline{g}_2 \geq 0$, $\underline{\mu}_1, \overline{\mu}_2 \in [0, r)$, $\underline{\mu}_2, \overline{\mu}_1 > 0$ and $\overline{\gamma}, \underline{\gamma}, \overline{m}, \underline{m} \in \mathbb{R}$ such that:

$$\begin{cases} \underline{g}_1 X + \underline{\gamma} \\ -\underline{g}_2 X + \underline{\gamma} \end{cases} \leq \; g^*(X,z) \leq \; \begin{cases} -\overline{g}_1 X + \overline{\gamma} & \text{if } X < 0 \\ \overline{g}_2 X + \overline{\gamma} & \text{if } X \geq 0 \end{cases}$$

$$\begin{cases} \underline{\mu}_1 X + \underline{m} \\ -\underline{\mu}_2 X + \underline{m} \end{cases} \leq \; \mu^*(X,z) \leq \; \begin{cases} -\overline{\mu}_1 X + \overline{m} & \text{if } X < 0 \\ \overline{\mu}_2 X + \overline{m} & \text{if } X \geq 0 \end{cases}$$

and

$$\begin{cases} \left(\underline{g}_1 - \frac{\overline{\mu}_1}{r} z\right) X + \underline{\gamma} + \frac{\overline{m}}{r} z \\ \left(-\underline{g}_2 + \frac{\overline{\mu}_2}{r} z\right) X + \underline{\gamma} + \frac{\overline{m}}{r} z \\ \left(\underline{g}_1 + \frac{\underline{\mu}_1}{r} z\right) X + \underline{\gamma} + \frac{\underline{m}}{r} z \\ -\left(\underline{g}_2 + \frac{\underline{\mu}_2}{r} z\right) X + \underline{\gamma} + \frac{\underline{m}}{r} z \end{cases} \leq \psi(X,z) \leq \begin{cases} \left(-\overline{g}_1 + \frac{\underline{\mu}_1}{r} z\right) X + \overline{\gamma} + \frac{\underline{m}}{r} z & \text{if } X < 0, z \leq 0 \\ \left(\overline{g}_2 - \frac{\underline{\mu}_2}{r} z\right) X + \overline{\gamma} + \frac{\underline{m}}{r} z & \text{if } X \geq 0, z \leq 0 \\ -\left(\overline{g}_1 + \frac{\overline{\mu}_1}{r} z\right) X + \overline{\gamma} + \frac{\overline{m}}{r} z & \text{if } X < 0, z \geq 0 \\ \left(\overline{g}_2 + \frac{\overline{\mu}_2}{r} z\right) X + \overline{\gamma} + \frac{\overline{m}}{r} z & \text{if } X \geq 0, z \geq 0 \end{cases}$$

Step 1b: Find conditions on $(\alpha_1, \beta_1, c_a, c_b)$ such that α, β are lower and upper solutions to (9) when $X \leq -1$. Note $\alpha''(X) = \beta''(X) = 0$, so this corresponds to showing $\psi(X, \alpha_1) \geq \alpha_1 X - c_a$ and $\psi(X, -\beta_1) \leq -\beta_1 X + c_b$. From the bound on ψ,

$$\begin{aligned} \psi(X, \alpha_1) \;&\geq\; \left(\underline{g}_1 + \frac{\underline{\mu}_1}{r}\alpha_1\right) X + \underline{\gamma} + \frac{\underline{m}}{r}\alpha_1 \\ \psi(X, -\beta_1) \;&\leq\; -\left(\overline{g}_1 + \frac{\underline{\mu}_1}{r}\beta_1\right) X + \overline{\gamma} - \frac{\underline{m}}{r}\beta_1. \end{aligned}$$

Therefore, showing α and β are lower and upper solutions requires

$$\begin{aligned} \alpha_1 \;&\geq\; \frac{r\underline{g}_1}{r - \underline{\mu}_1} \\ c_a \;&\geq\; -\underline{\gamma} - \frac{\underline{m}}{r}\alpha_1 \equiv c_a^1 \\ \beta_1 \;&\geq\; \frac{r\overline{g}_1}{r - \underline{\mu}_1} \\ c_b \;&\geq\; \overline{\gamma} - \frac{\underline{m}}{r}\beta_1 \equiv c_b^1. \end{aligned}$$

Step 1c: Find conditions on $(\alpha_1, \beta_1, c_a, c_b)$ such that α, β are lower and upper solutions to (9) when $X \geq 1$. This corresponds to showing $\psi(X, -\alpha_1) \geq -\alpha_1 X - c_a$ and $\psi(X, \beta_1) \leq$

$\beta_1 X + c_b$. From the bound on ψ,

$$\begin{aligned}\psi(X,-\alpha_1) &\geq -\left(\underline{g}_2 + \frac{\overline{\mu}_2}{r}\alpha_1\right)X + \underline{\gamma} - \frac{\overline{m}}{r}\alpha_1 \\ \psi(X,\beta_1) &\leq \left(\overline{g}_2 + \frac{\overline{\mu}_2}{r}\beta_1\right)X + \overline{\gamma} + \frac{\overline{m}}{r}\beta_1.\end{aligned}$$

Therefore, this requires

$$\begin{aligned}\alpha_1 &\geq \frac{r\underline{g}_2}{r-\overline{\mu}_2} \\ c_a &\geq -\underline{\gamma} + \frac{\overline{m}}{r}\alpha_1 \equiv c_a^2 \\ \beta_1 &\geq \frac{r\overline{g}_2}{r-\overline{\mu}_2} \\ c_b &\geq \overline{\gamma} + \frac{\overline{m}}{r}\beta_1 \equiv c_b^1.\end{aligned}$$

Step 1d: Find conditions on $(\alpha_1, \beta_1, c_a, c_b)$ such that α, β are lower and upper solutions to (9) when $X \in (-1,1)$. Note $\alpha''(X) = -\frac{3}{2}\alpha_1(1-X^2) \geq -\frac{3}{2}\alpha_1$ and $\alpha(X) \leq -\frac{3}{8}\alpha_1 - c_a$ and $\beta''(X) = \frac{3}{2}\beta_1(1-X^2) \leq \frac{3}{2}\beta_1$ and $\beta(X) \geq \frac{3}{8}\beta_1 + c_b$, so this is equivalent to showing

$$\begin{aligned}c_a &\geq \frac{3}{4}\left(\frac{|\sigma^*(X,\alpha')|^2}{r} - \frac{1}{2}\right)\beta_1 - \psi(X,\alpha') \\ c_b &\geq \frac{3}{4}\left(\frac{|\sigma^*(X,\alpha')|^2}{r} - \frac{1}{2}\right)\beta_1 + \psi(X,\beta')\end{aligned}$$

for $X \in (-1,1)$. Let $\overline{\sigma} = \sup_{X\in[0,1],z} |\sigma^*(X,z)|$, which exists since σ^* is Lipschitz continuous in X and bounded in z. First consider $X \in (-1,0]$, which means that $\beta' = \frac{1}{2}\beta_1 X(3-X^2) \in (-\beta_1, 0]$ and $a' = -\frac{1}{2}\alpha_1 X(3-X^2) \in [0,\alpha_1)$. From the bound on ψ,

$$\begin{aligned}\psi(X,\alpha') &\geq \left(\underline{g}_1 + \frac{\underline{\mu}_1}{r}\alpha'\right)X + \underline{\gamma} + \frac{\underline{m}}{r}\alpha' \geq -\underline{g}_1 + \underline{\gamma} - \frac{\underline{\mu}_1}{r}\alpha_1 + \frac{\alpha_1}{r}\min\{\underline{m},0\} \\ \psi(X,\beta') &\leq \left(-\overline{g}_1 + \frac{\underline{\mu}_1}{r}\beta'\right)X + \overline{\gamma} + \frac{\underline{m}}{r}\beta' \leq \overline{g}_1 + \overline{\gamma} + \frac{\underline{\mu}_1}{r}\beta_1 - \frac{\beta_1}{r}\min\{\underline{m},0\}.\end{aligned}$$

Therefore, this requires

$$\begin{aligned}c_a &\geq \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\alpha_1 + \underline{g}_1 - \underline{\gamma} + \frac{\underline{\mu}_1}{r}\alpha_1 - \frac{\alpha_1}{r}\min\{\underline{m},0\} \equiv c_a^3 \\ c_b &\geq \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\beta_1 + \overline{g}_1 + \overline{\gamma} + \frac{\underline{\mu}_1}{r}\beta_1 - \frac{\beta_1}{r}\min\{\underline{m},0\} \equiv c_b^3.\end{aligned}$$

Next consider $X \in [0,1)$, which means that $\beta' = \frac{1}{2}\beta_1 X\left(3-X^2\right) \in [0,\beta_1)$ and $a' = -\frac{1}{2}\alpha_1 X\left(3-X^2\right) \in (-\alpha_1, 0]$. From the bound on ψ,

$$\begin{aligned}
\psi\left(X,\alpha'\right) &\geq \left(-\underline{g}_2 + \frac{\overline{\mu}_2}{r}\alpha'\right)X + \underline{\gamma} + \frac{\overline{m}}{r}\alpha' \geq -\underline{g}_2 + \underline{\gamma} - \frac{\overline{\mu}_2}{r}\alpha_1 - \frac{\alpha_1}{r}\max\left\{\overline{m},0\right\} \\
\psi\left(X,\beta'\right) &\leq \left(\overline{g}_2 + \frac{\overline{\mu}_2}{r}\beta'\right)X + \overline{\gamma} + \frac{\overline{m}}{r}\beta' \leq \overline{g}_2 + \overline{\gamma} + \frac{\overline{\mu}_2}{r}\beta_1 + \frac{\beta_1}{r}\max\left\{\overline{m},0\right\}.
\end{aligned}$$

This requires

$$\begin{aligned}
c_a &\geq \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\alpha_1 + \underline{g}_2 - \underline{\gamma} + \frac{\overline{\mu}_2}{r}\alpha_1 + \frac{\alpha_1}{r}\max\left\{\overline{m},0\right\} \equiv c_a^4 \\
c_b &\geq \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\beta_1 + \overline{g}_2 + \overline{\gamma} + \frac{\overline{\mu}_2}{r}\beta_1 + \frac{\beta_1}{r}\max\left\{\overline{m},0\right\} \equiv c_b^4.
\end{aligned}$$

Step 1e: Compiling these conditions and choosing

$$\begin{aligned}
\alpha_1 &= \max\left\{\frac{r\underline{g}_1}{r-\underline{\mu}_1}, \frac{r\underline{g}_2}{r-\overline{\mu}_2}\right\} \\
\beta_1 &= \max\left\{\frac{r\overline{g}_1}{r-\underline{\mu}_1}, \frac{r\overline{g}_2}{r-\overline{\mu}_2}\right\}
\end{aligned}$$

yields $\alpha_1, \beta_1 \geq 0$ that satisfy the slope conditions in steps 1b-1d, and choosing

$$c_a = \max\left\{0, c_a^1, c_a^2, c_a^3, c_a^4\right\} \text{ and } c_b = \max\left\{0, c_b^1, c_b^2, c_b^3, c_b^4\right\}$$

yields c_a, c_b that satisfy the intercept conditions in steps 1b-1d. Conclude that α and β are lower and upper solutions to (9).

Step 2: Assume g is bounded. Show that there exist $\alpha_1, \beta_1, c_a, c_b \geq 0$ such that α, β are lower and upper solutions to (9). Note this step places no restrictions on the relationship between the growth rate of μ_x and r. Define $\overline{g} \equiv \sup_{(a,b,X)\in A\times E} g(a,b,X)$ and $\underline{g} \equiv \inf_{(a,b,X)\in A\times E} g(a,b,X)$, which exist since g is bounded. Let $\alpha_1 = 0$ and $c_a = -\underline{g}$. Then $\psi(X,\alpha'(X)) = g^*(X,0)$, so $\alpha - \psi(X,\alpha') = \underline{g} - g^*(X,0) \leq 0$ and $\alpha\left(X\right) = \underline{g}$ is a lower solution. Similarly, let $\beta_1 = 0$ and $c_b = \overline{g}$. Then $\psi(X,\beta') = g^*(X,0)$, so $\beta - \psi(X,\beta'(X)) = \overline{g} - g^*(X,0) \geq 0$ and $\beta\left(X\right) = \overline{g}$ is an upper solution.

Step 3: Show that the Nagumo condition (21), which is a growth condition on $f(X,U,U')$ that Theorem 5 uses to establish existence of a solution to the boundary value problem, is satisfied. Given a compact proper subset $I \subset \mathcal{X}$, there exists a $K_I > 0$ such that

$$|f(X,U,U')| = \left|\frac{2r}{|\sigma^*(X,U')|^2}\left(U - g^*(X,U') - \frac{U'}{r}\mu^*(X,U')\right)\right| \leq K_I(1+|U'|)$$

for all $(X,U,U') \in \{I \times \mathbb{R}^2 \text{ s.t. } \alpha(X) \leq U \leq \beta(X)\}$. This follows directly from the fact that $X \in I$, $\alpha(X)$ and $\beta(X)$ are bounded on I, $\alpha(X) \leq U \leq \beta(X)$, g^*, μ^* are bounded on $(X,U') \in I \times \mathbb{R}$ and the lower bound on σ_x. Define $H_I(z) = K_I(1+z)$. Therefore, $\int_0^\infty \frac{sds}{H_I(s)} = \infty$.

Conclude that $f(X,U(X),U'(X))$ has at least one $\mathcal{C}^2$ solution U such that for all $X \in \mathbb{R}$, $\alpha(X) \leq U(X) \leq \beta(X)$. If α and β are bounded, then U is bounded. ☐

Case 2: Bounded State Space. I use standard existence results from Coster and Habets (2006) and an extension in Faingold and Sannikov (2011), which is necessary because (9) is undefined at $\{\underline{X}, \overline{X}\}$. The result applied to the current setting is reproduced below.

Lemma 4 (Faingold Sannikov (2011))**.** *Let $E = \left\{(t,u,v) \in (\underline{t},\overline{t}) \times \mathbb{R}^2\right\}$ and $f : E \to \mathbb{R}$ be continuous. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$ and $f(t,\alpha,0) \leq 0 \leq f(t,\beta,0)$ for all $t \in \mathbb{R}$. Assume also that for any closed interval $I \subset (\underline{t},\overline{t})$, there exists a $K_I > 0$ such that $|f(t,u,v)| \leq K_I(1+|v|))$ for all $(t,u,v) \in I \times [\alpha,\beta] \times \mathbb{R}$. Then the differential equation $U'' = f(t,U(t),U'(t))$ has at least one $\mathcal{C}^2$ solution U on $(\underline{t},\overline{t})$ such that $\alpha \leq U(t) \leq \beta$.*

Proof. When $\mathcal{X}$ is compact, the feasible payoff set for the long-run player is bounded, since g is Lipschitz continuous and A is compact. Define $\underline{g} \equiv \inf_{(a,b,X)\in A\times E} g(a,b,X)$ and $\overline{g} \equiv \sup_{(a,b,X)\in A\times E} g(a,b,X)$. ☐

Lemma 5. *Suppose $\mathcal{X}$ is compact. Then (9) has at least one $\mathcal{C}^2$ solution U that lies in the range of feasible payoffs for the long-run player, $\underline{g} \leq U(X) \leq \overline{g}$.*

Proof. Suppose $\mathcal{X}$ is compact. Then (9) is continuous on the set $D = \left\{(X,U,U') \in (\underline{X},\overline{X}) \times \mathbb{R}^2\right\}$. For any closed interval $I \subset (\underline{X},\overline{X})$, there exists a $K_I > 0$ such that

$$\left|\frac{2r}{|\sigma^*(X,U')|^2}\left(U - g^*(X,U') - \frac{U'}{r}\mu^*(X,U')\right)\right| \leq K_I(1+|U'|)$$

for all $(X,U,U') \in I \times \left[\underline{g},\overline{g}\right] \times \mathbb{R}$. This follows directly from the fact that $X \in I$, $U \in \left[\underline{g},\overline{g}\right]$, g^*, μ^* are bounded on $\mathcal{X} \times \mathbb{R}$, and the lower bound on σ_x. Also note that

$$f(X,\underline{g},0) = \frac{2r}{|\sigma^*(X,0)|^2}\left(\underline{g} - g^*(X,0)\right) \leq 0 \leq f(X,\overline{g},0) = \frac{2r}{|\sigma^*(X,0)|^2}\left(\overline{g} - g^*(X,0)\right)$$

for all $X \in \mathcal{X}$. By Lemma 4, (9) has at least one C^2 solution U such that $\underline{g} \leq U(X) \leq \overline{g}$ for all $X \in \mathcal{X}$. □

Construct a Markov equilibrium. Suppose the state variable initially starts at X_0 and U is a solution to (9). The action profile $(a^*, \overline{b}^*) = S^*(X, 0, U'(X)/r)$ is unique and Lipschitz continuous. Given X_0, U and $(a_t^*, \overline{b}_t^*)_{t\geq 0}$, the state variable evolves according to the unique strong solution $(X_t)_{t\geq 0}$ to the stochastic differential equation

$$dX_t = \mu^*(X_t, U'(X_t))dt + \sigma^*(X_t, U'(X_t))\, dZ_t$$

which exists since μ^* and σ^* are Lipschitz continuous. The continuation value evolves according to:

$$\begin{aligned} dU(X_t) &= U'(X_t)\mu^*(X_t, U'(X_t))dt + \frac{1}{2}U''(X_t)\left|\sigma^*(X_t, U'(X_t))\right|^2 dt + U'(X_t)\sigma^*(X_t, U'(X_t))\, dZ_t \\ &= r\left(U(X_t) - g^*(X_t, U'(X_t))\right) dt + U'(X_t)\sigma^*(X_t, U'(X_t))\, dZ_t. \end{aligned}$$

This process satisfies (5). Additionally, $(a_t^*, \overline{b}_t^*)_{t\geq 0}$ satisfies (6) and (7) given process $(\beta_t)_{t\geq 0}$ with $\beta_t = (0, U'(X_t))$. Thus, the strategy profile $(a_t^*, \overline{b}_t^*)_{t\geq 0}$ is a PPE yielding equilibrium payoff $U(X_0)$.

A.3 Proof of Theorem 2

Let X_0 be the initial state, and let $\overline{U}$ be the upper envelope of the set of solutions to the optimality equation (9). Suppose there is a PPE $S = (a_t, \overline{b}_t)_{t\geq 0}$ that yields an equilibrium payoff $W_0 > \overline{U}(X_0)$. The continuation value in this equilibrium must evolve according to

$$dW_t(S) = r\left(W_t(S) - g(a_t, \overline{b}_t, X_t)\right) dt + r\beta_{yt}^{\intercal}\left[dY_t - \mu_y(a_t, \overline{b}_t, X_t)dt\right] + r\beta_{xt}\left[dX_t - \mu_x(a_t, \overline{b}_t, X_t)dt\right] \tag{22}$$

for some process $(\beta_t)_{t\geq 0}$. By sequential rationality, $(a_t, \overline{b}_t) = S^*(X_t, \beta_{yt}, \beta_{xt})$ for all t, and by Assumption 4, these actions are unique for each (X, β_y, β_x). Define

$$\begin{aligned} \widehat{g}(X, \beta_y, \beta_x) &\equiv g(S^*(X, \beta_y, \beta_x), X) \\ \widehat{\mu}(X, \beta_y, \beta_x) &\equiv \mu(S^*(X, \beta_y, \beta_x), X) \\ \widehat{\sigma}(X, \beta_y, \beta_x) &\equiv \sigma(S^*(X, \beta_y, \beta_x), X) \end{aligned}$$

which are Lipschitz continuous, given g, μ, σ and S^* are Lipschitz. The state $(X_t)_{t\geq 0}$ evolves according to (2), given PPE action profile $S = (a_t, \overline{b}_t)_{t\geq 0}$. By Ito's formula, the process

$(\overline{U}(X_t))_{t\geq 0}$ evolves according to

$$d\overline{U}(X_t) = \overline{U}'(X_t)\widehat{\mu}_x(X_t,\beta_{yt},\beta_{xt})dt+\frac{1}{2}\overline{U}''(X_t)\left|\widehat{\sigma}_x\left(X_t,\beta_{yt},\beta_{xt}\right)\right|^2 dt+\overline{U}'(X_t)\widehat{\sigma}_x\left(X_t,\beta_{yt},\beta_{xt}\right)dZ_t. \tag{23}$$

Define a process $D_t \equiv W_t(S) - \overline{U}(X_t)$ with initial condition $D_0 = W_0(S) - \overline{U}(X_0) > 0$. Then D_t evolves according to $dD_t = dW_t(S) - d\overline{U}(X_t)$. Plugging in (22) and (23), the process has drift $rD_t + d(X_t,\beta_{yt},\beta_{xt})$, where

$$\begin{aligned} d(X,\beta_y,\beta_x) &\equiv r\left(\overline{U}(X) - \widehat{g}(X,\beta_y,\beta_x)\right) - \overline{U}'(X)\widehat{\mu}_x(X,\beta_y,\beta_x) - \frac{1}{2}\overline{U}''(X)\left|\widehat{\sigma}_x\left(X,\beta_y,\beta_x\right)\right|^2 \\ &= r\left(\widehat{g}(X,\mathbf{0},\overline{U}'(X)/r) - \widehat{g}(X,\beta_y,\beta_x)\right) + \overline{U}'(X)\left(\widehat{\mu}_x(X,\mathbf{0},\overline{U}'(X)/r) - \widehat{\mu}_x(X,\beta_y,\beta_x)\right) \\ &\quad + \frac{1}{2}\overline{U}''(X)\left(|\widehat{\sigma}_x(X,\mathbf{0},U'(X)/r)|^2 - |\widehat{\sigma}_x(X,\beta_y,\beta_x)|^2\right), \end{aligned}$$

where the second line follows from substituting (9) for $\overline{U}$, and volatility

$$f(X,\beta_y,\beta_x) \equiv r\beta_y^{\intercal}\widehat{\sigma}_y\left(X,\beta_y,\beta_x\right) + (r\beta_x - \overline{U}'(X))\widehat{\sigma}_x\left(X,\beta_y,\beta_x\right).$$

Lemma 6. *If $|f(X,\beta_y,\beta_x)| = 0$, then $d(X,\beta_y,\beta_x) = 0$.*

Proof. Suppose $|f(X,\beta_y,\beta_x)| = 0$ for some (X,β_y,β_x). Then $\beta_y = \mathbf{0}$ and $r\beta_x = \overline{U}'(X)$. The action profile associated with $(X,\mathbf{0},\overline{U}'(X)/r)$ corresponds to the actions played in a Markov equilibrium at state X. Therefore, $d(X,\beta_y,\beta_x) = 0$. □

Lemma 7. *For every $\varepsilon > 0$, there exists a $\eta > 0$ such that either $d(X,\beta_y,\beta_x) > -\varepsilon$ or $|f(X,\beta_y,\beta_x)| > \eta$.*

Proof. Suppose the state space is unbounded, $\mathcal{X} = \mathbb{R}$. Fix $\varepsilon > 0$ and suppose $d(X,\beta_y,\beta_x) \leq -\varepsilon$. Show that there exists a $\eta > 0$ such that $|f(X,\beta_y,\beta_x)| > \eta$ for all $(X,\beta) \in \mathcal{X} \times \mathbb{R}^d$.

Step 1. Show $\exists M > 0$ such that this is true for $(X,\beta) \in \Omega_a \equiv \{\mathcal{X} \times \mathbb{R}^d : |\beta| > M\}$.
$\overline{U}'$ is bounded, by Assumption 1, σ_x is bounded away from 0 and by Assumption 2, there exists a $c > 0$ such that $|\sigma_{yy} \cdot y| \geq c|y|$ for all $(b,X) \in E$ and $y \in \mathbb{R}^{d-1}$, which bounds σ_{yy} away from 0. Therefore, there exists an $M > 0$ and $\eta_1 > 0$ such that $|f(X,\beta_y,\beta_x)| > \eta_1$ for all $|\beta| > M$, regardless of d.

Step 2. Show $\exists X^* > 0$ such that this is true for $(X,\beta) \in \Omega_b \equiv \{\mathcal{X} \times \mathbb{R}^d : |\beta| \leq M,\ |X| > X^*\}$.
Consider the set $\Phi_b \subset \Omega_b$ with $d(X,\beta_y,\beta_x) \leq -\varepsilon$. It must be that (β_x,β_y) is bounded away from $(\overline{U}'(X)/r, 0)$ on Φ_b. Suppose not. Then either (i) there exists some $(X,\beta) \in \Phi_b$ with

$\beta_x = \overline{U}'(X)/r$ and $\beta_y = \mathbf{0}$, which implies $|f(X, \beta_y, \beta_x)| = 0$ and therefore $d(X, \beta_y, \beta_x) = 0$, a contradiction, or (ii) as X becomes large, the boundary of the set Φ_b approaches $(\beta_x, \beta_y) = (\overline{U}'(X)/r, \mathbf{0})$, which implies that for any $\delta_1 > 0$, there exists an $(X, \beta) \in \Phi_b$ with $\max\{r\beta_x - \overline{U}'(X), \beta_y\} < \delta_1$. Choose δ_1 so that $|\widehat{g}(X, 0, \overline{U}'(X)/r) - \widehat{g}(X, \beta_y, \beta_x)| < \varepsilon/4r$ and $|\widehat{\mu}(X, 0, \overline{U}'(X)/r) - \widehat{\mu}(X, \beta_y, \beta_x)| < \varepsilon/4k$, where $|\overline{U}'(X)| \leq k$ is the bound on $\overline{U}'$. Then $|d(X, \beta_y, \beta_x)| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$ which is a contradiction. Therefore, there exists a η_2 such that $|f(X, \beta_y, \beta_x)| > \eta_2$ on Φ_b. Then on the set Ω_b, if $d(X, \beta_y, \beta_x) \leq -\varepsilon$ then $|f(X, \beta_y, \beta_x)| > \eta_2$.

Step 3. Show this is true for $(X, \beta) \in \Omega_c \equiv \{\mathcal{X} \times \mathbb{R}^d : |\beta| \leq M \text{ and } |X| \leq X^*\}$.
Consider the set $\Phi_c \subset \Omega_c$ where $d(X, \beta_y, \beta_x) \leq -\varepsilon$. The function d is continuous and Ω_c is compact, so Φ_c is compact. The function $|f|$ is also continuous, and therefore achieves a minimum η_3 on Φ_c. If $\eta_3 = 0$, then $d = 0$ by Lemma 6, a contradiction. Therefore, $\eta_3 > 0$ and $|f(X, \beta_y, \beta_x)| > \eta_3$ for all $(X, \beta) \in \Phi_c$.

Take $\eta = \min\{\eta_1, \eta_2, \eta_3\}$. Then when $d(X, \beta_y, \beta_x) \leq -\varepsilon$, $|f(X, \beta_y, \beta_x)| > \eta$. The proof for a bounded state space is analogous, omitting step 2b. □

Lemma 8. *Any PPE payoff W_0 is such that $\underline{U}(X_0) \leq W_0 \leq \overline{U}(X_0)$ where $\overline{U}$ and $\underline{U}$ are the upper and lower envelope of the set of solutions to (9).*

Proof. Lemma 7 implies that whenever the drift of D_t is less than $rD_t - \varepsilon$, the volatility is greater than η. Take $\varepsilon = rD_0/4$ and suppose $D_t \geq D_0/2$. Then whenever the drift is less than $rD_t - \varepsilon > rD_0/2 - rD_0/4 = rD_0/4 > 0$, there exists a η such that $|f(X, \beta_y, \beta_x)| > \eta$. Thus, whenever $D_t \geq D_0/2 > 0$, it has either positive drift or positive volatility, and grows arbitrarily large with positive probability, irrespective of X_t. This is a contradiction, since by Lemma 1, D_t is the difference of two processes that are bounded with respect to X_t. Thus, it cannot be that $D_0 > 0$ and it must be the case that $W_0 \leq \overline{U}(X_0)$. Similarly, if $\underline{U}$ is the lower envelope of the set of solutions to (9), it is not possible to have $D_0 < 0$ and therefore it must be the case that $W_0 \geq \underline{U}(X_0)$. □

The proof of Theorem 2 follows directly from Lemma 8, and the fact that at any state $X \in \mathcal{X}$, it is possible for the long-run player to achieve any payoff in the convex hull of the set of Markov equilibrium payoffs at state X by using randomization.

Proof of Corollary 1. Existence of a Markov equilibrium follows from Theorem 1. When μ_x is independent of a, the sequential rationality condition (6) in a Markov equilibrium collapses to maximizing the static flow payoff, and the long-run player plays the unique static Nash action profile $S^*(X, \mathbf{0}, 0)$ in each state. Therefore, the measure over the state is independent of the solution to (9). Any solution to (9) must satisfy

$$U(X_t) = E_t \left[r \int_t^\infty e^{-rs} g^*(X_s, 0) dt \right]. \tag{24}$$

Given the the RHS of (9) is independent of U, (9) must have a unique solution and there is a unique Markov equilibrium. By Theorem 2, this is also the unique PPE. The solution to (9) evaluated at the current state X_t analytically characterizes the RHS of (24).

A.4 Proof of Theorems 3 and 4

I prove Theorems 3 and 4 simultaneously. The proof proceeds in three steps:

1. Any solution to the optimality equation has the same boundary conditions.
2. If all solutions have the same boundary conditions, then there is a unique linear growth (bounded) solution.
3. When there is a unique solution, then there is a unique PPE.

As before, let $\psi(X, z) \equiv g^*(X, z) + z\mu^*(X, z)/r$. All intermediate theorems and lemmas assume Assumptions 1-4 and Assumption 5 or 5′.

Step 1: Boundary Conditions. Theorems 6, 7 and 8 characterize the boundary conditions for (i) unbounded $\mathcal{X}$ and g, (ii) unbounded $\mathcal{X}$ and bounded g and (iii) bounded $\mathcal{X}$, respectively, to establish step 1.

Step 1a: Boundary Conditions for Unbounded $\mathcal{X}$ and g (Theorem 3).

Theorem 6. *Suppose $\mathcal{X} = \mathbb{R}$ and g is unbounded and assume Assumptions 1-5. Then any solution U of (9) with linear growth satisfies*

$$\lim_{X \to p} U(X) - y^L(X) = g_2(z_p) + z_p \mu_2(z_p)/r$$
$$\lim_{X \to p} U'(X) = z_p$$
$$\lim_{X \to p} XU''(X) = 0$$

for $p \in \{-\infty, \infty\}$*, where*

$$
\begin{aligned}
y^L(X) &= -f(X) \int \frac{rg_1(X)}{f(X)\mu_1(X)} dX \\
f(X) &= \exp\left(\int \frac{r}{\mu_1(X)} dX\right) \\
z_p &= \lim_{X \to p} \frac{rg_1(X)/X}{r - \mu_1(X)/X}.
\end{aligned}
$$

The proof proceeds by a series of lemmas. Define $\overline{\psi}(X,z) \equiv \psi(X,z)/X$, $\overline{U}(X) \equiv U(X)/X$, and ψ' and $\overline{\psi}'$ refer to the partial derivative with respect to X where, given Assumption 5, $\psi'(X,z) = g_1'(X) + z\mu_1'(X)/r$.

Lemma 9. *Given* $p \in \{-\infty, \infty\}$*, there exists a* $\psi_p : \mathbb{R} \to \mathbb{R}$ *such that for all* $z \in \mathbb{R}$*,* $\psi_p(z) \equiv \lim_{X \to p} \overline{\psi}(X,z) = \lim_{X \to p} \psi'(X,z)$ *and* $\overline{\psi}(\cdot, z)$ *is monotone for large* $|X|$*.*

Proof. Let $p \in \{-\infty, \infty\}$ and fix $z \in \mathbb{R}$. Given that g^* and μ^* are Lipschitz continuous, there exists $M, \delta > 0$ such that $|\overline{\psi}(X,z)| \leq M(1 + |z| + z^2)$ for all $|X| > \delta$. Therefore, for fixed z, $\overline{\psi}(\cdot, z)$ is bounded in $|X|$ for $|X| > \delta$. By Assumption 5, $\psi'(\cdot, z)$ is monotone for large X, and therefore, $\overline{\psi}(\cdot, z)$ is monotone for large X (Lemma 35). Therefore, by Lemma 33, $\psi_p(z) \equiv \lim_{X \to p} \overline{\psi}(X,z)$ exists and $\lim_{X \to p} X\overline{\psi}'(X,z) = 0$. By Lemma 34, $\lim_{X \to p} \psi'(X,z) = \psi_p(z)$. □

Lemma 10. *Suppose* U *is a solution of* (9) *with linear growth. Then* $\exists \delta > 0$ *such that for* $|X| > \delta$*,* U' *and* $\overline{U}$ *are monotone and either both increasing or both decreasing. For* $p \in \{-\infty, \infty\}$*, there exists a* $U_p' \in \mathbb{R}$ *such that* $\lim_{X \to p} \overline{U}(X) = \lim_{X \to p} U'(X) = U_p'$*.*

Proof. Let $p \in \{-\infty, \infty\}$. Suppose U' is not monotone for large $|X|$. Then by the continuity of U', for any $\delta > 0$, there exists a z and a $|X_n|, |X_m| > \delta$ such that $U'(X_n) = z$ and $U''(X_n) \leq 0$ and $U'(X_m) = z$ and $U''(X_m) \geq 0$. From (9), this implies $U(X_n) \leq \psi(X_n, z)$ and $\psi(X_m, z) \leq U(X_m)$. Thus, the oscillation of $\psi'(\cdot, z)$ is at least as large as the oscillation of U'. By Assumption 5, $\psi'(\cdot, z)$ is monotone for large X, so U' cannot be non-monotone for large X. This is a contradiction; therefore, U' is monotone for large X.

By Lemma 35, if U' is monotone for large $|X|$, then $\overline{U}$ is monotone for large $|X|$. Given that U has linear growth, $\overline{U}$ is bounded. Therefore, $\lim_{X \to p} \overline{U}(X)$ exists and by Lemma 33, $\lim_{X \to p} X\overline{U}'(X) = 0$. By Lemma 34, $\lim_{X \to p} U'(X) = \lim_{X \to p} \overline{U}(X)$. Let U_p' denote this limit. □

Lemma 11. *Suppose* U *is a solution of* (9) *with linear growth. Then* $\lim_{X \to p} \overline{\psi}(X, U'(X)) = \psi_p(U_p')$ *for* $p \in \{-\infty, \infty\}$*, where* $U_p' \equiv \lim_{X \to p} U'(X)$*.*

Proof. Let $p \in \{-\infty, \infty\}$ and U be a solution of (9) with linear growth. Given μ^* and g^* are Lipschitz continuous, there exists a $\delta, M_1, M_2, M_3, c > 0$ such that for $|X| > \delta$,

$$|\psi(X, z_1) - \psi(X, z_2)| \leq M_1|z_1 - z_2| + M_2|z_1||z_1 - z_2| + M_3|z_1 - z_2|(|X| + |z_2|)$$

From Lemma 10, there exists a $U'_p \in \mathbb{R}$ such that $\lim_{X\to p} U'(X) = U'_p$. Therefore, for $|X| > \delta$,

$$\begin{aligned}
& \lim_{X\to p} |\overline{\psi}(X, U'(X)) - \overline{\psi}(X, U'_p)| \\
= \quad & \lim_{X\to p} \frac{|\psi(X, U'(X)) - \psi(X, U'_p)|}{|X|} \\
\leq \quad & \lim_{X\to p} \frac{M_1|U'(X) - U'_p| + M_2|U'(X)||U'(X) - U'_p| + M_3|U'(X) - U'_p|(|X| + |U'_p|)}{|X|} \\
= \quad & 0
\end{aligned}$$

Therefore, $\lim_{X\to p} \overline{\psi}(X, U'(X)) = \lim_{X\to p} \overline{\psi}(X, U'_p)$. From Lemma 9, $\lim_{X\to p} \overline{\psi}(X, U'_p) = \psi_p(U'_p)$. Therefore, $\lim_{X\to p} \overline{\psi}(X, U'(X)) = \psi_p(U'_p)$. □

Lemma 12. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has linear growth. Then any solution U of (9) with linear growth satisfies*

$$\lim_{X\to p} \inf |f(X)|U''(X) \leq 0 \leq \lim_{X\to p} \sup |f(X)|U''(X)$$

for $p \in \{-\infty, \infty\}$.

Proof. Let $p \in \{-\infty, \infty\}$. Suppose f has linear growth and $\lim_{X\to p} \inf |f(X)|U''(X) > 0$. There exists an $\delta_1, M > 0$ such that when $|X| > \delta_1$, $|f(X)| \leq M|X|$ Given $\lim_{X\to p} \inf |f(X)|U''(X) > 0$, there exists a $\delta_2, \varepsilon > 0$ such that when $|X| > \delta_2$, $|f(X)|U''(X) > \varepsilon$. Take $\delta = \max\{\delta_1, \delta_2\}$. Then for $|X| > \delta$, $U''(X) > \frac{\varepsilon}{|f(X)|} \geq \frac{\varepsilon}{M|X|}$. Then the antiderivative of $\frac{\varepsilon}{M|X|}$ is $\frac{\varepsilon}{M} \ln |X|$ which converges to ∞ as $X \to p$. Therefore, U' must grow unboundedly large as $X \to p$, which violates the linear growth of U. Therefore $\lim_{X\to p} \inf |f(X)|U''(X) \leq 0$. The proof is analogous for $\lim_{X\to p} \sup |f(X)|U''(X) \geq 0$. □

Lemma 13. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has linear growth. Then any solution U of (9) with linear growth satisfies $\lim_{X\to p} f(X)U''(X) = 0$ for $p \in \{-\infty, \infty\}$.*

Proof. Let $p \in \{-\infty, \infty\}$. Suppose that $\lim_{X\to p} \sup |X|U''(X) > 0$. By Lemma 10, there exists a $\delta > 0$ such that for $|X| > \delta$, U' is monotone. Then for $|X| > \delta$, $|X|U''(X)$ doesn't change sign. Therefore, if $\lim_{X\to p} \sup |X|U''(X) > 0$, then $\lim_{X\to p} \inf |X|U''(X) > 0$. This is a contradiction, given Lemma 12. Thus, $\lim_{X\to p} \sup |X|U''(X) = 0$. By similar reasoning, $\lim_{X\to p} \inf |X|U''(X) = 0$, and therefore $\lim_{X\to p} |X|U''(X) = 0$. Suppose f has linear growth.

Then there exists an $\delta_1, M > 0$ such that when $|X| > \delta_1$, $|f(X)| \leq M|X|$. Thus, for $|X| > \delta_1$, $|f(X)U''(X)| \leq M|XU''(X)| \to 0$. Note this also implies that $\lim_{X\to p} U''(X) = 0$. □

Lemma 14. *Suppose U is a solution of (9) with linear growth. Then U'_p is a fixed point of ψ_p, $U'_p = \psi_p(U'_p)$ for $p \in \{-\infty, \infty\}$.*

Proof. Let $p \in \{-\infty, \infty\}$ and U be a solution of (9) with linear growth. Given that σ^* is Lipschitz continuous, there exists an $\delta, M > 0$ such that for $|X| > \delta$,

$$|\sigma^*(X, z)|^2/|X| \leq M|(X, z)|^2/|X| = M(|X| + z^2/|X|).$$

Then

$$\lim_{X\to p} \frac{|\sigma^*(X, U'(X))|^2}{|X|}|U''(X)| \leq \lim_{X\to p} M|XU''(X)| + M|U'(X)^2|U''(X)/X| = 0.$$

where the equality follows from Lemmas 10 and 13. Plugging this into (9) yields $\lim_{X\to p} \overline{U}(X) - \overline{\psi}(X, U'(X)) = 0$ and therefore, by Lemma 10, $\lim_{X\to p} \overline{\psi}(X, U'(X)) = U'_p$. From Lemma 11, $\lim_{X\to p} \overline{\psi}(X, U'(X)) = \psi_p(U'_p)$. Combining these equations, $U'_p = \psi_p(U'_p)$, and U'_p must be a fixed point of ψ_p. □

Lemma 15. *Suppose U is a solution of (9) with linear growth. Then for $p \in \{-\infty, \infty\}$, the unique fixed point of ψ_p is $r\overline{g}_p/(r - \overline{\mu}_p)$, where $\overline{g}_p \equiv \lim_{X\to p} g^*(X, z)/X$ and $\overline{\mu}_p \equiv \lim_{X\to p} \mu^*(X, z)/X$ for all $z \in \mathbb{R}$, and $\overline{\mu}_p < r$. Therefore, $U'_p = r\overline{g}_p/(r - \overline{\mu}_p)$.*

Proof. Let $p \in \{-\infty, \infty\}$. Given $\overline{\psi}(\cdot, z)$ is monotone for large $|X|$ and μ^* and g^* are additively separable in (X, z), $g^*(X, z)/X$ and $\mu^*(X, z)/X$ are monotone for large $|X|$. Since g^* and μ^* are Lipschitz continuous, $g^*(X, z)/X$ and $\mu^*(X, z)/X$ are bounded. Therefore, the limits $\overline{g}_p \equiv \lim_{X\to p} g^*(X, z)/X$ and $\overline{\mu}_p \equiv \lim_{X\to p} \mu^*(X, z)/X$ exist. By additive separability, these limits are independent of z. Therefore,

$$\psi_p(z) = \lim_{X\to p} \frac{g^*(X, z) + z\mu^*(X, z)/r}{X} = \overline{g}_p + z\overline{\mu}_p/r.$$

At a fixed point z^*, $\psi_p(z^*) = z^*$. The unique fixed point of ψ_p is $z^* = \frac{r\overline{g}_p}{r - \overline{\mu}_p}$.

By Lemma 14, for any linear growth solution U of (9) and $p \in \{-\infty, \infty\}$, U'_p is a fixed point of ψ_p. Thus, the final statement follows.

□

Lemma 16. *Suppose y is a solution to the ODE*

$$y'(x) - (r/\mu_1(x))y(x) = 0 \tag{25}$$

with linear growth. Then for $p \in \{-\infty, \infty\}$, $\lim_{x \to p} y(x) = 0$.

Proof. The general solution to (25) is

$$y(x) = c \exp\left(\int \frac{r}{\mu_1(x)} dx\right) \tag{26}$$

where $c \in \mathbb{R}$ is a constant. Trivially, there always exists a solution with linear growth because $y(x) = 0$ is a solution. Consider $p = \infty$. By Assumption 3, μ_1 has linear growth with rate slower than r. By Assumption 5, μ_1 is monotone for large x. Therefore, there exists a $\delta > 0$ such that for $x > \delta$, either (i) there exists a $k \in (0, r)$ such that $\mu_1(x) \in (0, kx]$ or (ii) there exists a $k > 0$ such that $\mu_1(x) \in [-kx, 0)$.

Case (i): Suppose there exists a $k \in (0, r)$ and $\delta > 0$ such that for $x > \delta$, $\mu_1(x) \in (0, kx]$. Then $\frac{1}{\mu_1(x)} \geq \frac{1}{kx}$. But $\exp\left(\int \frac{r}{kx} dx\right) = \exp\left(\frac{r}{k} \ln x\right) = x^{r/k}$ is not in $O(x)$ since $r/k > 1$. Therefore, $\exp\left(\int \frac{r}{\mu_1(x)} dx\right)$ is not in $O(x)$. Therefore, any solution to (26) that has linear growth must have $c = 0$. The unique solution with linear growth is $y(x) = 0$, which trivially satisfies $\lim_{x \to \infty} y(x) = 0$.

Case (ii): Suppose there exists a $k, \delta > 0$ such that for $x > \delta$, $\mu_1(x) \in [-kx, 0)$. Then $\frac{1}{\mu_1(x)} \leq -\frac{1}{kx}$. But $\exp\left(\int -\frac{r}{kx} dx\right) = \exp\left(-\frac{r}{k} \ln x\right) = x^{-r/k}$ and $\lim_{x \to \infty} x^{-r/k} \to 0$. Therefore, $\lim_{x \to \infty} \exp\left(\int \frac{r}{\mu_1(x)} dx\right) = 0$. Therefore, for all c, $\lim_{x \to \infty} y(x) = 0$ and any solution to (26) satisfies this property.

The case for $p = -\infty$ is analogous. ∎

Lemma 17. *Suppose* U *and* V *are solutions of* (9) *with linear growth. Then for* $p \in \{-\infty, \infty\}$, $\lim_{X \to p} U(X) - V(X) = 0$.

Proof. Let $p \in \{-\infty, \infty\}$, U and V be solutions of (9) with linear growth. Then

$$\begin{aligned}
\Rightarrow \lim_{X \to p} \quad & U(X) - g_1(X) - U'(X)\mu_1(X)/r - g_2(U'(X)) - U'(X)\mu_2(U'(X))/r \\
& -U''(X)|\sigma^*(X, U'(X))|^2/2r = 0 \\
\Rightarrow \lim_{X \to p} \quad & U(X) - g_1(X) - U'(X)\mu_1(X)/r = g_2(z_p) + z_p\mu_2(z_p)/r
\end{aligned} \tag{27}$$

where the first line follows from (9) and the additive separability of g^* and μ^*, and the second line follows from the Lipschitz continuity of $|\sigma^*|^2$, $\lim_{X \to p} U'(X) = z_p$ and the Lipschitz continuity of g_2 and μ_2. By Lemma 15, $\lim_{X \to p} U'(X) = \lim_{X \to p} V'(X) = z_p$.

Define $D = U - V$. Then $D' = U' - V'$, D has linear growth since U and V have linear growth and

$$\lim_{X\to p} D(X) - \mu_1(X)D'(X)/r = 0$$
$$\lim_{X\to p} D'(X) = 0$$

where the first line follows from (27). Therefore, there exists a solution y to (25) with linear growth such that $\lim_{X\to p} D(X) - y(X) = 0$. By Lemma 16, $\lim_{X\to p} y(X) = 0$ for any solution y with linear growth. Therefore, $\lim_{X\to p} D(X) = 0$ and any two solutions U and V with linear growth have the same boundary conditions, $\lim_{X\to p} U(X) - V(X) = 0$. ☐

Lemma 18. *Suppose y is a solution to the ODE*

$$y(x) - g_1(x) - \mu_1(x)y'(x)/r = 0 \tag{28}$$

with linear growth. Then for $p \in \{-\infty, \infty\}$, $\lim_{x\to p} y(x) - y^L(x) = 0$, where

$$y^L(x) \equiv -\phi(x)\int\left(\frac{1}{\phi(x)}\right)\frac{rg_1(x)}{\mu_1(x)}dx \tag{29}$$

is a solution with linear growth and $\phi(x) \equiv \exp\left(\int r/\mu_1(x)dx\right)$.

Proof. The general solution to (28) is

$$y(x) = -\phi(x)\int\left(\frac{1}{\phi(x)}\right)\frac{rg_1(x)}{\mu_1(x)}dx - \phi(x)c \tag{30}$$

where ϕ is as defined above and $c \in \mathbb{R}$ is a constant. Consider $p = \infty$. By Assumption 3, μ_1 has linear growth with rate slower than r. By Assumption 5, μ_1 is monotone for large x. Therefore, there exists a $\delta > 0$ such that for $x > \delta$, either (i) there exists a $k \in (0, r)$ such that $\mu_1(x) \in (0, kx]$ or (ii) there exists a $k > 0$ such that $\mu_1(x) \in [-kx, 0)$.

Case (i): Suppose there exists a $k \in (0, r)$ and $\delta > 0$ such that for $x > \delta$, $\mu_1(x) \in (0, kx]$. Then $\frac{1}{\mu_1(x)} \geq \frac{1}{kx}$. But $\exp\left(\int \frac{r}{kx}dx\right) = \exp\left(\frac{r}{k}\ln x\right) = x^{r/k}$ is not in $O(x)$ since $r/k > 1$. Therefore, $\exp\left(\int \frac{r}{\mu_1(x)}dx\right)$ is not in $O(x)$ and ϕ doesn't have linear growth. Therefore, any solution to (30) that has linear growth must have $c = 0$. The unique solution with linear growth is (29), which trivially satisfies $\lim_{x\to\infty} y(x) - y^L(x) = 0$.

Case (ii): Suppose there exists a $k, \delta > 0$ such that for $x > \delta$, $\mu_1(x) \in [-kx, 0)$. Then $\frac{1}{\mu_1(x)} \leq -\frac{1}{kx}$. But $\exp\left(\int -\frac{r}{kx}dx\right) = \exp\left(-\frac{r}{k}\ln x\right) = x^{-r/k}$ and $\lim_{x\to\infty} x^{-r/k} = 0$. Therefore,

$\lim_{x\to\infty}\phi(x) = \lim_{x\to\infty}\exp\left(\int \frac{r}{\mu_1(x)}dx\right) = 0$ and c does not affect the limit properties of a solution y. Therefore, for all c, $\lim_{x\to\infty} y(x) - y^L(x) = 0$ and any solution to (30) satisfies this property.

The case for $p = -\infty$ is analogous. □

Lemma 19. *Suppose U is a solution of (9) with linear growth. Then for $p \in \{-\infty, \infty\}$, $\lim_{X\to p} U(X) - y^L(X) = g_2(z_p) + z_p\mu_2(z_p)/r$, where y_L is defined by (29).*

Proof. Let $p \in \{-\infty, \infty\}$ and U be a solution of (9) with linear growth. Then

$$\begin{aligned}
\lim_{X\to p} \quad & U(X) - g_1(X) - U'(X)\mu_1(X)/r - U''(X)|\sigma^*(X, U'(X))|^2/2r \\
& -g_2(U'(X)) - U'(X)\mu_2(U'(X))/r = 0 \\
\Rightarrow \lim_{X\to p} \quad & U(X) - g_1(X) - U'(X)\mu_1(X)/r = g_2(z_p) + z_p\mu_2(z_p)/r
\end{aligned}$$

where the first line follows from (9) and the additive separability of g^* and μ^*, and the second line follows from the Lipschitz continuity of $|\sigma^*|^2$, $\lim_{X\to p} U'(X) = z_p$ and the Lipschitz continuity of g_2 and μ_2. Therefore, there exists a solution y to (28) with linear growth such that $\lim_{X\to p} U(X) - y(X) = g_2(z_p) + z_p\mu_2(z_p)/r$. By Lemma 18, $\lim_{X\to p} y(X) - y^L(X) = 0$. Therefore, $\lim_{X\to p} U(X) - y^L(X) = g_2(z_p) + z_p\mu_2(z_p)/r$ which establishes the boundary condition for U. □

Step 1b: Boundary Conditions for Unbounded $\mathcal{X}$ and Bounded g (Theorem 4).

Theorem 7. *Suppose $\mathcal{X} = \mathbb{R}$ and g is bounded and assume Assumptions 1-4 and Assumption 5′. Then any bounded solution U of (9) satisfies*

$$\begin{aligned}
&\lim_{X\to p} U(X) = g_p \\
&\lim_{X\to p} XU'(X) = 0 \\
&\lim_{X\to p} X^2U''(X) = 0
\end{aligned}$$

for $p \in \{-\infty, \infty\}$, where $g_p \equiv \lim_{X\to p} g^*(X, 0)$.

The proof proceeds by a series of lemmas. Note g_p exists given g bounded and $g^*(\cdot, 0)$ monotone for large $|X|$.

Lemma 20. *If U is a bounded solution of (9), then there exists a $\delta > 0$ such that for $|X| > \delta$, U is monotone and for $p \in \{-\infty, \infty\}$, there exists a $U_p \in \mathbb{R}$ such that $\lim_{X\to p} U(X) = U_p$.*

Proof. Let $p \in \{-\infty, \infty\}$. Suppose U is not monotone for large X. Then for all $\delta > 0$, there exists a $|X_n| > \delta$ that corresponds to a local maximum of U, so $U'(X_n) = 0$ and $U''(X_n) \leq 0$ and there exists a $|X_m| > \delta$ that corresponds to a local minimum of U, so $U'(X_m) = 0$ and $U''(X_m) \geq 0$, by the continuity of U. Given the incentives for the long-run player, a static Nash equilibrium is played at any X such that $U'(X) = 0$, yielding flow payoff $g^*(X, 0)$. From (9), this implies $g^*(X_n, 0) \geq U(X_n)$ at the maximum and $g^*(X_m, 0) \leq U(X_m)$ at the minimum. Thus, the oscillation of $g^*(\cdot, 0)$ is at least as large as the oscillation of U. This is a contradiction, as $g^*(\cdot, 0)$ is monotone for large $|X|$. Thus, there exists a δ such that for $|X| > \delta$, U is monotone. The existence of $\lim_{X \to p} U(X)$ follows from U is bounded and monotone for large $|X|$. □

Lemma 21. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has linear growth. Then any bounded solution U of* (9) *satisfies*

$$\begin{aligned} \liminf_{X \to p} |f(X)| U'(X) &\leq 0 \leq \limsup_{X \to p} |f(X)| U'(X) \\ \liminf_{X \to p} f(X)^2 U''(X) &\leq 0 \leq \limsup_{X \to p} f(X)^2 U''(X) \end{aligned}$$

for $p \in \{-\infty, \infty\}$.

Proof. Let $p \in \{-\infty, \infty\}$. Suppose f has linear growth and $\lim_{X \to p} \inf |f(X)| U'(X) > 0$. Then there exists an $\delta_1, M > 0$ such that when $|X| > \delta_1, |f(X)| \leq M|X|$. Given $\lim_{X \to p} \inf |f(X)| U'(X) > 0$, there exists a $\delta_2, \varepsilon > 0$ such that when $|X| > \delta_2$, $|f(X)| U'(X) > \varepsilon$. Take $\delta = \max\{\delta_1, \delta_2\}$. Then for $|X| > \delta$, $|U'(X)| > \frac{\varepsilon}{|f(X)|} \geq \frac{\varepsilon}{M|X|}$. Then the antiderivative of $\frac{\varepsilon}{M|X|}$ is $\frac{\varepsilon}{M} \ln |X|$ which converges to ∞ as $|X| \to \infty$. This violates the boundedness of U. Therefore $\lim_{X \to p} \inf |f(X)| U'(X) \leq 0$. The proof is analogous for $\lim_{X \to p} \sup |f(X)| U'(X) \geq 0$.

Suppose f has linear growth and $\lim_{X \to \infty} \inf f(X)^2 U''(X) > 0$. Then there exists a $M, \delta_1 > 0$ such that when $|X| > \delta_1, |f(X)| \leq M|X|$ and therefore, $f(X)^2 \leq M^2 X^2$. There also exists a $\delta_2, \varepsilon > 0$ such that when $|X| > \delta_2$, $f(X)^2 U''(X) > \varepsilon$. Take $\delta = \max\{\delta_1, \delta_2\}$. Then for $|X| > \delta$, $|U''(X)| > \frac{\varepsilon}{f(X)^2} > \frac{\varepsilon}{M^2 X^2}$. The antiderivative of $\frac{\varepsilon}{M^2 X^2}$ is $\frac{-\varepsilon}{M^2} \ln |X|$ which converges to $-\infty$ as $|X| \to \infty$. This violates the boundedness of U. Therefore $\lim_{X \to p} \inf f(X)^2 U''(X) \leq 0$. The proof is analogous for $\lim_{X \to p} \sup f(X)^2 U''(X) \geq 0$. □

Lemma 22. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has linear growth. Then for $p \in \{-\infty, \infty\}$, any bounded solution U of* (9) *satisfies* $\lim_{X \to p} f(X) U'(X) = 0$.

Proof. Let $p \in \{-\infty, \infty\}$. Suppose that $\lim_{X \to p} \sup |X| U'(X) > 0$. By Lemma 20, there exists a $\delta > 0$ such that U is monotone for $|X| > \delta$. Then for $|X| > \delta$, $XU'(X)$ doesn't change sign. Therefore, if $\lim_{X\to p}\sup|X|U'(X) > 0$, then $\lim_{X\to p}\inf|X|U'(X) > 0$. This is a contradiction. Thus, $\lim_{X\to p}\sup|X|U'(X) = 0$. By similar reasoning, $\lim_{X\to p}\inf|X|U'(X) = 0$, and therefore $\lim_{X\to p}|X|U'(X) = 0$. Suppose f has linear growth. Then there exists an $M, \delta_1 > 0$ such that when $|X| > \delta_1, |f(X)| \le M|X|$. Thus, for $|X| > \delta_1$, $|f(X)U'(X)| \le M|XU'(X)| \to 0$. Therefore, $\lim_{X\to p} f(X)U'(X) = 0$. This also implies that $\lim_{X\to p} U'(X) = 0$. □

Lemma 23. *Let U be a bounded solution of (9). Then for $p \in \{-\infty, \infty\}$, $\lim_{X\to p} U(X) = g_p$.*

Proof. Let $p \in \{-\infty, \infty\}$. Suppose $U_p < g_p$. Given μ^* is Lipschitz continuous, there exists a $\delta, M > 0$ such that for $|X| > \delta$,

$$|\mu^*(x,z)| \le M|(x,z)| \le M(|x| + |z|).$$

Therefore,

$$\lim_{X\to\infty} |U'(X)\mu^*(X,U'(X))| \le \lim_{x\to\infty} |U'(X)|M(|X| + |U'(X)|) = 0$$

where the equality follows from Lemma 22. Similarly,

$$\lim_{X\to p} g^*(X,U'(X)) = \lim_{X\to p} g^*(X,0) = g_p.$$

since g^* is Lipschitz continuous. Plugging these limits into (9),

$$\begin{aligned}\limsup_{X\to p} \frac{1}{2}\left|\sigma^*(X,U'(X))\right|^2 U''(X) &= \limsup_{X\to p} \left(rU(X) - rg^*(X,U'(X)) - \mu^*(X,U'(X))U'(X)\right) \\ &= r(U_p - g_p) < 0.\end{aligned}$$

But by Lemma 21, $\limsup_{X\to p}\left|\sigma^*(X,U'(X))\right|^2 U''(X) > 0$ since σ^* is Lipschitz continuous, a contradiction. Thus, $U_p \ge g_p$. A similar contradiction holds for $U_p > g_p$. Therefore, $U_p = g_p$. □

Lemma 24. *Any bounded solution U of (9) satisfies $\lim_{X\to p}\left|\sigma^*(X,U'(X))\right|^2 U''(X) = 0$ for $p \in \{-\infty, \infty\}$.*

Proof. Let $p \in \{-\infty, \infty\}$. By Lemmas 22 and 23 and the squeeze theorem,

$$\lim_{X\to p}\frac{1}{2}\left|\left|\sigma^*(X,U'(X))\right|^2 U''(X)\right| = \lim_{X\to p}|rU(X) - rg^*(X,U'(X)) - \mu^*(X,U'(X))U'(X)| = 0.$$

If $\lim_{X\to p}|\sigma^*(X,U'(X))|^2 > 0$, this implies that $\lim_{X\to p} U''(X) = 0$. □

Step 1c: Boundary Conditions for Bounded $\mathcal{X}$ (Theorem 4).

Theorem 8. *Suppose $\mathcal{X}$ is compact and assume Assumptions 1-4 and Assumption 5′. Then any bounded solution U of (9) satisfies*

$$\lim_{X\to p} U(X) = g^*(p,0)$$
$$\lim_{X\to p} (X-p)U'(X) = 0$$
$$\lim_{X\to p} (X-p)^2 U''(X) = 0.$$

for $p \in \{-\underline{X}, \overline{X}\}$.

The proof proceeds by a series of lemmas.

Lemma 25. *Any bounded solution U of (9) has bounded variation.*

Proof. Suppose U has unbounded variation. Then there exists a sequence $(X_n)_{n\in N}$ that correspond to local maxima of U, so $U'(X_n) = 0$ and $U''(X_n) \leq 0$. Given (6), a static Nash equilibrium is played at any X such that $U'(X) = 0$, yielding flow payoff $g^*(X, 0)$. From (9), this implies $g^*(X_n, 0) \geq U(X_n)$. Likewise, there exists a sequence $(X_m)_{m\in N}$ that correspond to local minima of U, so $U'(X_m) = 0$ and $U''(X_m) \geq 0$. This implies $g^*(X_m, 0) \leq U(X_m)$. Thus, $g^*(\cdot, 0)$ has unbounded variation. This is a contradiction, since $g^*(\cdot, 0)$ is Lipschitz continuous. □

Lemma 26. *Suppose $f : \mathcal{X} \to \mathbb{R}$ is Lipschitz continuous with $f(\overline{X}) = f(\underline{X}) = 0$. Then any bounded solution U of (9) satisfies*

$$\begin{aligned} \lim_{X\to p} \inf f(X)U'(X) &\leq 0 \leq \lim_{X\to p} \sup f(X)U'(X) \\ \lim_{X\to p} \inf f(X)^2 U''(X) &\leq 0 \leq \lim_{X\to p} \sup f(X)^2 U''(X) \end{aligned}$$

for $p \in \left\{\underline{X}, \overline{X}\right\}$.

Proof. Let $p \in \{-\underline{X}, \overline{X}\}$ and $f : \mathcal{X} \to \mathbb{R}$ be Lipschitz continuous with $f(p) = 0$. Then f is $O(p - X)$, so there exists an $M, \delta_1 > 0$ such that when $|p - X| < \delta_1, |f(X)| \leq M|p - X|$.

Suppose $\lim_{X\to p} \inf |f(X)|U'(X) > 0$. Then there exists a $\delta_2, \varepsilon > 0$ such that when $|p - X| < \delta_2$, $|f(X)|U'(X) > \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then for $|p - X| < \delta$, $U'(X) > \frac{\varepsilon}{|f(X)|} \geq \frac{\varepsilon}{M|p-X|}$. Then the antiderivative of $\frac{\varepsilon}{M|p-X|}$ is $\frac{\varepsilon}{M} \ln|p - X|$ which diverges to $-\infty$ as $X \to p$. This violates the boundedness of U. Therefore $\lim_{X\to p} \inf |f(X)|U'(X) \leq 0$. The proof is analogous for $\lim_{X\to p} \sup |f(X)|U'(X) \geq 0$.

Suppose $\lim_{X \to p} \inf f(X)^2 U''(X) > 0$. There exists a $\delta_3, \varepsilon_2 > 0$ such that when $|p - X| < \delta_3$, $f(X)^2 U''(X) > \varepsilon_2$. Take $\delta = \min\{\delta_1, \delta_3\}$. Then for $|p - X| < \delta$, $U''(X) > \frac{\varepsilon_2}{f(X)^2} > \frac{\varepsilon_2}{M^2(p-X)^2}$. The second antiderivative of $\frac{\varepsilon_2}{M^2(p-X)^2}$ is $\frac{-\varepsilon_2}{M^2} \ln |p - X|$ which converges to $-\infty$ as $X \to p$. This violates the boundedness of U. Therefore $\lim_{X \to \infty} \inf f(X)^2 U''(X) \leq 0$. The proof is analogous for $\lim_{X \to p} \sup f(X)^2 U''(X) \geq 0$. □

Lemma 27. *Suppose $f : \mathcal{X} \to \mathbb{R}$ is Lipschitz continuous, with $f(\overline{X}) = f(\underline{X}) = 0$. Then any bounded solution U of (9) satisfies $\lim_{X \to p} f(X)U'(X) = 0$ for $p \in \{\underline{X}, \overline{X}\}$.*

Proof. Let $p \in \{-\underline{X}, \overline{X}\}$. Suppose that $\limsup_{X \to p} |p - X| U'(X) > 0$. By Lemma 26, $\lim_{X \to p} \inf |p - X| U'(X) \leq 0$. Then there exist constants $k, K > 0$ such that $|p - X|U'(X)$ crosses the interval (k, K) infinitely many times as X approaches p. Additionally, there exists an $L > 0$ such that

$$
\begin{aligned}
|U''(X)| &= \left| \frac{2r\left[U(X) - g^*(X, U'(X))\right] - 2\mu^*(X, U'(X))U'(X)}{|\sigma^*(X, U'(X))|^2} \right| \leq \left| \frac{L_1 - L_2 \left|p - X\right| U'(X)}{(p - X)^2} \right| \\
&\leq \left| \frac{L_1 - L_2 k}{(p - X)^2} \right| = \frac{L}{(p - X)^2}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\left|[(p - X)U'(X)]'\right| &\leq |U'(X)| + |(p - X)U''(X)| = \left(1 + \left|(p - X)\frac{U''(X)}{U'(X)}\right|\right) |U'(X)| \\
&\leq \left(1 + \frac{L}{k}\right) |U'(X)|
\end{aligned}
$$

where the first line follows from differentiating $(p - X)\, U'(X)$ and the subadditivity of the absolute value function, the next line follows from rearranging terms, the third line follows from the bound on $|U''(X)|$ and $(p - X)U'(X) \in (k, K)$. Then

$$
U'(X) \geq \frac{\left|[(p - X)\, U'(X)]'\right|}{\left(1 + \frac{L}{k}\right)}
$$

Therefore, the total variation of U is at least $\frac{K-k}{\left(1+\frac{L}{k}\right)}$ on the interval $|p - X|U'(X) \in (k, K)$, which implies that U has unbounded variation near p. This is a contradiction. Thus, $\lim_{X \to p} \sup(p - X)U'(X) = 0$. Likewise, $\lim_{X \to p} \inf(p - X)U'(X) = 0$. Therefore $\lim_{X \to p}(p - X)U'(X) = 0$.

Suppose f is $O(p - X)$. Then there exists an $M, \delta > 0$ such that for $|p - X| < \delta$, $|f(X)| \leq M|\,(p - X)\,| \to 0$. Thus for $|p - X| < \delta$, $|f(X)U'(X)| \leq M|\,(p - X)\, U'(X)| \to 0$. Therefore $\lim_{X \to p} f(X)U'(X) = 0$. □

Lemma 28. *Let U be a bounded solution of (9). Then for $p \in \{\underline{X}, \overline{X}\}$, $\lim_{X \to p} U(X) = g^*(p, 0)$.*

Proof. Let $p \in \{-\underline{X}, \overline{X}\}$. Given that U is continuous, bounded and has bounded variation, $U_p \equiv \lim_{X \to p} U(X)$ exists. Suppose $U_p < g^*(p, 0)$. By Lemma 27, the Lipschitz continuity of μ^* and the assumption that $\mu_x(a, b, p) = 0$ for all $(a, b) \in A \times B(p)$,

$$\lim_{X \to p} \mu^*(X, U'(X))U'(X) = 0.$$

If X is an absorbing state, then $S^*(X, z_y, z_x) = S^*(X, 0, 0)$ for all $(z_y, z_x) \in \mathbb{R}^d$. Therefore, $g^*(X, z) = g^*(X, 0)$ for all $z \in \mathbb{R}$. By the Lipschitz continuity of g^*,

$$\lim_{X \to p} g^*(X, U'(X)) = g^*(p, 0).$$

Plugging these limits into (9),

$$\begin{aligned} \limsup_{X \to \underline{X}} \left|\sigma^*(X, U'(X))\right|^2 U''(X)/2r &= \limsup_{X \to p} \left[U(X) - g^*(X, U'(X)) - \mu^*(X, U'(X))U'(X)/r\right] \\ &= 2r\left(U_p - g^*(p, 0)\right) < 0. \end{aligned}$$

But by Lemma 26, $\limsup_{X \to p} \left|\sigma^*(X, U'(X))\right|^2 U''(X) > 0$ since σ^* is Lipschitz continuous and $|\sigma(b, p)| = 0$ for all $b \in B(p)$. This is a contradiction. Thus, $U_p \geq g^*(p, 0)$. A similar contradiction holds for $U_p > g^*(p, 0)$. Therefore, $U_p = g^*(p, 0)$. □

Lemma 29. *Any bounded solution U of (9) satisfies*

$$\lim_{X \to p} \left| \left|\sigma^*(X, U'(X))\right|^2 U''(X) \right| = 0$$

for $p \in \{-\underline{X}, \overline{X}\}$.

Proof. Let $p \in \{-\underline{X}, \overline{X}\}$. Applying Lemmas 27 and 28 and the squeeze theorem,

$$\lim_{X \to p} \left| \left|\sigma^*(X, U'(X))\right|^2 U''(X)/2r \right| = \lim_{X \to p} \left|U(X) - g^*(X, U'(X)) - \mu^*(X, U'(X))U'(X)/r\right| = 0.$$

□

Step 2: Uniqueness of Solution to Optimality Equation (Theorems 3 and 4).

Lemma 30. *If U and V are two bounded solutions of (9) such that $U(X_0) \leq V(X_0)$ and $U'(X_0) \leq V'(X_0)$, with at least one strict inequality, then $U(X) < V(X)$ and $U'(X) < V'(X)$*

for all $X > X_0$. Similarly if $U(X_0) \leq V(X_0)$ and $U'(X_0) \geq V'(X_0)$, with at least one strict inequality, then $U(X) < V(X)$ and $U'(X) > V'(X)$ for all $X < X_0$.

Proof. This follows directly from Lemma C.7 in Faingold and Sannikov (2011), defining $X_1 = \inf \left\{ X \in [X_0, \overline{X}) : U'(X) \geq V'(X) \right\}$. ☐

Lemma 31. *There exists a unique linear growth (bounded) solution U to (9).*

Proof. Suppose U and V are both solutions to (9). Suppose $V(X) > U(X)$ for some $X \in \mathcal{X}$. Let X^* be the point where $V(X) - U(X)$ is maximized, which is well-defined given U and V are continuous functions and $\lim_{X \to p} U(X) - V(X) = 0$ for $p \in \left\{ \underline{X}, \overline{X} \right\}$. Then $U'(X^*) = V'(X^*)$ and $V(X^*) > U(X^*)$. By Lemma 30, $V'(X) > U'(X)$ for all $X > X^*$, and $V(X) - U(X)$ is strictly increasing, a contradiction since X^* maximizes $U(X) - V(X)$. ☐

Step 3: Uniqueness of PPE (Theorems 3 and 4).

Lemma 32. *There exists a unique PPE.*

Proof. By Lemma 31, there is a unique linear growth (bounded) solution to (9). It is obvious to see that this implies that there is a unique Markov equilibrium, by Theorem 1. It remains to show that there are no other PPE. When there is a unique Markov equilibrium, Theorem 2 implies that in any PPE with continuation values $(W_t)_{t \geq 0}$, $W_t = U(X_t)$ for all t. Therefore, the volatility of the two continuation values are equal, otherwise they both cannot be equal to $U(X_t)$. Given equal volatilities, actions are uniquely specified by $S^*(X, 0, U'(X)/r)$. ☐

A.5 Proofs from Section 5

Proof of Proposition 3.

Proof. Suppose g is bounded. Then U is continuous and bounded on a closed set. Therefore, U either attains its maximum on $\mathcal{X}$, in which case $\overline{W} = U(X_H)$ for some X_H, or if $\mathcal{X}$ is unbounded, $\overline{W} = \limsup_{X \to X_H} U(X)$ for $X_H \in \{-\infty, \infty\}$. Suppose U attains a maximum at an interior state X_H. Then $U'(X_H) = 0$ and $U''(X_H) \leq 0$. From (9),

$$U''(X_H) = \frac{2r \left(\overline{W} - g^*(X_H, 0) \right)}{\left| \sigma^* \left(X_H, 0 \right) \right|^2} \leq 0.$$

and therefore $\overline{W} \leq g^*(X_H, 0)$. Suppose $\mathcal{X}$ is unbounded and U doesn't attain a maximum at an interior state. By Lemma 21, the local minima and maxima of U are bounded by the local minima and maxima of $g^*(\cdot, 0)$. Therefore, for any sequence of local maxima

$\{U_k\}$ converging to $\overline{W}$, there exists a sequence of states $\{X_k\}$ such that $U_k = U(X_k)$ and $\overline{W} \leq \lim_{k\to\infty} g^*(X_k, 0) = \limsup_{X\to\infty} g^*(X, 0)$. Suppose $\mathcal{X}$ is bounded and $\overline{W} = U(\overline{X})$. By the definition of $\overline{X}$, $\mu^*(\overline{X}, U'(\overline{X})) \leq 0$ and $\sigma^*\left(\overline{X}, U'(\overline{X})\right) = 0$. Also, $U'(\overline{X}) \geq 0$. From (9),

$$\overline{W} - g^*(\overline{X}, 0) = \frac{1}{r}U'(\overline{X})\mu^*(\overline{X}, U'(\overline{X})) \leq 0$$

The proof for $\overline{W} = U(\underline{X})$ and the lower bound $\underline{W}$ are analogous. □

Proof of Proposition 2.

Proof. Suppose $\mathcal{X}$ is bounded. At a state X corresponding to an interior extremum, $U'(X) = 0$. From (9), if X is a minimum, $U(X) \geq g^*(X, 0)$, and if X is a maximum, $U(X) \leq g^*(X, 0)$. Given an interval $I \subset \mathcal{X}$, order the states that correspond to strict interior extrema of U as $X_1 < X_2 < ... < X_{n_U(I)}$.

Part 1: Follows directly from Lemma 21 and Lipschitz continuity of g^*.

Part 2: Suppose $g^*(\cdot, 0)$ is constant on an interval $I \subset \mathcal{X}$ and $n_U(I) > 1$. If X_1 is a minimum and X_2 is a maximum, then from (9), $g^*(X_1, 0) \leq U(X_1) < U(X_2) \leq g^*(X_2, 0)$. This is a contradiction, because $g^*(\cdot, 0)$ is constant on I. The same logic holds if X_1 is a maximum and X_2 is a minimum. Therefore, $n_U(I) \leq 1$. If $\underline{X} \in I$ and X_1 is a minimum, then $g^*(X_1, 0) \leq U(X_1) < g^*(\underline{X}, 0) = U(\underline{X})$. This is a contradiction, because $g^*(X, 0)$ is constant. Similarly, it's not possible for X_1 to be a maximum. Therefore, if I contains a boundary point, $n_U(I) = 0$.

Part 3: Suppose $g^*(\cdot, 0)$ is strictly increasing on an interval $I \subset \mathcal{X}$ and $n_U(I) > 2$. If X_i is a strict maximum and X_{i+1} is a strict minimum, then from (9), $g^*(X_{i+1}, 0) \leq U(X_{i+1}) < U(X_i) \leq g^*(X_i, 0)$. This is a contradiction, because $g^*(\cdot, 0)$ is increasing on I. Therefore, it's not possible to have a maximum followed by a minimum. If X_1 is a maximum, $n_U(I) = 1$ and if X_1 is a minimum, $n_U(I) \leq 2$. Suppose $\underline{X} \in I$. By the boundary conditions, $g^*(\underline{X}, 0) = U(\underline{X})$. If X_1 is a strict minimum then $g^*(X_1, 0) \leq U(X_1) < g^*(\underline{X}, 0) = U(\underline{X})$. This is a contradiction, because $g^*(X, 0)$ is increasing. Therefore, $n_U(I) = 0$ or $n_U(I) = 1$ and X_1 is a maximum. Similarly, if $\overline{X} \in I$, it's not possible to have a maximum. Either $n_U(I) = 0$ or $n_U(I) = 1$ and X_1 is a minimum.

Suppose U is constant on an interval $I \subset \mathcal{X}$. There exists a constant c such that $U(X) = c$ for all $X \in I$. Then $U'(X) = 0$ and $U''(X) = 0$ for all $X \in I$. From (9), $U(X) = g^*(X, 0)$ for all $X \in I$. Therefore, $g^*(X, 0) = c$ for all $X \in I$ and $g^*(\cdot, 0)$ is constant on I, a contradiction. The proof for when $g^*(X, 0)$ is decreasing is analogous.

For $\mathcal{X}$ unbounded, replace $g^*(p, 0)$ with $\lim_{X \to p} g^*(X, 0)$ and $U(p)$ with $\lim_{X \to p} U(X)$ for $p \in \{\underline{X}, \overline{X}\}$. □

Proof of Proposition 1.

Proof. Suppose $\mathcal{X}$ is bounded.

Part 3: Suppose $g^*(\cdot, 0)$ is constant on $\mathcal{X}$. Then there exists a $c \in \mathbb{R}$ such that $g^*(\cdot, 0) = c$ for all X and $n_g = 0$. By Proposition 2.1, $n_U = 0$. By the boundary conditions, $U(\underline{X}) = c$ and $U(\overline{X}) = c$, which implies $U(\underline{X}) = U(\overline{X})$. Combined with $n_U = 0$, this implies that U is constant on $\mathcal{X}$.

I prove the contrapositive to establish that if U is constant on $\mathcal{X}$, then $g^*(\cdot, 0)$ is constant on $\mathcal{X}$. Suppose $g^*(\cdot, 0)$ is not constant on $\mathcal{X}$. Then there exists an interval $I \subset \mathcal{X}$ such that $g^*(\cdot, 0)$ is strictly increasing or decreasing on I. By Proposition 2.3, U is not constant on I. Therefore, U is not constant on $\mathcal{X}$.

Part 1: Suppose $g^*(\cdot, 0)$ is monotonically increasing on $\mathcal{X}$, but U is not monotonically increasing. Then $U'(X) < 0$ for some $X \in \mathcal{X}$. By Proposition 2.1, $n_U(\mathcal{X}) = 0$. Therefore, it must be that $U'(X) \leq 0$ for all $X \in \mathcal{X}$, and $U(\underline{X}) > U(\overline{X})$. By the boundary conditions, $U(\underline{X}) = g^*(\underline{X}, 0)$ and $U(\overline{X}) = g^*(\overline{X}, 0)$, and by monotonicity, $g^*(\underline{X}, 0) \leq g^*(\overline{X}, 0)$, which implies $U(\underline{X}) \leq U(\overline{X})$, a contradiction. Therefore, U is monotonically increasing. If $g^*(\cdot, 0)$ is strictly increasing on $\mathcal{X}$, then by Proposition 2.3, U is not constant on any $I \subset \mathcal{X}$, so U is also strictly increasing. The proof for U monotonically decreasing is analogous.

Part 2: Suppose $n_g = 1$, $g^*(\cdot, 0)$ has a unique interior maximum at X^* and $g^*(\underline{X}, 0) = g^*\left(\overline{X}, 0\right)$. Then $g^*(X, 0)$ is monotonically increasing for $X < X^*$ and monotonically decreasing for $X > X^*$. By Proposition 2.1, $n_U \leq 1$. By Proposition 2.3, if $n_U|_{[\underline{X}, X^*]} = 1$, then the extremum is a maximum and if $n_U|_{[X^*, \overline{X}]} = 1$, then the extremum is a maximum. Therefore, if $n_U = 1$, the extremum is a maximum.

Suppose $n_U = 0$. Since $g^*(\underline{X}, 0) = g^*\left(\overline{X}, 0\right)$, by the boundary conditions, $U(\underline{X}) = U\left(\overline{X}\right)$ and U is constant on $\mathcal{X}$. By Part 3, this implies that $g^*(\cdot, 0)$ is constant on $\mathcal{X}$, a contradiction. Therefore, it must be the case that $n_U = 1$ and U is single-peaked with a maximum. The proof for U single-peaked with a minimum is analogous.

For $\mathcal{X}$ unbounded, replace $g^*(p,0)$ with $\lim_{X\to p} g^*(X,0)$ and $U(p)$ with $\lim_{X\to p} U(X)$ for $p \in \{\underline{X}, \overline{X}\}$. $\square$

B Intermediate Results

Lemma 33. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, differentiable and there exists a $\delta > 0$ such that for $|x| > \delta$, f is monotone. Then for $p \in \{-\infty, \infty\}$, $\lim_{x\to p} xf'(x) = 0$.*

Proof. Suppose that $\lim_{x\to\infty} \inf |x|f'(x) > 0$. Then there exists a $\delta_2, \varepsilon > 0$ such that when $|x| > \delta_2$, $|x|f'(x) > \varepsilon$. Then for $|x| > \delta_2$, $f'(x) > \frac{\varepsilon}{|x|}$. The antiderivative of $\frac{\varepsilon}{|x|}$ is $\varepsilon \ln |x|$ which converges to ∞ as $|x| \to p$. This violates the boundedness of f. Therefore $\lim_{x\to p} \inf |x|f'(x) \leq 0$. Similarly, $\lim_{x\to p} \sup |x|f'(x) \geq 0$.

Suppose that $\lim_{x\to p} \sup |x|f'(x) > 0$. For $|x| > \delta$, f is monotone and therefore $|x|f'(x)$ doesn't change sign. Therefore, if $\lim_{x\to p} \sup |x|f'(x) > 0$, then $\lim_{x\to p} \inf |x|f'(x) > 0$. This is a contradiction. Thus, $\lim_{x\to p} \sup |x|f'(x) = 0$. By similar reasoning, $\lim_{x\to p} \inf |x|f'(x) = 0$, and therefore $\lim_{x\to p} |x|f'(x) = 0$. Note this result also implies that $\lim_{x\to p} f'(x) = 0$. $\square$

Lemma 34. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and define $\overline{f}(x) = f(x)/x$. For $p \in \{-\infty, \infty\}$, if $\lim_{x\to p} \overline{f}(x) = c$ and $\lim_{x\to p} x\overline{f}'(x) = 0$, then $\lim_{x\to p} f'(x) = c$.*

Proof. Suppose $\lim_{x\to p} \overline{f}(x) = c$ and $\lim_{x\to p} x\overline{f}'(x) = 0$. Given $\overline{f}' = (f' - \overline{f})/x$, $\lim_{x\to p} f' = \lim_{x\to p}(x\overline{f}' + \overline{f}) = c$. $\square$

Lemma 35. *If $f : \mathbb{R} \to \mathbb{R} \in \mathcal{C}^2$ and there exists a $\delta > 0$ such that for $|x| > \delta$, f' is monotone increasing (decreasing), then there exists a δ_2 such that for $|x| > \delta_2$, $\overline{f}(x) \equiv f(x)/x$ is monotone increasing (decreasing).*

Proof. Note that $\overline{f}' = (1/x)(f' - \overline{f})$, $\overline{f}'' = (1/x)(f'' - 2\overline{f}')$ and if $f' = \overline{f}$, then $\overline{f}' = 0$ with a maximum if $f'' \leq 0$ and a minimum if $f'' \geq 0$. Let f' be monotone increasing for $|x| > \delta$ i.e. $f'' \geq 0$ for all $|x| > \delta$. From $\overline{f}'' = (1/x)(f'' - 2\overline{f}')$, if $\overline{f}' < 0$ and $f'' \geq 0$, then $\overline{f}'' > 0$ and $\overline{f}'$ is increasing. Suppose there exists a $\delta_2 > \delta$ such that $\overline{f}'(\delta_2) \geq 0$. Then, by continuity of $\overline{f}'$ and the fact that $\overline{f}' < 0$ and $f'' \geq 0 \Rightarrow \overline{f}'' > 0$, it is not possible to have $\overline{f}' < 0$ for $|x| > \delta_2$. Therefore, $\overline{f}\,' \geq 0$ for all $|x| > \delta_2$ and $\overline{f}$ is monotonically increasing for all $|x| > \delta_2$. Otherwise, $\overline{f}' < 0$ for all $|x| > \delta$, and therefore $\overline{f}$ is monotonically decreasing for all $x > \delta$. The proof is analogous when f' is monotone decreasing. $\square$

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